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Note on operational quantities and Mil'man isometry spectrum

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ABSTRACT

In this note, using Mil'man isometry spectrum and some Tsirelson-like examples of Banach spaces, due to Figiel and Johnson, and Schlumprecht, we show that two operational quantities that have been used to characterize upper semi-Fredholm operators, are not equivalent.

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INTRODUCTION

Let X,Y be infinite dimensional Banach spaces and let $L(X,Y)$ be the class of all (linear continuous) operators acting between X and Y.

V. D. Mil'man [7] introduced the *isometry spectrum* $I(T)$ of $T\in L(X,Y)$ in the following way:

$$I(T) := \{\alpha\geq 0 : \forall\varepsilon>0,\ \exists M\in S_\infty(X),\ \forall x\in S_M,\ |\,\|Tx\|-\alpha\,|<\varepsilon\} ,$$

where $S_\infty(X)$ is the set of all infinite dimensional (closed) subspaces of X and $S_M := \{x\in M : \|x\|=1\}$ is the unit sphere of $M\in S_\infty(X)$. Clearly $I(T)$ is closed and for every $M\in S_\infty(X)$ we have $I(TJ_M)\subset I(T)$, where J_M is the canonical inclusion of M in X.

We shall consider the norm of T, $n(T) := \|T\|$, the injection modulus of T, $j(T) := \inf\{\|Tx\| : \|x\|=1\}$, and the following operational quantities:

$$in(T) := \inf\{n(TJ_M) : M\in S_\infty(X)\} \quad [8] ,$$

$$sin(T) := \sup\{in(TJ_M) : M\in S_\infty(X)\} \quad [8] ,$$

$$sj(T) := \sup\{j(TJ_M) : M\in S_\infty(X)\} \quad [8] ,$$

$$isj(T) := \inf\{sj(TJ_M) : M\in S_\infty(X)\} \quad [4,6] .$$

These operational quantities have been used to characterize the classes SF_+ of all upper semi-Fredholm operators (operators with closed range and finite dimensional kernel) and SS of all strictly singular operators (operators whose restriction to an infinite dimensional subspace is never an isomorphism) in the following way:

$$T\in SF_+ \iff in(T)>0 \iff isj(T)>0 \quad [8;4,6] ,$$

$$T\in SS \iff sin(T)=0 \iff sj(T)=0 \quad [8] .$$

It is not difficult to derive from the definitions the following inequalities:

$$isj \leq in \leq sin \ ,$$

$$isj \leq sj \leq sin \ .$$

We relate the above quantities and the Mil'man isometry spectrum. We prove that

$$0 \in I(T) \Leftrightarrow T \notin SF_+ \ ,$$

$$I(T) \subset [in(T), sj(T)] \ ,$$

$$sin(T) = sj(T) \Leftrightarrow sin(T) = max \ I(T) \ ,$$

$$in(T) = isj(T) \Leftrightarrow isj(T) = min \ I(T) \ .$$

Using these results and some Tsirelson-like examples of Banach spaces, due to Figiel and Johnson [3] and Schlumprecht [9], we show that the quantities isj and in do not coincide, although they characterize the class SF_+ . Moreover, they are not equivalent, in the sense that there is no positive number $\delta > 0$ such that $\delta . in \leq isj$. Analogously, the quantities sj and sin are not equivalent. Hence we obtain that the quantities in and sj are not comparable.

THE RESULTS

Let X,Y be infinite dimensional Banach spaces and $T \in L(X,Y)$.

The first result assures that $I(T)$ is non-empty if $T \notin SF_+$; in particular if $T \in SS$.

1 PROPOSITION. $0 \in I(T) \iff in(T)=0$.

Proof. It is enough to note that $0 \in I(T)$ if and only if for every $\varepsilon>0$ there exists $M \in S_\infty(X)$ such that

$$0 \le j(TJ_M) \le n(TJ_M) \le \varepsilon \ . \qquad \blacksquare$$

O. J. Beucher [1,2] shows that $I(T) \subset [in(T), sin(T)]$. In the following proposition we give a better result:

2 PROPOSITION. $I(T) \subset [in(T), sj(T)]$

Proof. Note that $\alpha \in I(T)$ if and only if for every $\varepsilon>0$ there exists $M \in S_\infty(X)$ such that

$$\alpha-\varepsilon \le j(TJ_M) \le n(TJ_M) \le \alpha+\varepsilon \ .$$

Consequently, if $\alpha \in I(T)$, then

$$\alpha \le sj(T) \quad \text{and} \quad in(T) \le \alpha \ ;$$

hence $I(T) \subset [in(T), sj(T)]$. $\blacksquare$

From Propositions 1 and 2 we obtain

3 COROLLARY. $T \in SS \Rightarrow I(T)=\{0\}$.

In the Introduction we noted that $I(T)$ is closed and bounded, hence it has a maximum and a minimum if it is non-empty.

