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La Distribución de Rathie - (p,q)

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Resumen

En este trabajo generalizamos la función asociada de *Rathie* e introducimos una v.a X definida con base en esta función asociada de *Rathie* generalizada. Se procede así mismo a calcular $E(X^m)$, su función de distribución, la función generadora de momentos y la distribución que sigue X/Y cuando ambas son v.a de *Rathie* generalizadas e independientes.

Palabras claves : variable aleatoria, función de *Rathie* y *Kámpe de Fériet*

The *Rathie's* Distribution - (p,q)

In this work we generalize the function associated of *Rathie* and we introduce an random variable defined with base in this function. We proceed likewise to calculate $E(X^m)$, the distribution function, the generating func-

tion of moments and the X/Y distribution, for X and Y being random variable of widespread and independent *Rathie*.

Introducción

En este trabajo generalizamos los resultados obtenidos por *Sarabia*, en [11], para la distribución de *Rathie* de parámetros - $(\alpha, \beta; \lambda, \mu; \nu, \gamma)$. Así en [11] se define a la variable aleatoria de *Rathie* de parámetros - $(\alpha, \beta; \lambda, \mu; \nu, \gamma)$ como aquella v.a de función de densidad :

$$f(\alpha, \beta; \lambda, \mu; \nu, \gamma; t) = \begin{cases} Ae^{-\alpha t} t^{\beta} \, {}_1Q_2(\lambda : \mu, \nu; \gamma t) & \text{para} \quad t > 0 \\ 0 & \text{e.o.c} \end{cases} \qquad (1)$$

Donde $\alpha > 0$; $\beta, \gamma \geq 0$; $\lambda, \mu, \nu > -1$ (2) y ${}_1Q_2(\lambda : \mu, \nu; x)$ es la función asociada de *Rathie*, definida así :

$${}_1Q_2(\lambda : \mu, \nu; x) = \frac{1}{\Gamma(\mu+1)\,\Gamma(\nu+1)} \cdot {}_2F_3 \left(\begin{matrix} \frac{\lambda+1}{2}, \frac{\lambda+2}{2} \\ \lambda+1, \mu+1, \nu+1 \end{matrix} ; 4x \right) \qquad (3a)$$

$$A = \frac{\Gamma(\mu+1)\,\Gamma(\nu+1)\,\alpha^{\beta+1}}{\Gamma(\beta+1)\,S} \; (3b) \quad \text{y} \quad S = {}_3F_3 \left(\begin{matrix} \frac{\lambda+1}{2}, \frac{\lambda+2}{2}, \beta+1 \\ \lambda+1, \mu+1, \nu+1 \end{matrix} ; \frac{4\gamma}{\alpha} \right) \qquad (3c)$$

Sean $p, q \in \mathbb{N}$ con $p \geq 2$ y $p < q$, a la función :

$${}_{p-1}Q_{q-1}(\lambda; \lambda_1, \ldots, \lambda_{p-2} : \mu, \nu; \beta_1, \ldots, \beta_{q-3}; x)$$

$$= B \cdot {}_pF_q \left(\begin{matrix} \frac{\lambda+1}{2}, \frac{\lambda+2}{2}, \lambda_1, \ldots, \lambda_{p-2} \\ \lambda+1, \mu+1, \nu+1, \beta_1, \ldots, \beta_{q-3} \end{matrix} ; 4x \right) \qquad (4)$$

Donde $B = \dfrac{1}{\Gamma(\mu+1)\Gamma(\nu+1)\Gamma(\beta_1)\cdots\Gamma(\beta_{q-3})}$; $\quad 2 \leq p < 3$;

$\lambda, \mu, \nu > -1$; $\quad \lambda_1, \ldots, \lambda_{p-2}, \beta_1, \ldots, \beta_{q-3} > 0$ (5); la denominaremos función - (p,q) asociada generalizada de *Rathie* o simplemente R*-(p,q), la que es una evidente generalización de Q en [10], cuando $p = 2$, $q = 3$

De acuerdo a (14) y (15) en [4,p.p. 174 - 175], (4) se puede escribir en términos de la función G de *Meijer* (Vea [5])

Así, usando la fórmula de duplicación para la función gamma, tenemos

$$ {}_{p-1}Q_{q-1}(\lambda;\lambda_1,\ldots,\lambda_{p-2}:\mu,\nu;\beta_1,\ldots,\beta_{q-3};x) \;=\; \frac{2^{\lambda}}{\pi^{1/2}\Gamma(\lambda_1)\cdots\Gamma(\lambda_{p-2})}. $$

$$ G^{1,p}_{p,q+1}\left(-4x\left|\begin{array}{c}\frac{1-\lambda}{2}\,,\;-\frac{\lambda}{2}\,,\;1-\lambda_1,\ldots,1-\lambda_{p-2}\\ 0\,,\;-\lambda\,,\;-\mu\,,\;-\nu\,,\;1-\beta_1,\ldots,1-\beta_{q-3}\end{array}\right.;4x\right) \tag{6} $$

Usando (6) podemos tener una aproximación asintótica para la función R-(p,q), la cual es de utilidad en el estudio de la existencia de la moda para la v.a de *Rathie*-(p,q).

