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Unified Theory of Obtaining a more General Class of Generating Relations from a given Class of Quasi-Bilateral Generating Relations involving some Special Functions by Group Theoretic Method

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Abstract

In this paper, the author has made a modest attempt to present a unified theory of obtaining a more general class of generating relations starting from a given class of quasi-bilateral generating relations involving some special functions from the group theoretic view point. A good number of theorems in connection with the extensions of bilateral generating functions for various special functions are obtained while applying our theorem on various special functions.

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1 Introduction

In their paper [1], Chakraborti and Chatterjea introduced the term "quasi-bilateral generating relations" as follows:

$$G(x,z,w)=\sum_{n=0}^{\infty} a_n w^n p_n^{(\alpha)}(x) q_m^{(n)}(z), \tag{1.1}$$

where the coefficients a_n's are arbitrary and $p_n^{(\alpha)}(x)$ and $q_m^{(n)}(z)$ are two special functions of order n and m and of parameters α and n, respectively.

In the present article, the author has discussed a group theoretic method for deriving a unified presentation of a more general class of generating relations from a given class of quasi-bilateral generating relations involving some special functions subject to the condition of construction of one parameter continuous transformations group for the special functions under consideration. Furthermore, it is interesting to mention that while applying our theorem on various special functions, we get a good number of theorems in connection with the extensions of bilateral generating functions for the special functions under consideration found derived in the literature and in recent works. The detailed discussion is given below.

2 Group-Theoretic Discussion

We first consider a quasi-bilateral generating relation:

$$G(x,u,w)=\sum_{n=0}^{\infty} a_n p_n^{(\alpha)}(x) q_m^{(n)}(u) w^n. \tag{2.1}$$

Replacing w by $wztv$ and then multiplying both sides of (2.1) by y^α, we get

$$y^\alpha G(x,u,wztv)=\sum_{n=0}^{\infty} a_n (p_n^{(\alpha)}(x) y^\alpha z^n)(q_m^{(n)}(u) t^n)(wv)^n. \tag{2.2}$$

Let us now suppose that for the above special functions $p_n^{(\alpha)}(x)$ and $q_m^{(n)}(u)$, it is possible to define two linear partial differential operators each of which generates one parameter continuous transformations group as follows:

$$R_1 = \zeta_1(x,y,z)\frac{\partial}{\partial x} + \zeta_2(x,y,z)\frac{\partial}{\partial y} + \zeta_3(x,y,z)\frac{\partial}{\partial z} + \zeta_0(x,y,z)$$

and

$$R_2 = \eta_1(u,t)\frac{\partial}{\partial u} + \eta_2(u,t)\frac{\partial}{\partial t} + \eta_0(u,t)$$

such that

$$(2.3)\qquad \begin{cases} R_1(p_n^{(\alpha)}(x)y^\alpha z^n) &= C'_n p_{n+1}^{(\alpha-1)}(x)y^{\alpha-1}z^{n+1} \\ \text{and} & \\ R_2(q_m^{(n)}(u)t^n) &= C''_n q_m^{(n+1)}(u)t^{n+1}. \end{cases}$$

So we can assume that the groups generated by R_1 and R_2 are as follows:

$$(2.4)\qquad \begin{cases} e^{wR_1}f(x,y,z) &= \Omega'(x,y,z)f(g_1(x,y,z),h_1(x,y,z),k(x,y,z)) \\ \text{and} & \\ e^{wR_2}f(u,t) &= \Omega''(u,t)f(g_2(u,t),h_2(u,t)). \end{cases}$$

Operating both sides of (2.2) by $e^{wR_1}e^{wR_2}$, we get

$$(2.5)\qquad e^{wR_1}e^{wR_2}(y^\alpha G(x,u,wztv)) = e^{wR_1}e^{wR_2}\sum_{n=0}^{\infty} a_n(p_n^{(\alpha)}(x)y^\alpha z^n)(q_m^{(n)}(u)t^n)(wv)^n.$$

Now the left hand side of (2.5), with the help of (2.4), becomes

$$(2.6)\qquad \Omega'(x,y,z)\Omega''(u,t)(h_1(x,y,z))^\alpha G(g_1(x,y,z),g_2(u,t),wvh_2(u,t)k(x,y,z)).$$

The right hand side of (2.5), with the help of (2.3), becomes

$$(2.7)\qquad \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_n v^n \frac{w^{n+r+s}}{r!s!} C'_n C'_{n+1}\cdots C'_{n+r-1} p_{n+r}^{(\alpha-r)}(x)$$
$$\times y^{\alpha-r}z^{n+r} C''_n C''_{n+1}\cdots C''_{n+s-1} q_m^{(n+s)}(u)t^{n+s}.$$

Now equating (2.6) and (2.7) and then putting $y=z=t=1$, we get

$$\Omega'(x,1,1)\Omega''(u,1)(h_1(x,1,1))^\alpha \times G(g_1(x,1,1),g_2(u,1),vwh_2(u,1)k(x,1,1))$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_n C_1 C_2 v^n \frac{w^{n+r+s}}{n!s!} p_{n+r}^{(\alpha-r)}(x) q_m^{(n+s)}(u)$$

where

$C_1 = \prod_{i=0}^{r-1} C'_{n+i}$

and

$C_2 = \prod_{j=0}^{s-1} C''_{n+j}$.

