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Prigogine dynamical systems based on Bernoulli shifts of real functions

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ABSTRACT

The baker-transform and the related dynamical system is usually studied with the unit-square as its state-space. In this paper we define the baker-transform and related discrete-time dynamical system with the non-negative real numbers as state-space. Furthermore we consider the extension of the so defined baker transform to real-valued functions. As a mathematical instrument we use the generalized Walsh-Functions as originally introduced by Fine (1950). The Λ-transform of Prigogine is proven to be identical to dyadic convolution. It is shown that the method of Prigogine to construct irreversible processes can also in this case be applied successfully.

0. Introduction

The baker-transform is extensively studied in mathematics. Its representation as a Bernoulli shift operation is wellknown [1], [2]. I. Prigogine and his coworkers have shown how the baker-transform and the related discrete time dynamical system (baker-Prigogine dynamical system) can be used to construct irreversible processes. The essential mathematical tool for this is provided by the so-called Λ-transform [3], [4]. In a former report it was shown by the author that by use of Walsh Fourier analysis, the Λ-transform can be considered as a discrete dyadic convolution [5]. A similar approach has been suggested in [6]. In this paper we study the baker transform on the non-negative real numbers $\mathbf{R}^+$. It is shown that in this case the generalized Walsh functions as originally defined by N.J. Fine [7] can be used to study irreversibility in the line of Prigogine's work. The Λ-transform is equivalent in this case to the dyadic convolution defined on the non-negative real numbers $[0,\infty)$.

It is shown that dyadic differentiation as introduced by Butzer in analysis provides means to construct specific examples of the Λ-transform.

1. Preliminary Consideration

Prigogine (1977 Nobel price for chemistry) and his group explore complex phenomena by bottom up methods starting at deterministic microscopic models with conservative dynamics to be lifted up to macroscopic models with dissipative dynamics and collective variables for state definition. The hope is, to explore possible emerging properties of attracting chaos which can be observed in macroscopic patterns of behavior. Besides of physics and chemistry the field of biology and socio-economics is considered to be promising for the application of this approach in modeling.

In the work of Prigogine ([3], [4]) several examples serve for the demonstration of the method. Besides of the horse-shoe map of Smale the baker transform, a specific case of a K-system (K stands for Kolmogoroff) is used there to define conservative discrete-time dynamical systems with chaotic behaviors. For the case of the state space $L_2[0,1)^2$, consisting of the hilbert space of square-integrable functions on the unit-square $[0,1) \times [0,1)$, Prigogine is able to construct a unitary transformation Λ which allows to transform a conservative (reversible) dynamic process into a dissipative (and therefor irreversible) dynamic process and vice versa.

2. Representation of the baker transform on $[0,\infty)$

By **D**$[0,\infty)$ we denote the subgroup of the dyadic group **D** which consists of all sequences u which are 0-periodic to the left. Between $[0,\infty)$ and **D**$[0,\infty)$ we define the following (natural) map $e:[0,\infty)\rightarrow D[0,\infty)$. For $s\in[0,\infty)$ with $s=\sum_{i=-N}^{\infty} s_i 2^{-i}$ we define e(s):=u with $u(i)=s_i$ for all i with $-N\leq i<\infty$ and u(i):=0 for all i with $i<-N$. The map e is bijective and its inverse e^{-1}:**D**$[0,\infty) \rightarrow [0,\infty)$ therefor exists. It is easy to confirm that the Bernoulli-shiftoperation $\rightarrow$ in **D**$[0,\infty)$ corrresponds by e in $[0,\infty)$ to a multiplication by $2^{-1}s$.

$$e^{-1}(u \rightarrow 1) = 2^{-1}s \qquad (1)$$

From this result it is evident, that the baker-transform B – if restricted to the domain $Q_2[0,1)\times[0,1)$ – results in $[0,\infty)$ as an ordinary multiplication by 2^{-1}. In detail this can be seen as follows: For $s \in [0,\infty)$ and the baker-transform B we define the following map $B_\infty:[0,\infty)\to[0,\infty)$: For $s\in[0,\infty)$ the value $B_\infty(s)$ is given by