En efecto de (3), (4) y (16) en [4,p.p. 207 - 209] tenemos :

$$ {}_{p-1}Q_{q-1}(\lambda;\lambda_1,\ldots,\lambda_{p-2}:\mu,\nu;\beta_1,\ldots,\beta_{q-3};x) \backsim \frac{2^{\lambda}}{\pi^{1/2}\Gamma(\lambda_1)\cdots\Gamma(\lambda_{p-2})}.K_{p,q}(4x) \tag{7} $$

cuando $x \to \infty$. Donde $\beta = p+q-1$; $B_1 = \lambda + \frac{3}{2} + \lambda_1 + \ldots + \lambda_{p-2}$; $C_1 = \lambda+\mu+\nu+3+\beta_1+\ldots+\beta_{q-3}$; $\gamma = \frac{\beta-1}{2} + B_1 - C_1$, y

$$ K_{p,q}(z) = \frac{(2\pi)^{(1-\beta)/2}}{\beta^{1/2}}.\exp(\beta z^{1/\beta}).z^{\gamma} \tag{8} $$

Finalmente utilizando el teorema 28 en [7,p.p. 85 - 86] tenemos una representación integral de la función R*-(p,q), por tener las condiciones dadas en (5)

$$ Q(\lambda;\lambda_1,\ldots,\lambda_{p-2}:\mu,\nu;\beta_1,\ldots,\beta_{q-3};x) \;=\; L.\int_0^1 t^{(\lambda-1)/2}(1-t)^{(\lambda+1)/2}. $$

$$ {}_{p-1}F_{q-1}\left(\begin{array}{c}\frac{\lambda+2}{2}\,,\;\lambda_1,\ldots,\lambda_{p-2}\\ \mu+1\,,\;\nu+1\,,\;\beta_1,\ldots,\beta_{q-3}\end{array};4xt\right)dt \tag{9a} $$

Donde : $$ L = \frac{1}{\Gamma(\mu+1)\Gamma(\nu+1)\Gamma(\beta_1)\cdots\Gamma(\beta_{q-3})\mathbf{B}(\frac{\lambda+1}{2},\frac{\lambda+1}{2})} \tag{9b} $$

La variable aleatoria R-(p,q)

Denominamos variable aleatoria R-(p,q) de parámetros - ($\lambda;\lambda_1,\ldots,\lambda_{p-2}:\mu,\nu;\beta_1,\ldots,\beta_{q-3};\alpha,\beta,\gamma$) a la v.a cuya función de densidad viene dada por :

$$f\,(\overline{\Lambda},\overline{\Omega};\lambda,\beta,\gamma;t\,) = \begin{cases} Ae^{-\alpha t}\,t^{\beta}\;{}_{p-1}Q_{q-1}\,(\,\overline{\Lambda},\overline{\Omega};\gamma\,t\,) & \text{para} \quad t>0 \\ 0 & \text{e.o.c} \end{cases} \qquad (10a)$$

Donde $\overline{\Lambda} = (\,\lambda;\lambda_1,\ldots,\lambda_{p-2}\,)$; $\overline{\Omega} = (\,\mu,\nu;\beta_1,\ldots,\beta_{q-3}\,)$; $\alpha > 0$, $\beta,\gamma \geq 0$; $\lambda,\mu,\nu > -1$; $\beta_1,\ldots,\beta_{q-3} > 0$; $\lambda_1,\ldots,\lambda_{p-2} > 0$; $2 \leq p < q$ $(10b)$ (En adelante suponemos que $(10b)$ se cumple y $(10a)$ lo abreviaremos por $f(\,t\,)$). Claramente cuando $p=2,\ q=3,\ (10a)$ se reduce a (11) en [11], por lo que $(10a)$ también constituye una generalización de la v.a gamma - $(\,\alpha,\beta\,)$

Por ser $f(\,t\,)$ función de densidad, ya que la integral :

$$\int_0^{\infty} e^{-\alpha t}\,t^{\beta}\;{}_{p-1}Q_{q-1}\,(\,\overline{\Lambda},\overline{\Omega};\gamma\,t\,)\,dt$$

converge para las condiciones $(10b)$, y por el teorema 14.31 en [1], tenemos por intercambio de la integral con la serie y propiedades de la función gamma que :

$$A = \frac{\Gamma(\mu+1)\Gamma(\nu+1)\Gamma(\beta_1)\cdots\Gamma(\beta_{q-3})\;\alpha^{\beta+1}}{\Gamma(\beta+1)\cdot\;{}_{p+1}F_q\left(\begin{matrix}\dfrac{\lambda+1}{2}\,,\;\dfrac{\lambda+2}{2}\,,\;\lambda_1,\ldots,\lambda_{p-2},\;\beta+1 \\ \lambda+1\;,\;\mu+1\;,\;\nu+1\;,\;\beta_1,\ldots,\beta_{q-3}\end{matrix}\;;\dfrac{4\gamma}{\alpha}\right)} \qquad (11)$$

De acuerdo a $(9a)$ y $(9b)$, tenemos :

$$A = \frac{\alpha^{\beta+1}}{\Gamma(\beta+1)\cdot\;{}_{p}Q_{q-1}\left(\lambda;\lambda_1,\ldots,\lambda_{p-2},\beta+1:\mu,\nu;\beta_1,\ldots,\beta_{q-3};\dfrac{\gamma}{\alpha}\right)} \qquad (11a)$$