Thus we arrive at the following theorem:

Theorem 1 If

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n p_n^{(\alpha)}(x) q_m^{(n)}(u) w^n$$

then

$$\Omega'(x,1,1)\Omega''(u,1)(h_1(x,1,1))^{\alpha} \times G(g_1(x,1,1), g_2(u,1), vwh_2(u,1)k(x,1,1))$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_n C_1 C_2 v^n \frac{w^{n+r+s}}{r!s!} p_{n+r}^{(\alpha-r)}(x) q_m^{(n+s)}(u)$$

where

$C_1 = \prod_{i=0}^{r-1} C'_{n+i}$

and

$C_2 = \prod_{j=0}^{s-1} C''_{n+j}$

which is our desired result.

A good number of applications are given below.

3 Applications

(a) At first we take

$$p_n^{(\alpha)}(x) = L_{a,b,m,n}(x) \text{ with } \alpha = m, \text{ and}$$
$$q_m^{(n)}(u) = L_{a,b,n,l}(u) \text{ with } m = l.$$

Then from [2,3] we see that

$$R_1 = bxy^{-1}z\frac{\partial}{\partial x} + zb\frac{\partial}{\partial y} - y^{-1}z(ax+b)$$

$$R_2 = \frac{b}{a}t\frac{\partial}{\partial u} - t$$

$$C'_1 = (n+1)$$

$$C''_2 = -1$$

$$\Omega'(x,y,z) = (1 + \frac{bwz}{y})^{-1} exp(\frac{-axzw}{y})$$

$$\Omega''(u,t) = exp(-wt)$$

$$g_1(x,y,z) = x(1 + \frac{bwz}{y}), \quad h_1(x,y,z) = y(1 + \frac{bwz}{y}), \quad k(x,y,z) = z,$$

$$g_2(u,t) = u + \frac{b}{a}wt, \quad h_2(u,t) = t.$$

Then by the application of our theorem, we at once get the following result involving modified Laguerre polynomials:

Theorem 2 If

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n L_{a,b,m,n}(x) L_{a,b,n,l}(u) w^n$$

then

$$exp(-w - axw)(1 + bw)^{m-1} G(x(1 + bw), u + \frac{b}{a}w, wz)$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_n v^n \frac{(-1)^s (n+1)_r}{s!r!} w^{n+r+s} L_{a,b,m-r,n+r}(x) L_{a,b,n+s,l}(u).$$

Corollary 1 If we put $l = 0$, in the above theorem we get the result found derived in [4].

(b) We now take

$$p_n^{(\alpha)}(x) = C_n^{\lambda}(x) \quad \text{with} \quad \alpha = \lambda,$$
$$q_m^{(n)}(u) = C_m^{n}(u).$$

Then from [5,6] we see that

$$R_1 = (x^2 - 1) y^{-1} z \frac{\partial}{\partial x} + 2xz \frac{\partial}{\partial y} - xy^{-1} z,$$

$$R_2 = ut \frac{\partial}{\partial u} + 2t^2 \frac{\partial}{\partial t} + mt$$

$$C_n' = \frac{(n + 2\lambda - 1)(n+1)}{2(\lambda - 1)}$$

$$C_n'' = 2n$$

$$\Omega'(x,y,z) = \{1 + 2wx\frac{z}{y} + w^2(x^2 - 1)\frac{z^2}{y^2}\}^{-\frac{1}{2}}$$

$$\Omega''(u,t) = (1 - 2wt)^{-\frac{m}{2}}$$

$$g_1(x,y,z) = x + w(x^2-1)\frac{z}{y}$$

$$h_1(x,y,z) = y\{1 + 2wx\frac{z}{y} + w^2(x^2-1)\frac{z^2}{y^2}\}$$

$$k_1(x,y,z) = z$$

$$g_2(u,t) = \frac{u}{(1-2wt)^{\frac{1}{2}}}$$

$$h_2(u,t) = \frac{t}{1-2wt}.$$

Then by the application of our theorem, we at once obtain the following result involving Gegen bauer polynomials:

Theorem 3 If

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n C_n^{\lambda}(x) C_m^n(u) w^n$$

then

$$(1-2w)^{-m/2}\{1+2wx+w^2(x^2-1)\}^{\lambda-\frac{1}{2}} \times G(x+w(x^2-1), u/(1-2w)^{1/2}, wv/(1-2w))$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(-n-2\lambda+1)_r (n+1)_r}{(-\lambda+1)_r \ r!} \frac{(n)_s}{s!} 2^{s-r} a_n v^n w^{n+r+s} C_{n+r}^{\lambda-r}(x) C_m^{n+s}(u).$$

Corollary 2 If we put $m = 0$, in the above theorem we get the result found derived in [6].