$B_\infty(s):=r(B(r^{-1}(s)))$, ·(2)

where r is the bijective map $r:Q_2[0,1)\times[0,1) \to [0,\infty)$ given by r(x,y)=s with $s=\sum_{i=-N}^{\infty} s_i 2^{-i}$ and

$s_i=x_{-i+1}$ for i=0,-1,-2,...,-N and $s_i=y_i$ for i=1,2,3,.... , where $x=\sum_{i=1}^{n} x_i 2^{-i}$ and y= $y=\sum_{i=1}^{\infty} y_i 2^{-i}$.

Since the baker transform B defined on $Q_2[0,1) \times [0,1)$ is represented in D[0, ∞) by a 1-step shift to the "right" (Bernoulli shift) it follows from (1) that

$B_\infty(s)=2^{-1}s$. (3)

We call B_∞ the *real baker-transform*. Although B_∞ looks rather trivial it can be considered as the representation of the (not so trivial) dyadic baker-transform on [0,∞). Next we study the effect of the real baker-transform for function $sf\in L^2[0,\infty)$. To begin with, we compute first the real baker-transform of the generalized Walshfunctions $\psi_\omega, \omega\in[0,\infty)$, as defined by Fine[7]. Here we repeat this definition.

For $\omega=\sum_{i=-N}^{\infty}\omega_i 2^{-i}$ and $s=\sum_{i=-M}^{\infty} s_i 2^{-i}$ the generalized Walshfunction $\psi_\omega:[0,1)\to \mathbf{R}$ is defined by

$$\psi_\omega(s)=\begin{cases} +1 & if \ \sum_i \omega_i s_{1-i} \ \ even \\ -1 & if \ \sum_i \omega_i s_{1-i} \ \ odd \end{cases} \quad (4)$$

In line with our earlier definition of B_∞ we define now the real baker-transform b_∞ of a function $f\in L^2[0,\infty)$ by $b_\infty f(s)=f(B_\infty(s))$. Using (3) we have in general

$b_\infty f(s)=f(2^{-1}s)$ (5)

For the case of a generalized Walshfunctions ψ_ω this results in

$b_\infty(\psi_\omega)(s)=\psi_\omega(2^{-1}s)=\psi_{2^{-1}\omega}(s)$ since in general we have $\psi_\omega(2^m s)=\psi_{2^m\omega}(s)$ for $m\in\mathbb{Z}$.

We see that for all (ω∈ [∞)) we have

$b_\infty(\psi_\omega)=\psi_{2^{-1}\omega}$ (6)

This result is analogous to the result which was derived in [5] for the dyadic Walshfunctions.

The theory of the generalized Walshfunctions as developed originally by Fine [7] shows that each function $f \in L^2[0,\infty)$ can be represented by an integral in the following form

$$f(s) = \int_0^\infty F(\omega)\psi_\omega(s)d\omega \tag{7}$$

Here the function F denotes the generalized Walsh-Fouriertransform which is given by

$$F(\omega) = \int_0^\infty f(s)\psi_\omega(s)ds \tag{8}$$

Using the representation (7) we get for $b_\infty(f)$ the following Walsh-Fourier representation

$$b_\infty(f)(s) = f(B_\infty)s)) = \int_o^\infty F(\omega)\psi_\omega(B_\infty(s))d\omega = \int_0^\infty F(\omega)\psi_\omega(2^{-1}s)d\omega = \int_0^\infty F(\omega)\psi_{2^{-1}\omega}(s)d\omega$$

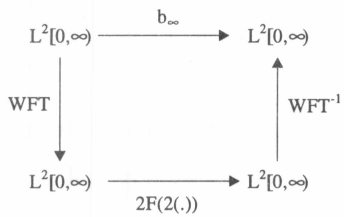
Substituting $2^{-1}\omega \mapsto \omega$ gives

$$b_\infty(f)(s) = \int_0^\infty 2F(2\omega)\psi_\omega(s)d\omega \tag{9}$$

We see that the generalized WF-transform of $b_\infty(f)$ is given by

$$WFT(b_\infty(f))(\omega) = 2F(2\omega) \tag{10}$$

The formula (10) allows an effective computation of the real baker-transform $b_\infty(f)$ by means of the generalized WF-transform by "chasing" the following commutative diagram. The computational efficiency is assured by the use of the existing fast algorithm for computing the generalized WF-transform (WFFT-algorithm).