Relación de la variable aleatoria R*-(p,q) con otras v.a

Ya dijimos que si $p=2,\ q=3,\ \gamma=0$ entonces $(10a)$ se reduce a

$$f(\,\alpha,\beta;t\,) = \frac{\alpha^{\beta+1}}{\Gamma(\beta+1)}e^{-\alpha t}\,t^{\beta}\,, \quad t>0 \quad \text{y} \quad 0 \quad \text{e.o.c}$$

Otra relación viene expresada en el teorema siguiente :

Teorema 1 . *Sea X una v.a cuya distribución condicional con respecto a una v.a de Rathie generalizada -* $(\,\lambda;\lambda_1,\ldots,\lambda_{p-2}:\mu,\nu;\beta_1,\ldots,\beta_{q-3};\xi,\delta,\gamma\,)$

es una distribución gamma - (α, β), *entonces la v.a* X *es* $(p+1, q)$ - ***hiper-**geométrica.*

Demostración:

Sea $h(x)$ la función de densidad de X, y sea $g(\alpha)$ la función de densidad de *Rathie* generalizada, es decir :

$$g(\alpha) = Ae^{-\xi\alpha}\alpha^{\delta} \cdot {}_{p-1}Q_{q-1}(\lambda; \lambda_1, \ldots, \lambda_{p-2} : \mu, \nu; \beta_1, \ldots, \beta_{q-3}; \gamma\alpha)$$

Entonces, para $x > 0$:

$$h(x) = \int_0^{\infty} f\,(\,x\,|\,\alpha\,)\,g(\alpha)d\alpha$$

$$h(x) = \int_0^{\infty} \frac{\alpha^{\beta+1}}{\Gamma(\beta+1)} e^{-\alpha x} x^{\beta} Ae^{-\xi\alpha}\alpha^{\delta} \cdot {}_{p-1}Q_{q-1}(\lambda{:}\lambda_1,\ldots,\lambda_{p-2}{:}\mu,\nu;\beta_1,\ldots,\beta_{q-3};\gamma\alpha)d\alpha$$

De acuerdo a (4) e intercambiando la integral con la serie nos queda :

$$h(x) = \frac{A\,x^{\beta}}{\Gamma(\beta+1)\Gamma(\mu+1)\Gamma(\nu+1)\Gamma(\beta_1)\cdots\Gamma(\beta_{q-3})} \cdot$$

$$\sum_{n=0}^{\infty} S(\,\lambda, \lambda_1, \ldots, \lambda_{p-2}, \mu, \nu; \beta_1, \ldots, \beta_{q-3}; n\,).$$

$$(4\gamma)^n \int_0^{\infty} \alpha^{n+\delta+\beta+1}\, e^{-(x+\xi)\alpha}\, d\alpha \qquad (12a)$$

Donde :

$$S(\,\lambda, \lambda_1, \ldots, \lambda_{p-2}, \mu, \nu; \beta_1, \ldots, \beta_{q-3}; n\,)$$

$$= \frac{\left(\dfrac{\lambda+1}{2}\right)_n \left(\dfrac{\lambda+2}{2}\right)_n (\,\lambda_1\,)_n \cdots (\,\lambda_{p-2}\,)_n}{(\,\lambda+1\,)_n\,(\,\mu+1\,)_n\,(\,\nu+1\,)_n\,(\,\beta_1\,)_n \cdots (\,\beta_{q-3}\,)_n\; n!} \qquad (12b)$$

Luego de $(12a)$, tenemos

$$h(x) = \frac{A\,\Gamma(\beta+\delta+2)}{\Gamma(\beta+1)\Gamma(\mu+1)\Gamma(\nu+1)\Gamma(\beta_1)\cdots\Gamma(\beta_{q-3})} \cdot \frac{x^{\beta}}{(x+\xi)^{\beta+\delta+2}} \cdot$$

$${}_{p+1}F_q \left(\begin{array}{c} \dfrac{\lambda+1}{2}\,,\ \dfrac{\lambda+2}{2}\,,\ \lambda_1, \ldots, \lambda_{p-2},\ \delta+\beta+2 \\ \lambda+1\,,\ \mu+1\,,\ \nu+1\,,\ \beta_1, \ldots, \beta_{q-3} \end{array} ; \frac{4\gamma}{x+\xi} \right) \qquad (13)$$

Para $x > 0$; y $h(x) = 0$ para $x \leq 0$. Bajo la condición (5), y donde A viene dada por (11) o ($11a$).