(c) We now take

$$p_n^{(\alpha)}(x) = Y_n^{(\alpha)}(x), \quad q_m^{(n)}(u) = Y_m^{(n)}(u).$$

Then from [7,8], we notice that

$$R_1 = x^2 y^{-1} z \frac{\partial}{\partial x} + xz\frac{\partial}{\partial y} + xy^{-1}z^2\frac{\partial}{\partial z} + y^{-1}z(\beta - x)$$

$$R_2 = ut\frac{\partial}{\partial u} + u^2\frac{\partial}{\partial t} + (m-1)u$$

$$C_1' = \beta, C_2'' = (m+n-1)$$

$$\Omega'(x,y,z) = (1 - wxy^{-1}z)exp(\beta wy^{-1}z)$$

$$\Omega''(u,t) = (1-wt)^{-m+1}$$

$$g_1(x,y,z) = \frac{x}{1-wxy^{-1}z}$$
$$h_1(x,y,z) = \frac{y}{1-wxy^{-1}z}$$
$$k(x,y,z) = \frac{z}{1-wxy^{-1}z}$$
$$g_2(u,t) = \frac{u}{1-wt}$$
$$h_2(u,t) = \frac{t}{1-wt}.$$

Then by the application of our theorem, we get the following result involving generalised Bessel polynomials:

Theorem 4 If

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n Y_n^{(\alpha)}(x) Y_m^{(n)}(u) w^n$$

then

$$exp(\beta w)(1-w)^{-m+1}(1-wx)^{-\alpha+1} G(\frac{x}{1-wx}, \frac{u}{1-w}, \frac{wv}{(1-w)(1-wx)})$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{\beta^r}{r!}\frac{(m+n-1)_s}{s!} a_n v^n w^{n+r+s} Y_{n+r}^{(\alpha-r)}(x) Y_m^{(n+s)}(u).$$

Corollary 3 If we put $m = 0$, in the above theorem we get the result found derived in [9].

(d) We now take

$$p_n^{(\alpha)}(x) = P_n^{(\alpha,\beta)}(x), \quad q_m^{(n)}(u) = P_m^{(n,\beta)}(u).$$

Then from [10,11], we notice that

$$R_1 = (1-x^2)y^{-1}z\frac{\partial}{\partial x} - z(1+x)\frac{\partial}{\partial y} + z^2y^{-1}(1-x)\frac{\partial}{\partial z} + (1-\beta)(1-x)y^{-1}z$$

$$R_2 = (1+u)t\frac{\partial}{\partial u} + t^2\frac{\partial}{\partial t} + (1+\beta+m)t$$
$$C_n' = -2(n+1), \quad C_n'' = (1+n+\beta+m)$$
$$\Omega'(x,y,z) = (\frac{y}{y+w(x-1)z})^{\beta+1}$$
$$\Omega''(u,t) = (1-wt)^{-1-\beta-m}$$
$$g_1(x,y,z) = \frac{xy-w(x-1)z}{y+w(x-1)z}, \quad h_1(x,y,z) = \frac{y(y-2wz)}{y+w(x-1)z}, \quad k = \frac{yz}{y+w(x-1)z}$$

$$g_2(u,t)=\frac{u+wt}{1-wt},\quad h_2(u,t)=\frac{t}{1-wt}.$$

Then by the application of our theorem, we get the following result on Jacobi polynomials:

Theorem 5 If

$$G(x,u,w)=\sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(x) P_m^{(n,\beta)}(u) w^n$$

then

$$(1-w)^{-1-\beta-m}(1-2w)^{\alpha}\{1+w(x-1)\}^{-1-\alpha-\beta}$$

$$\times G\Big(\frac{x-w(x-1)}{1+w(x-1)},\frac{u+w}{1-w},\frac{wv}{(1-w)\{1+w(x-1)\}}\Big)$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-2)^r(n+1)_r}{r!}\frac{(1+n+\beta+m)_s}{s!}a_n v^n w^{n+r+s}\times P_{n+r}^{(\alpha-r,\beta)}(x)P_m^{(n+s,\beta)}(u).$$

Corollary 4 If we put $m=0$, in the above theorem we get the result found derived in [12].