(11)

This concludes the mathematical analysis of the real baker-transform. The next chapters will deal with systemstheoretical problems.

3. Baker-Prigogine Dynamical Systems on $L^2[0,\infty)$

We have seen, that the baker-transform *B*, if restricted to dyadic points, can be represented on $[0,\infty)$, using the real baker transform B_∞ by multiplication by 2^{-1}; $B_\infty(s)=2^{-1}s$ for $s \in [0,\infty)$. The inverse baker-transformation B^{-1} is then represented by B_∞^{-1} on $[0,\infty)$ by multiplication by 2; $B_\infty^{-1}(s) = 2s$. For the real baker-transformation of functions from $L^2[0,\infty)$ we have shown by (6) that $b_\infty \psi_\omega = \psi_{2^{-1}\omega}$. Similarly we have

$$b_\infty^{-1}\psi_\omega = \psi_{2\omega} \tag{12}$$

Furthermore we showed in (10) that $WFT(b_\infty f)(\omega) = 2F(2\omega)$. In a similar way it can be shown that for $f \in L^2[0,\infty)$ we have

$$WFT(b_\infty^{-1} f)(\omega) = \frac{1}{2}F(\frac{1}{2}\omega). \tag{13}$$

3.1 Real baker-dynamics

We define now in direct analogy to the definition of the baker-Gibbs dynamics which has been developed in [5] the appropriate dynamics for the state-space $L^2[0,\infty)$. The place of the evolution operator $u=b^{-1}$ of [5] is then taken by $U_\infty = b_\infty^{-1}$. A distribution function $\rho:\mathbf{E}\to\mathbf{R}$ considered in the book of Nicolis-Prigogine [4] corresponds in our dynamics to a function $\rho \in L^2[0,\infty)$.

Let the elements of $L^2[0,\infty)$ be considered as states of a discrete-time autonomous dynamical system. For the state ρ_t of this dynamics at time t, the state ρ_{t+1} at time t+1 is computed by

$$\rho_{t+1} = U_\infty \rho_t \tag{14}$$

The global state-transition function $\varphi(U_\infty)$ is then given by

$$\rho_t = U_\infty^{t-t_0}\rho_0 \tag{15}$$

We call the so defined dynamical system $rb := (\varphi(U_\infty), \mathbf{L}^2[0,\infty), [0,\infty))\cdot$ the *real baker-dynamics*. By means of the Walsh-Fourierintegral representation of a state ρ_t at time t, which is given by

$$\rho_t(s) = \int_0^\infty R_t(\omega)\psi_\omega(s)d\omega \tag{16}$$

we compute the next state ρ_{t+1} by

$$\rho_{t+1}(s) = (U_\infty \rho_t)(s) = \int_0^\infty R_t(\omega)\psi_{2\omega}(s)d\omega = \int_0^\infty \frac{1}{2}R_t(\frac{1}{2}\omega)\psi_\omega(s)dw$$

This means, that (as expected by (13))

$$R_{t+1}(\omega) = \frac{1}{2}R_t(\frac{1}{2}\omega) \tag{17}$$

Consequently, the WF-transform R_t *of* ρ_t can be computed from the initial state ρ_0 by

$$R_t(\omega) = 2^{-(t-t_0)} R_0(2^{-(t-t_0)}\omega) \tag{18}$$

Next we want to investigate the "energy-function" $H_\infty : L^2[0,\infty) \to [0,\infty)$of a real baker-dynamics rb.