4 PROPOSITION.

(1) $sin(T) = sj(T) \iff sin(T) = max\ I(T)$.

(2) $in(T) = isj(T) \iff isj(T) = min\ I(T)$.

Proof. (1) Let $\alpha:=sin(T)=sj(T)$. For every $\varepsilon>0$ there exists $M\in S_\infty(X)$ such that

$$j(TJ_M) > sj(T)-\varepsilon = \alpha-\varepsilon .$$

Because $in(T)\leq sin(T)=\alpha$, there exists $N\in S_\infty(M)$ such that

$$n(TJ_N) < in(TJ_M)+\varepsilon \leq sin(T)+\varepsilon = \alpha+\varepsilon .$$

Hence

$$\alpha-\varepsilon < j(TJ_M) \leq j(TJ_N) \leq n(TJ_N) < \alpha+\varepsilon .$$

Consequently $sj(T)\in I(T)$ and it coincides with $max\ I(T)$.

Conversely, assume $sin(T)\in I(T)$. From $I(T) \subset [in(T),sj(T)]$ we obtain $sin(T)\leq sj(T)$. Because $sj(T)\leq sin(T)$ we have $sj(T)=sin(T)$.

(2) The proof is analogous to (1). Let $\alpha:=isj(T)=in(T)$. For every $\varepsilon>0$ there exists $M\in S_\infty(X)$ such that

$$\alpha = in(T) \leq n(TJ_M) < in(T)+\varepsilon = \alpha+\varepsilon .$$

Because $isj(T)\leq sj(TJ_M)$, there exists $N\in S_\infty(M)$ such that

$$\alpha-\varepsilon = isj(T)-\varepsilon \leq sj(TJ_M)-\varepsilon < j(TJ_N).$$

Hence

$$\alpha-\varepsilon < j(TJ_N) \leq n(TJ_N) \leq n(TJ_M) < \alpha+\varepsilon .$$

Consequently $in(T)\in I(T)$ and coincides with $min\ I(T)$.

Conversely, assume $isj(T)\in I(T)$. From $I(T) \subset [in(T),sj(T)]$ we obtain $in(T)\leq isj(T)$. Because $isj(T)\leq in(T)$ we have that $isj(T)=in(T)$. ■

Next we present some examples showing the non-equivalence of quantities mentioned in the introduction.

T. Schlumprecht [9] introduced the following concept of λ-distortable space:

5. DEFINITION. *Let $(X,\|.\|)$ be an infinite dimensional Banach space, and $\lambda>1$. X is called λ-distortable if there exists an equivalent norm $|.|$ on X such that for each infinite dimensional subspace M of X we have*

$$sup \left\{ \frac{|x|}{|y|} : x,y \in M \ , \ \|x\|=\|y\|=1 \right\} \ \geq \ \lambda \ .$$

6. OBSERVATION. Mil'man proved that every uniformly convex space containing no distortable subspaces must contain a copy of l_p $(1<p<\infty)$ [7;sect.3; p.145]. Hence the uniformly convex Banach space containing no isomorphic copies of l_p $(1<p<\infty)$ given in [3] has a λ-distortable subspace M.

7. EXAMPLE. We give an operator T such that $I(T)=\emptyset$; hence, by Proposition 4, $isj(T)<in(T)$ and $sj(T)<sin(T)$.

Suppose X is λ-distortable Banach space, for some $\lambda>1$. We consider the isomorphism

$$T: (X,\|.\|) \longrightarrow (X,|.|) \quad , \quad Tx := x \ ,$$

where $|.|$ is the equivalent norm given in Definition 5.

For every $M\in S_\infty(X)$ we have

$$\lambda \ j(TJ_M) \leq n(TJ_M) \ ,$$

hence $I(T)=\emptyset$. ■

8. EXAMPLE. There exists an operator S such that $sj(S)<in(S)$.

If $in(S)\leq sj(S)$ for every S, we would have for every $M\in S_\infty(X)$,

$$in(TJ_M) \leq sj(TJ_M) \ ,$$

with T the operator in Example 7.