Expresando (13) por medio de Q tenemos al usar (4) :

$$h(x) = \frac{\Gamma(\beta+\delta+2)\,\xi^{\delta+1}\,x^{\beta}}{\Gamma(\beta+1)\Gamma(\delta+1)\,(x+\xi)^{\beta+\delta+2}} \cdot$$

$$\frac{{}_pQ_{q-1}\left(\lambda;\lambda_1,\ldots,\lambda_{p-2},\beta+\delta+2:\mu,\nu;\beta_1,\ldots,\beta_{q-3};\dfrac{\gamma}{x+\xi}\right)}{{}_pQ_{q-1}\left(\lambda;\lambda_1,\ldots,\lambda_{p-2},\delta+1:\mu,\nu;\beta_1,\ldots,\beta_{q-3};\dfrac{\gamma}{\xi}\right)} \qquad (13a)$$

Teorema 2 . *Sea X una v.a Rice - (a, α, ν), con $a > 0$, $\alpha \geq 1$ y $\nu \geq 0$ entonces $Y = X^2$ es una v.a de Rathie - $(\nu : \frac{\nu-1}{2}, \frac{\nu}{2}; \frac{1}{2}, \frac{\alpha+\nu-1}{2}, \frac{a^2}{16})$*

Demostración:

Vea teorema 1 en [11], donde $\lambda, \mu, \nu, \alpha, \beta$ y γ son respectivamente ν, $\frac{\nu-1}{2}$, $\frac{\nu}{2}$, $\frac{1}{2}$, $\frac{\alpha+\nu-1}{2}$ y $\frac{a^2}{16}$.

Momento de orden m

Teorema 3 . *Sea X una v.a generalizada de Rathie de parámetros - $(\lambda; \lambda_1, \ldots, \lambda_{p-2} : \mu, \nu; \beta_1, \ldots, \beta_{q-3}; \alpha, \beta, \gamma)$, bajo la condición (5) y $m > -(\beta+1)$. Entonces $E(X^m)$ existe y :*

$$E(X^m) = \frac{\Gamma(m+\beta+1)}{\Gamma(\beta+1)\,\alpha^m} \cdot$$

$$\frac{{}_pQ_{q-1}\left(\lambda;\lambda_1,\ldots,\lambda_{p-2},m+\beta+1:\mu,\nu;\beta_1,\ldots,\beta_{q-3};\dfrac{\gamma}{\alpha}\right)}{{}_pQ_{q-1}\left(\lambda;\lambda_1,\ldots,\lambda_{p-2},\beta+1:\mu,\nu;\beta_1,\ldots,\beta_{q-3};\dfrac{\gamma}{\alpha}\right)} \qquad (14)$$

Demostración:

Bajo las condiciones de los parámetros, y de acuerdo a (7) y (8), tenemos que :

$$E(X^m) = A\int_0^{\infty} t^{\beta+m} e^{-\alpha t}\,{}_{p-1}Q_{q-1}(\lambda;\lambda_1,\ldots,\lambda_{p-2}:\mu,\nu;\beta_1,\ldots,\beta_{q-3};\gamma t)\,dt \qquad (15)$$

converge pues :

$$\lim_{t\to\infty}\left[\frac{t^{m+\beta}\,e^{-\alpha t}\,{}_{p-1}Q_{q-1}(\lambda;\lambda_1,\ldots,\lambda_{p-2}:\mu,\nu;\beta_1,\ldots,\beta_{q-3};\gamma t)}{e^{-\alpha t}\,t^{m+\beta}\,e^{4\sqrt{t}}}\right] = c \neq 0$$

y $\int_0^{\infty} t^{\beta+m} e^{-t(\alpha-4t^{-1/2})}\,dt$ converge, luego $E(X^m)$ existe para $m > -(\beta+1)$

Por intercambio de la integral con la serie, de (15) tenemos :

$$E(X^m) = \frac{A\Gamma(m+\beta+1)}{\Gamma(\mu+1)\Gamma(\nu+1)\Gamma(\beta_1)\cdots\Gamma(\beta_{q-3})\,\alpha^{m+\beta+1}}\cdot$$

$$_{p+1}F_q\left(\begin{array}{c}\frac{\lambda+1}{2},\ \frac{\lambda+2}{2},\ \lambda_1,\ldots,\lambda_{p-2},\ m+\beta+1\\ \lambda+1\ ,\ \mu+1\ ,\ \nu+1\ ,\ \beta_1,\ldots,\beta_{q-3}\end{array};\frac{4\gamma}{\alpha}\right)$$

Y recordando (4),(11) y (11a), tenemos

$$E(X^m) = \frac{\Gamma(m+\beta+1)}{\Gamma(\beta+1)\,\alpha^m}\cdot$$

$$\frac{_pQ_{q-1}\left(\lambda;\lambda_1,\ldots,\lambda_{p-2},m+\beta+1:\mu,\nu;\beta_1,\ldots,\beta_{q-3};\frac{\gamma}{\alpha}\right)}{_pQ_{q-1}\left(\lambda;\lambda_1,\ldots,\lambda_{p-2},\beta+1:\mu,\nu;\beta_1,\ldots,\beta_{q-3};\frac{\gamma}{\alpha}\right)} \qquad (14)$$

Función generadora de momentos y de distribución

Teorema 4 *La función generadora de momentos para la v.a generalizada de Rathie, del teorema 3 tiene función generadora de momentos para* $x < \alpha$, *y*

$$m_X(x) = \left(\frac{\alpha}{\alpha-x}\right)^{\beta+1}\cdot$$

$$\frac{M\left(\lambda;\lambda_1,\ldots,\lambda_{p-2},\mu,\nu,\beta_1,\ldots,\beta_{q-3};\beta+1;\frac{4\gamma}{(\alpha-x)}\right)}{M\left(\lambda;\lambda_1,\ldots,\lambda_{p-2},\mu,\nu,\beta_1,\ldots,\beta_{q-3};\beta+1;\frac{4\gamma}{\alpha}\right)} \qquad (15)$$

donde M *viene dada por* (15b).