We have in general

$$H_\infty(\rho_t) = \int_0^\infty \rho_t^2(s)ds \,.$$

For the difference $H_\infty(\rho_{t+1}) - H_\infty(\rho_t)$ we compute using (16), (17) and the parseval's equation

$$\int_0^\infty \rho_{t+1}^2(s)ds - \int_0^\infty \rho_t^2(s)ds = \int_0^\infty R_{t+1}^2(\omega)d\omega - \int_0^\infty R_t^2(\omega)d\omega = \int_0^\infty \frac{1}{2}R_t^2(\frac{1}{2}\omega)d\omega - \int_0^\infty R_t^2(\omega)d\omega =$$

$$\int_0^\infty R_t^2(\omega)d\omega - \int_0^\infty R_t^2(\omega)d\omega = 0$$

This result shows that the real baker-dynamics rb constitutes, as expected, a conservative dynamical system.

3.2 Λ-Transformation for the real baker-dynamics

Following the methods, as developed by Prigogine et al. [3], [4], now we want to assign to a real baker-dynamics rb an irreversible dynamical system. Let the real baker-dynamics rb as before be given by rb=($\varphi(U_\infty), L^2[0,\infty), [0,\infty)$).

Let Λ denote a dyadic convolution operator $\Lambda : L^2[0,\infty) \to L^2[0,\infty)$ which is generated by a weighting function l, that means that $\Lambda = l \otimes$. Now we require from Λ the following properties:

For the WF-transform L of l we want that for

$L(0) = 1$ and $0 \le \omega_1 < \omega_2 < \infty$ it should follow that $1 \ge L(\omega_1) > L(\omega_2) > 0$ (19)

In other words, by (19) we require that L starts with value 1 and is a positive monotonically decreasing function in the hilbert space $L^2[0,\infty)$.

In analogy to our approach in [5] we use Λ as a state assignment map to establish a dynamorphic image of the given baker-dynamics rb.

$$\tilde{\rho}_t = \Lambda\rho_t \quad (t \in [0,\infty) \tag{20}$$

The condition (19) as required in our case for a dyadic convolution operator Λ assures the existence of the inverse operator Λ^{-1}. For the evolution-operator W of the assigned dynamorphic system we have therefor

$$W_\infty = \Lambda U_\infty \Lambda^{-1} \tag{21}$$

Similarly as above we define for the state space $L^2[0,\infty)$ an associated discrete-time autonomous dynamical system with local state transition function W_∞. $\varphi(W_\infty)$ denotes the associated global state transition function derived by iterating W_∞.

We call the dynamical system $rbP := (\varphi(W_\infty), L^2[0,\infty), [0,\infty))$ defined in this way, the real baker-Prigogine dynamics.

Since $U_\infty = \Lambda^{-1} W_\infty \Lambda$ and $\rho_t = \Lambda^{-1}\tilde{\rho}_t$ we have a one to one correspondence between the real baker-dynamics rb and the associated real baker-Prigogine dynamics rbP.

3.3 Irreversibility of the real baker-Prigogine dynamics

As in the case of the baker-Prigogine dynamics which is investigated in [5] there is also in the case of a real baker-Prigogine dynamics rbP the proof of its irreversibility of most importance. We approach this problem by investigating its "energy-function" H_∞ which is given by

$$H_\infty(\tilde{\rho}_t) := \int_0^\infty \tilde{\rho}_t^2(s)ds \tag{22}$$

Let $t > t_0 \geq 0$. We want to compute $H_\infty(\tilde{\rho}_t) - H_\infty(\tilde{\rho}_0)$. We have

$$H_\infty(\tilde{\rho}_t) = \int_0^\infty \tilde{\rho}_t^2(s)ds = \int_0^\infty \tilde{R}_t^2(\omega)d\omega = \int_0^\infty L^2(\omega)R_t^2(\omega)d\omega = \int_0^\infty L^2(\omega)2^{t_0-t}R_0^2(2^{t_0-t}\omega)d\omega$$