Hence $in(T)\leq isj(T)$, a contradiction, and we conclude that there exists $M\in S_\infty(X)$ such that $sj(TJ_M)<in(TJ_M)$. ■

9. EXAMPLE. For every $n= 1, 2, \ldots$ we give an operator T_n such that

$$\frac{1}{\log_2(n+1)} = j(T_n) = sj(T_n)$$

and

$$in(T_n) = n(T_n) = 1 .$$

Hence the quantities *isj* and *in* are not equivalent.

There exists a Banach space $(X,\|.\|)$ such that for every $n= 1, 2, \ldots$ there is an equivalent norm $|.|_n$ such that

$$\frac{1}{\log_2(n+1)} \|x\| \leq |x|_n \leq \|x\| ,$$

and for every $\varepsilon>0$ and each $M\in S_\infty(X)$ there exists $x,y\in M$, $\|x\|=\|y\|=1$, with

$$|x|_n > 1\text{-}\varepsilon \quad \text{and} \quad |y|_n \leq \frac{1+\varepsilon}{\log_2(n+1)}$$

[9;Theor.3]. Clearly the isomorphisms

$$T_n: (X,\|.\|) \longrightarrow (X,|.|) \quad , \quad T_n x := x ,$$

verify the announced result. ■

We observe that the non-emptiness of Mil'man isometry spectrum is related with the distortion problem [5;2.e.2] (whether a space which contains an isomorphic copy of some l_p $(1<p<\infty)$ must actually contain almost ismoetric copies of this space) which is still open. Recall that a Banach space X contains l_p almost-isometrically if for every $\varepsilon>0$ there exists a sequence $(x_n)\subset X$ such that

$$(1\text{-}\varepsilon) \|(a_i)\|_p \leq \|\sum a_i x_i\| \leq (1+\varepsilon) \|(a_i)\|_p$$

for any finite sequence of scalars (a_i).

10 PROPOSITION. *If there exists an isomorphic embedding $T\in L(l_p,X)$ $(1\leq p<\infty)$ with $I(T)\neq\emptyset$, then X contains l_p almost-isometrically.*

Proof. Since T is an isomorphism, $0 \notin I(T)$; hence there exists $0<\alpha\in I(T)$. Now for any $0<\varepsilon<min\{1,\alpha\}$ we can find an infinite dimensional subspace M of l_p such that

$$\big|\, \|Tx\| - \alpha\|x\| \,\big| \leq \frac{\varepsilon\alpha}{3}\|x\|$$

for any $x\in M$.

Moreover, denoting

$$[e_i]_{i>n} := \{(\xi_n)\in l_p : \xi_1=\ldots=\xi_n=0\}$$

it is clear that $M \cap [e_i]_{i>n} \neq \emptyset$ for every n. Using a standard perturbation argument [5;1.a.11], given $\varepsilon>0$ we construct a block basis (z_k),

$$z_k = \sum_{i=n_{k-1}}^{n_k - 1} a_i e_i \quad \text{with} \quad \|z_k\| = 1 ,$$

and $(y_k)\subset M$ such that $\|z_k-y_k\|<\varepsilon.2^{-(k+1)}$. The sequence (z_k) generates a subspace isometric to l_p [5;2.a.1], and we have

$$(1-\frac{\varepsilon}{2})\,\|(a_i)\|_p \leq \|\Sigma a_i y_i\| \leq (1+\frac{\varepsilon}{2})\,\|(a_i)\|_p$$

for any finite sequence of scalars (a_i).

Now taking $x_i=\alpha^{-1}Ty_i$, for every finite sequence of scalars we have

$$\big|\,\|\Sigma a_i x_i\| - \|(a_i)\|_p\big| = \big|\alpha^{-1}\|\Sigma a_i Ty_i\| - \|(a_i)\|_p\big| \leq$$

$$\leq \big|\alpha^{-1}\|T\Sigma a_i y_i\| - \alpha^{-1}\alpha\|\Sigma a_i y_i\|\big| + \big|\,\|\Sigma a_i y_i\| - \|(a_i)\|_p\big| \leq$$

$$\leq \frac{\varepsilon}{3}\|\Sigma a_i y_i\| + \frac{\varepsilon}{2}\|(a_i)\|_p \leq \left(\frac{\varepsilon}{3}(1+\frac{\varepsilon}{2})+\frac{\varepsilon}{2}\right)\|(a_i)\|_p < \varepsilon\,\|(a_i)\|_p \,. \;\blacksquare$$

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