Demostración:

De acuerdo a (10a), tenemos :

$$m_X(x) = A\int_0^{\infty} e^{xt}e^{-\alpha t}t^{\beta}\,Q(\lambda;\lambda_1,\ldots,\lambda_{p-2}:\mu,\nu;\beta_1,\ldots,\beta_{q-3};\gamma t)\,dt \qquad (15a)$$

De acuerdo a (7) y (8), y como

$$\int_0^{\infty} e^{-(\alpha-x)t}\,t^{\beta}\,dt \quad \text{converge, para} \quad x<\alpha,$$

tenemos que $m_X(x)$ existe para $x < \alpha$.

Así mismo por el intercambio de la integral con la serie en $(15a)$, tenemos que :

$$m_X(x) = \frac{A\,\Gamma(\beta+1)}{\Gamma(\mu+1)\Gamma(\nu+1)\Gamma(\beta_1)\cdots\Gamma(\beta_{q-3})\,(\alpha-x)^{\beta+1}}\cdot$$

$$_{p+1}F_q\left(\begin{array}{c}\frac{\lambda+1}{2},\ \frac{\lambda+2}{2},\ \lambda_1,\ldots,\lambda_{p-2},\ \beta+1\\ \lambda+1,\ \mu+1,\ \nu+1,\ \beta_1,\ldots,\beta_{q-3}\end{array};\frac{4\gamma}{\alpha-x}\right) \qquad (15b)$$

Finalmente, usando (11) y $(11a)$, resulta :

$$m_X(x) = \left(\frac{\alpha}{\alpha-x}\right)^{\beta+1}\cdot$$

$$\frac{_{p+1}F_q\left(\begin{array}{c}\frac{\lambda+1}{2},\ \frac{\lambda+2}{2},\ \lambda_1,\ldots,\lambda_{p-2},\ \beta+1\\ \lambda+1,\ \mu+1,\ \nu+1,\ \beta_1,\ldots,\beta_{q-3}\end{array};\frac{4\gamma}{\alpha-x}\right)}{_{p+1}F_q\left(\begin{array}{c}\frac{\lambda+1}{2},\ \frac{\lambda+2}{2},\ \lambda_1,\ldots,\lambda_{p-2},\ \beta+1\\ \lambda+1,\ \mu+1,\ \nu+1,\ \beta_1,\ldots,\beta_{q-3}\end{array};\frac{4\gamma}{\alpha}\right)} \qquad (15c)$$

y finalmente

$$m_X(x) = \left(\frac{\alpha}{\alpha-x}\right)^{\beta+1}\cdot$$

$$\frac{_pQ_{q-1}\left(\lambda;\lambda_1,\ldots,\lambda_{p-2},\beta+1:\mu,\nu;\beta_1,\ldots,\beta_{q-3};\beta+1;\frac{\gamma}{\alpha-x}\right)}{_pQ_{q-1}\left(\lambda;\lambda_1,\ldots,\lambda_{p-2},\beta+1:\mu,\nu;\beta_1,\ldots,\beta_{q-3};\beta+1;\frac{\gamma}{\alpha}\right)} \qquad (15)$$

Teorema 5 . *Sea X una v.a de Rathie generalizada del teorema 3, entonces su función de distribución para $\omega > 0$, viene dada por : (En este caso en lugar de γ usamos δ)*

$$a)\ F_X(\omega) = \frac{A\alpha^{-(\beta+1)}}{\Gamma(\mu+1)\Gamma(\nu+1)\Gamma(\beta_1)\cdots\Gamma(\beta_{q-3})}\cdot$$

$$\sum_{n=0}^{\infty}\frac{\left(\frac{\lambda+1}{2}\right)_n\left(\frac{\lambda+2}{2}\right)_n(\lambda_1)_n\cdots(\lambda_{p-2})_n}{(\lambda+1)_n(\mu+1)_n(\nu+1)_n(\beta_1)_n\cdots(\beta_{q-3})_n\,n!}\cdot$$

$$\left(\frac{4\delta}{\alpha}\right)^n\gamma(\beta+n+1,\alpha\omega) \qquad (16)$$

Donde $\gamma\,(t,x) = \int_0 u^{t-1}\, e^{-u} du$ *(función gamma incompleta)*

$$b)\ F_X\,(\omega) = A\omega^{\beta+1} \sum_{n=0}^{\infty} \frac{(-\alpha\omega)^n\,\Gamma(\beta+n+1)}{n!} \cdot$$

$$_pQ_q\,(\lambda;\lambda_1,\ldots,\lambda_{p-2},\beta+n+1:\mu,\nu;\beta_1,\ldots,\beta_{q-3};\beta+n+2;\delta\omega) \quad (17)$$

$$c)\ F_X\,(\omega) = \frac{A\omega^{\beta+1}}{\Gamma(\mu+1)\Gamma(\nu+1)\Gamma(\beta_1)\cdots\Gamma(\beta_{q-3})(\beta+1)} \cdot$$

$$F_1\left(\begin{array}{c|c|c} 1 & \beta+1 & \\ 0,p & -\,;\ \dfrac{\lambda+1}{2}\,,\ \dfrac{\lambda+2}{2}\,,\ \lambda_1,\ldots,\lambda_{p-2} & -\,\alpha\omega, 4\delta\omega \\ 1 & \beta+2 & \\ 0,q & -\,;\ \lambda+1\,,\ \mu+1\,,\ \nu+1\,,\ \beta_1,\ldots,\beta_{q-3} & \end{array}\right) \quad (18)$$