Substitution of $2^{t_0-t}\omega \to \omega$ gives finally $H_\infty(\tilde{\rho}_t) = \int_0^\infty L^2(2^{t-t_0}\omega)R_0^2(\omega)d\omega$. With this result we have then

$$H_\infty(\tilde{\rho}_t) - H_\infty(\tilde{\rho}_{t_0}) = \int_0^\infty (L^2(2^{t-t_0}\omega) - L^2(\omega))R_0^2(\omega)d\omega < 0$$

since for $t>t_0$ follows by (19) $L^2(2^{t-t_0}\omega) - L^2(\omega) < 0$.

This shows that the "energy-function" H_∞ of a real baker-Prigogine dynamics rbP is a Ljapunov-functional and the transformation Λ^2 is a Ljapunov-variable. Therefor the dynamical system $rbP = (\varphi(W_\infty), L^2[0,\infty), [0,\infty))$ is irreversible.

As a result we see, that the Λ–transform of Prigogine, which is in our case of real function analysis given by the dyadic convolution of real functions defined on [0,∞) provides a state isomorphisms to associate to a conservative reversible real baker-dynamics rb the corresponding dissipative irreversible real baker-Prigogine dynamics rbP.

3.4 Realization by dyadic differentiation

In this final section we investigate how the Λ–transformation of a real baker-dynamics can be realized by dyadic differentiation. For a function $f \in L^2[0,\infty)$ we denote by $f^{[1]}$ the (strong) Gibbs-derivative, which is given by (see for example [9], [10])

$$f^{[1]}(s) = \lim_{n\to\infty} \sum_{j=0}^{n} 2^{j-1}(f(s) - f(s \oplus 2^{-j-1})) \qquad (23)$$

It is known that for the generalized Walshfunctions ψ_ω, $(\omega \in [0,\infty))$, we have

$$\psi_\omega^{[1]} = \omega \psi_\omega \qquad (24)$$

(24) shows that the generalized Walshfunctions ψ_ω are the eigenfunctions of the Gibbs-differential operator $D^{[1]}$ which is defined by $D^{[1]} f = f^{[1]}$.

However, by (24) it is also evident that $D^{[1]}$ is a dyadic convolution operator, that means, that there exists a function $d \in L^2[0,\infty)$, such that $D^{[1]} = d \otimes$ and furthermore that $f^{[1]}$ can consequently be represented by the integral

$$f^{[1]}(s) = \int_0^\infty d(t \otimes s) f(t) dt \qquad (25)$$

Let D denote the WF-transform of d. Then by the dyadic convolution theorem and considering (24) we have

$$D(\omega) = \omega \qquad (26)$$

By means of D we define the transformation $L_D : [0,\infty) \to \mathbf{R}$ by

$$L_D := e^{-D} \qquad (27)$$

This menas, that we havé

$$L_D(\omega) = e^{-\omega} \qquad (28)$$

We see, that $L_D(0) = 1$ and furthermore that for $0 \le \omega_1 < \omega_2 < \infty$ we have

$1 \ge L_D(\omega_1) > L_D(\omega_2) > 0$.

This shows that L_D meets the requirements (19). Consequently $\Lambda_D = l_D \otimes$ where l_D is given by

$$l_D(s) = \int_0^\infty L_D(\omega)\psi_\omega(s)d\omega \qquad (29)$$

is a suitable Λ–transformation in the sense of Prigogine.

This means that $(\varphi(W_\infty(D)), L^2[0,\infty), [0,\infty)$, where

$$W_\infty(D) = \Lambda_D U_\infty \Lambda_D^{-1} \qquad (30)$$

is a specific real baker-Prigogine dynamics.