(e) We now take

$$P_n^{(\alpha)}(x)=P_n^{(k,\alpha)}(x),\quad q_m^{(n)}(u)=P_m^{(k,n)}(u).$$

Then from [13,14], we notice that

$$R_1=(1-x^2)y^{-1}z\frac{\partial}{\partial x}+(1-x)z\frac{\partial}{\partial y}-y^{-1}z^2(1+x)\frac{\partial}{\partial z}-(1+x)(1+k)y^{-1}z$$

$$R_2=(1-u)t\frac{\partial}{\partial u}-t^2\frac{\partial}{\partial t}-(1+k+m)t$$

$$C_n'=-2(n+1),\quad C_n''=-(1+k+n+m)$$

$$\Omega'(x,y,z)=\{1+w(1+x)y^{-1}z\}^{-1-k}$$

$$g_1(x,y,z)=\frac{x+w(1+x)y^{-1}z}{1+w(1+x)y^{-1}z},\quad h_1(x,y,z)=\frac{y(1+2wy^{-1}z)}{1+w(1+x)y^{-1}z},$$

$$k(x,y,z)=\frac{z}{1+w(1+x)y^{-1}z}$$

$$\Omega''(u,t)=(1+tw)^{-1-k-m}$$

$$g_2(u,t)=\frac{u+tw}{1-tw},\quad h_2(u,t)=\frac{t}{1-tw}$$

Then by the application of our theorem, we get the following result on Jacobi polynomials:

Theorem 6 If

$$G(x,u,w)=\sum_{n=0}^{\infty}a_nP_n^{(k,\alpha)}(x)P_m^{(k,n)}(u)w^n$$

then

$$(1+w)^{-1-k-m}(1+2w)^{\alpha}\{1+w(1+x)\}^{-1-k-\alpha}$$

$$\times G(\frac{x+w(1+x)}{1+w(1+x)},\frac{u+w}{1+w},\frac{wv}{(1+w)\{1+w(1+x)\}})$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}a_n\frac{w^{n+r+s}}{r!s!}v^n(-2)^r(n+1)_r(-1)^s(1+k+n+m)_s\times P_{n+r}^{(k,\alpha-r)}(x)P_m^{(k,n+s)}(u).$$

Corollary 5 If we put $m=0$, in the above theorem we get the result found derived in [15].

(f) We now take

$$p_n^{(\alpha)}(x)={}_2F_1(-n,\beta;\nu;x)\quad\text{with}\quad\alpha=\nu,$$
$$q_m^{(n)}(u)={}_2F_1(-m,\beta;n;u).$$

Then from [16,17], we notice that

$$R_1=x(1-x)y^{-1}z\frac{\partial}{\partial x}+z\frac{\partial}{\partial y}-(x\beta+1)y^{-1}z$$

$$R_2=(1-u)t\frac{\partial}{\partial u}+t^2\frac{\partial}{\partial t}+(m-\beta)t$$

$$C_n'=(\nu-1),\quad C_n''=\frac{(n+m)(n-\beta)}{n}$$

$$\Omega'(x,y,z)=(1+\frac{z}{y}w)^{-1}(1+x\frac{z}{y}w)^{-\beta}$$

$$\Omega''(u,t)=(1-wt)^{\beta-m}$$

$$g_1(x,y,z)=\frac{x(1+\frac{z}{y}w)}{1+x\frac{z}{y}w},\quad h_1(x,y,z)=y(1+\frac{z}{y}w),\quad k(x,y,z)=z$$

$$g_2(u,t)=u+(1-u)tw,\quad h_2(u,t)=\frac{t}{1-tw}$$

then by the application of our theorem, we get the following result on Hypergeometric polynomials:

Theorem 7 If

$$G(x,u,w)=\sum_{n=0}^{\infty}a_n{}_2F_1(-n,\beta;\nu;x){}_2F_1(-m,\beta;n;u)w^n$$

then

$$(1+w)^{\nu-1}(1-w)^{\beta-m}(1+xw)^{-\beta}G(\frac{x(1+w)}{1+xw},u+(1-u)w,\frac{wv}{1-w})$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}a_n(-1)^r(1-\nu)_r\frac{(n+m)_s(n-\beta)_s}{(n)_s}v^n\frac{w^{n+r+s}}{r!s!}$$
$$\times{}_2F_1(-(n+r),\beta,\nu-r;x){}_2F_1(-m,\beta;n+s;u).$$

Corollary 6 If we put $m=0$ in the above theorem we get the result found derived in [18].

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