Donde F_1 es la primera función generalizada de Kámpe de Fériet (*Vea* [3] *o* [9])

Demostración:

a) Recordemos que :

$$F_X\,(\omega) = \int_0^{\omega} f(t)dt$$

$$= A\int_0^{\omega} e^{-\alpha t}\, t^{\beta}\, Q\,(\,\lambda;\lambda_1,\ldots,\lambda_{p-2}:\mu,\nu;\beta_1,\ldots,\beta_{q-3};\delta t\,)\,dt$$

De acuerdo a ($3a,b,c$), desarrollando $_pF_q$, intercambiando la integral con la serie y recordando que

$$\int_0^{\omega} e^{-\alpha t}\, t^{\beta+n}\, dt = \frac{1}{\alpha^{\beta+n+1}}\,\gamma(\beta+n+1,\alpha\omega) \quad (16a)$$

tenemos :

$$F_X\,(\omega) = \frac{A\alpha^{-(\beta+1)}}{\Gamma(\mu+1)\Gamma(\nu+1)\Gamma(\beta_1)\cdots\Gamma(\beta_{q-3})} \cdot$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{\lambda+1}{2}\right)_n \left(\frac{\lambda+2}{2}\right)_n (\lambda_1)_n\cdots(\lambda_{p-2})_n}{(\lambda+1)_n\,(\mu+1)_n\,(\nu+1)_n\,(\beta_1)_n\cdots(\beta_{q-3})_n\, n!} \cdot$$

$$\left(\frac{4\delta}{\alpha}\right)^n \gamma(\beta+n+1,\alpha\omega) \quad (16)$$

b) Desarrollando $e^{-\alpha t}$, y usando el teorema 14.31 de [1], tenemos :

$$F_X(\omega) = \frac{A}{\Gamma(\mu+1)\Gamma(\nu+1)\Gamma(\beta_1)\cdots\Gamma(\beta_{q-3})}\cdot$$

$$\sum_{n=0}^{\infty}\frac{(-\alpha)^n}{n!}\int_0^{\omega} t^{\beta+n}\cdot$$

$${}_pF_q\left(\begin{matrix}\frac{\lambda+1}{2}\,,\,\frac{\lambda+2}{2}\,,\,\lambda_1,\ldots,\lambda_{p-2} \\ \lambda+1\,,\,\mu+1\,,\,\nu+1\,,\,\beta_1,\ldots,\beta_{q-3}\end{matrix};4\delta t\right)\cdot dt$$

Usando el teorema 28 en [7,p. 85], ya que $p < q$, tenemos:

$$F_X(\omega) = \frac{A\,\omega^{\beta+1}}{\Gamma(\mu+1)\Gamma(\nu+1)\Gamma(\beta_1)\cdots\Gamma(\beta_{q-3})}\cdot\sum_{n=0}^{\infty}\frac{(-\alpha\omega)^n}{(\beta+n+1)n!}\cdot$$

$${}_{p+1}F_{q+1}\left(\begin{matrix}\frac{\lambda+1}{2}\,,\,\frac{\lambda+2}{2}\,,\,\lambda_1,\ldots,\lambda_{p-2},\beta+n+1 \\ \lambda+1\,,\,\mu+1\,,\,\nu+1\,,\,\beta_1,\ldots,\beta_{q-3},\beta+n+2\end{matrix};4\delta\omega\right) \quad (17a)$$

De acuerdo a (4), resulta :

$$F_X(\omega) = A\omega^{\beta+1}\sum_{n=0}^{\infty}\frac{(-\alpha\omega)^n\,\Gamma(\beta+n+1)}{n!}\cdot$$

$${}_pQ_q\,(\lambda;\lambda_1,\ldots,\lambda_{p-2},\beta+n+1:\mu,\nu;\beta_1,\ldots,\beta_{q-3};\beta+n+2;\delta\omega) \quad (17)$$

c) En (17*a*), desarrollando ${}_{p+1}F_{q+1}$ en serie, y como :

$$\frac{1}{\beta+n+1} = \frac{(\beta+1)_n}{(\beta+1)(\beta+2)_n}\;;\quad (\beta+1)_{n+m} = (\beta+1)_n(\beta+1+n)_m$$

resulta :

$$F_X(\omega) = \frac{A\,\omega^{\beta+1}}{(\beta+1)\Gamma(\mu+1)\Gamma(\nu+1)\Gamma(\beta_1)\cdots\Gamma(\beta_{q-3})}\cdot$$