The following generalization of this specific example of a real baker-Prigogine dynamics is straightforward. For non-negative real numbers $c_1, c_2, \ldots, c_n$ we define by $D^{[1]}$ a polynomial operator by

$$p(D^{[1]}) := c_1 D^{[1]} + c_2 D^{[2]} + \ldots + c_n D^{[n]} \qquad (31)$$

Application of $p(D^{[1]})$ to ψ_ω gives

$$p(D^{[1]})\psi_\omega = (c_1 D^{[1]} + c_2 D^{[2]} + \ldots + c_n D^{[n]})\psi_\omega = (c_1\omega + c_2\omega^2 + \ldots + c_n\omega^n)\psi_\omega \qquad (32)$$

Together with $D^{[1]}$ also $p(D^{[1]})$ is a dyadic convolution operator. Therefor there exists a function $p(d) \in L^2[0,\infty)$ such that $p(D^{[1]}) = p(d)\otimes$. The WF-transform WFT(p(d)) of p(d) is by (8) given here by the polynomial

$$p(D) = c_1 D + c_2 D^2 + \ldots + c_n D^n \quad with \quad p(D)(\omega) = c_1\omega + c_2\omega^2 + \ldots + c_n\omega^n \qquad (33)$$

Let now $L_{p(D)}$ be defined by

$$L_{p(D)} := e^{-p(D)} \qquad (34)$$

Then we have

$$L_{p(D)}(\omega) := e^{-p(c_1\omega + c_2\omega_2 + \ldots + c_n\omega^n})} \qquad (35)$$

The function $L_{p(D)}$ meets in analogy to L_D the requirements (19) and we have with the dynamical system $(\varphi(W_\infty p(D)), L^2[0,\infty), [0,\infty)$ where

$$W_\infty(p(D)) = \Lambda_{p(D)} U_\infty \Lambda_{p(D)}^{-1} \qquad (36)$$

a further example for a real baker-Prigogine dynamics.

Its Λ–transformation $\Lambda_{p(D)}$ is given by

$$\Lambda_{p(D)} = l_{p(D)}\otimes \qquad (37)$$

The weighting function $l_{p(D)}$ can be computed again by the inverse WF-transformation of $L_{p(D)}$

$$l_{p(D)}(s) = \int_0^\infty L_{p(D)}(\omega)\psi_\omega(s)d\omega \tag{38}$$

4. Summary and Conclusion

Starting from the example of the wellknown 2-dimensional baker transform B as considered in Prigogine works [3], [4] which has been investigated further by the author in [5], the restriction of B on the set of dyadic rational numbers on the horizontal line is interpreted as the real baker transform b_∞ for the set $[0,\infty)$ of nonnegative real numbers. B_∞ is trivially given by the division of a real number $s \in [0,\infty)$ by 2; $B_\infty(s) = 2^{-1}s$. For real functions f the real baker-transform b_∞ is defined by $(b_\infty f)(s) := f(2^{-1}s)$. In consequence it follows that the real-valued Walshfunctions ψ_ω of Fine are with respect to b_∞ a closed set of functions since $b_\infty\psi_\omega = \psi_{2^{-1}\omega}$ for all $\omega \in [0,\infty)$.

With respect to the spectral domain of Walsh-Fourier analysis it follows, that the real baker transform of functions results in an inverse real baker transform of the associated Walsh-Fourier Spectrum. This property allows to prove that the energy function of the discrete time system rb with the real baker transform a state transition is conservative. However, if we use a dyadic invertible convolution operator l with non-decreasing spectral representation Λ to assign the states from rb to rbP (real baker Prigogine dynamics) we find that rbP is dissipative and irreversible.

The paper shows how the well known example of Prigogine for the Λ-transform can be demonstrated by the most simplest case with the interval $[0,\infty)$ as state space and where Λ is defined by dyadic convolution. Furthermore our paper demonstrates the construction of different possible Λ-transforms using the concept of dyadic differentiation. From our viewpoint it might be possible to find models for practical examples taken from physics, chemistry or biology which use the approach of Prigogine to associate a (microscopic) dynamical system of conservative and reversible kind to a (mesoscopic) dynamical system with collective variables which is dissipative and irreversible.

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