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\left(\frac{\lambda+1}{2}\right)_n\left(\frac{\lambda+2}{2}\right)_n(\lambda_1)_n\cdots(\lambda_{p-2})_n(\beta+1)_{n+m}}{(\lambda+1)_n\,(\mu+1)_n\,(\nu+1)_n\,(\beta_1)_n\cdots(\beta_{q-3})_n\,(\beta+2)_{n+m}\,n!\,m!}\cdot$$

$$(-\alpha\omega)^n\,(4\delta\omega)^m$$

luego :

$$F_X(\omega) = \frac{A\omega^{\beta+1}}{\Gamma(\mu+1)\Gamma(\nu+1)\Gamma(\beta_1)\cdots\Gamma(\beta_{q-3})(\beta+1)} \cdot$$

$$F_1\left(\begin{array}{c|c|c} 1 & \beta+1 & \\ 0,p & -\,;\ \frac{\lambda+1}{2},\ \frac{\lambda+2}{2},\ \lambda_1,\ldots,\lambda_{p-2} & -\alpha\omega, 4\delta\omega \\ 1 & \beta+2 & \\ 0,q & -\,;\ \lambda+1\ ,\ \mu+1\ ,\ \nu+1\ ,\ \beta_1,\ldots,\beta_{q-3} & \end{array}\right) \quad (18)$$

Distribución de la v.a X/Y, para X e Y R*-(p,q) independientes

Sea X v.a de *Rathie* generalizada de parámetros - ($\lambda; \lambda_1, \ldots, \lambda_{p-2} : \mu, \nu; \beta_1, \ldots, \beta_{q-3}; \alpha, \beta, \gamma$) e Y v.a del mismo tipo con parámetros - ($\lambda'; \lambda'_1, \ldots, \lambda'_{p-2} : \mu', \nu'; \beta'_1, \ldots, \beta'_{q-3}; \alpha', \beta', \gamma'$), ambas bajo la condición (5) y con $\alpha = \alpha'$. Además son independientes.

Como $E\left(X^{m-1}\right) = \int_0^\infty x^{m-1} f(x)dx = \mathcal{M}[f(x), m]$, donde $\mathcal{M}$ es la transformada de *Mellín*, es claro que si $E\left(X^{m-1}\right)$ cumple condiciones para la existencia de $\mathcal{M}^{-1}\left[E\left(X^{m-1}\right); x\right]$, (Vea [12,p.p. 272 - 274]).

En nuestro caso recordando (14) y como :

$$E\left((X/Y)^{m-1}\right) = E\left(X^{m-1}\right).E\left(Y^{1-m}\right) \quad (19)$$

tenemos :

$$E\left((X/Y)^{m-1}\right) =$$

$$\left[\frac{A\,\Gamma(m+\beta)}{\Gamma(\mu+1)\Gamma(\nu+1)\Gamma(\beta_1)\cdots\Gamma(\beta_{q-3})\,\alpha^{m+\beta}} \cdot \right.$$

$$\left. \sum_{n=0}^{\infty} \frac{\left(\frac{\lambda+1}{2}\right)_n \left(\frac{\lambda+2}{2}\right)_n (\lambda_1)_n \cdots (\lambda_{p-2})_n (m+\beta)_n}{(\lambda+1)_n\ (\mu+1)_n\ (\nu+1)_n\ (\beta_1)_n \cdots (\beta_{q-3})_n\ n!} \left(\frac{4\gamma}{\alpha}\right)^n \right] \cdot$$

$$\left[\frac{A'\,\Gamma(1-m+\beta')}{\Gamma(\mu'+1)\Gamma(\nu'+1)\Gamma(\beta'_1)\cdots\Gamma(\beta'_{q-3})\,\alpha^{1-m+\beta'}} \cdot \right.$$

$$\left. \sum_{k=0}^{\infty} \frac{\left(\frac{\lambda'+1}{2}\right)_k \left(\frac{\lambda'+2}{2}\right)_k (\lambda'_1)_k \cdots (\lambda'_{p-2})_k (1-m+\beta')_k}{(\lambda'+1)_k\ (\mu'+1)_k\ (\nu'+1)_k\ (\beta'_1)_k \cdots (\beta'_{q-3})_k\ k!} \left(\frac{4\gamma'}{\alpha'}\right)^k \right]$$

Para m cumpliendo con $\beta'+1 > m > -\beta$. (19a)

Luego :

$$E\left((X/Y)^{m-1}\right) =$$

$$\frac{A\,A'}{\left[\,\Gamma(\mu+1)\Gamma(\nu+1)\Gamma(\beta_1)\cdots\Gamma(\beta_{q-3})\,\right]\left[\,\Gamma(\mu'+1)\Gamma(\nu'+1)\Gamma(\beta_1')\cdots\Gamma(\beta_{q-3}')\,\right]\alpha^{\beta+\beta'+1}}\cdot$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\left(\frac{\lambda+1}{2}\right)_n\left(\frac{\lambda+2}{2}\right)_n(\lambda_1)_n\cdots(\lambda_{p-2})_n\,\Gamma(m+\beta+n)}{(\lambda+1)_n\,(\mu+1)_n\,(\nu+1)_n\,(\beta_1)_n\cdots(\beta_{q-3})_n}\cdot$$

$$\frac{\left(\frac{\lambda'+1}{2}\right)_k\left(\frac{\lambda'+2}{2}\right)_k(\lambda_1')_k\cdots(\lambda_{p-2}')_k\,\Gamma(1-m+\beta'+k)}{(\lambda'+1)_k\,(\mu'+1)_k\,(\nu'+1)_k\,(\beta_1')_k\cdots(\beta_{q-3}')_k\;n!k!}\cdot$$

$$\left(\frac{4}{\alpha}\right)^{n+k}\gamma^n\,\gamma'^{\,k}$$

Luego, por la aproximación de *Stirling* y criterios de convergencia uniforme, tenemos :

$$f_{X/Y}(u) =$$

$$\frac{A\,A'}{\left[\,\Gamma(\mu+1)\Gamma(\nu+1)\Gamma(\beta_1)\cdots\Gamma(\beta_{q-3})\,\right]\left[\,\Gamma(\mu'+1)\Gamma(\nu'+1)\Gamma(\beta_1')\cdots\Gamma(\beta_{q-3}')\,\right]\alpha^{\beta+\beta'+1}}\cdot$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\left(\frac{\lambda+1}{2}\right)_n\left(\frac{\lambda+2}{2}\right)_n(\lambda_1)_n\cdots(\lambda_{p-2})_n}{(\lambda+1)_n\,(\mu+1)_n\,(\nu+1)_n\,(\beta_1)_n\cdots(\beta_{q-3})_n}\cdot$$

$$\frac{\left(\frac{\lambda'+1}{2}\right)_k\left(\frac{\lambda'+2}{2}\right)_k(\lambda_1')_k\cdots(\lambda_{p-2}')_k}{(\lambda'+1)_k\,(\mu'+1)_k\,(\nu'+1)_k\,(\beta_1')_k\cdots(\beta_{q-3}')_k\;n!k!}\cdot$$

$$\left(\frac{4}{\alpha}\right)^{n+k}\gamma^n\,\gamma'^{\,k} \qquad (19b)$$

$$\mathcal{M}^{-1}\left[\,\Gamma(m+\beta+n)\Gamma(1-m+\beta'+k);u\,\right]$$

Usando 5.36 en [6,p.196], tenemos que :

$$\mathcal{M}^{-1}\left[\,\Gamma(m+\beta+n)\Gamma(1-m+\beta'+k);u\,\right]$$

$$= \Gamma(\beta+\beta'+n+k+1)\,\frac{u^{\beta+n}}{(1+u)^{\beta+\beta'+n+k+1}} \qquad (19c)$$

ya que se cumple (19a).

Luego :

$$f_{X/Y}(u) \;=$$

$$\frac{A\,A'\,\Gamma(\beta+\beta'+1)u^{\beta}(1+u)^{-(\beta+\beta'+1)}}{\left[\Gamma(\mu+1)\Gamma(\nu+1)\Gamma(\beta_1)\cdots\Gamma(\beta_{q-3})\right]\left[\Gamma(\mu'+1)\Gamma(\nu'+1)\Gamma(\beta'_1)\cdots\Gamma(\beta'_{q-3})\right]\alpha^{\beta+\beta'+1}}\cdot$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\beta+\beta'+1)_{n+k}\left(\frac{\lambda+1}{2}\right)_n\left(\frac{\lambda+2}{2}\right)_n(\lambda_1)_n\cdots(\lambda_{p-2})_n\left(\frac{\lambda'+1}{2}\right)_k\left(\frac{\lambda'+2}{2}\right)_k(\lambda'_1)_k\cdots(\lambda'_{p-2})_k}{(\lambda+1)_n(\mu+1)_n(\nu+1)_n(\beta_1)_n\cdots(\beta_{q-3})_n(\lambda'+1)_k(\mu'+1)_k(\nu'+1)_k(\beta'_1)_k\cdots(\beta'_{q-3})_k\,n!k!}\cdot$$

$$\left(\frac{4\gamma u}{\alpha(1+u)}\right)^{n}\cdot\left(\frac{4\gamma'}{\alpha(1+u)}\right)^{k}$$

Finalmente, tenemos :

$$f_{X/Y}(u) \;= C\cdot\;\frac{u^{\beta}}{(1+u)^{\beta+\beta'+1}}\cdot$$

$$F_1\left(\begin{array}{c|l|l} 1 & \beta+\beta'+1 & \\ p,p & \frac{\lambda+1}{2},\frac{\lambda+2}{2},\lambda_1,\ldots,\lambda_{p-2};\ \frac{\lambda'+1}{2},\frac{\lambda'+2}{2},\lambda'_1,\ldots,\lambda'_{p-2} & \frac{4\gamma u}{\alpha(1+u)},\frac{4\gamma'}{\alpha(1+u)} \\ 0 & \text{——} & \\ q,q & \lambda+1,\mu+1,\nu+1,\beta_1,\ldots,\beta_{q-3};\ \lambda'+1,\mu'+1,\nu'+1,\beta'_1,\ldots,\beta'_{q-3} & \end{array}\right) \quad (20)$$

Donde

$$C \;=\frac{A\,A'\,\Gamma(\beta+\beta'+1)}{\left[\Gamma(\mu+1)\Gamma(\nu+1)\Gamma(\beta_1)\cdots\Gamma(\beta_{q-3})\right]\left[\Gamma(\mu'+1)\Gamma(\nu'+1)\Gamma(\beta'_1)\cdots\Gamma(\beta'_{q-3})\right]\alpha^{\beta+\beta'+1}}$$

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