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ON G-LOCALLY TRIVIAL SPACES OVER FINITE G-COMPLEXES

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ABSTRACT. The purpose of this paper is to prove that a G-locally trivial G-space over a finite G-complex is an equivariant Serre-fibration. An application of this result to equivariant fixed point theory is described in [2].

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1. PRELIMINARIES

Let G be finite group and p: E⟶B an equivariant map of a G-space E into a finite G-complex B.

Definition 1.1 p: E⟶B is said to have the G-covering homotopy property (abbreviated G-CHP) for the G-space X if, for every equivariant map f: X⟶E and every equivariant homotopy ξ_t: X⟶B, $0 \leq t \leq 1$, of the map p∘f: X⟶B, there exists an equivariant homotopy $\hat{\xi}_t$: X⟶E of f which covers the homotopy ξ_t. p is said to be an equivariant Serre-fibration if it has the G-CHP for every finite G-complex X.

Definition 1.2 p: E⟶B is said to have the G-bundle property (abbreviated G-BP) if it is G-locally trivial, that is, if for each $b \in B$ there is an open G_x-invariant neighborhood U_b of b in B together with a G_x-equivariant homeomorphism (G_x=isotropy group of x) $\eta_{U_b} : U_b \times \bar{p}^1(b) \longrightarrow \bar{p}^1(U_b)$, satisfying the condition $p \circ \eta_{U_b}(x,y) = x$.

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for all $x \in U_b$, $y \in p^{-1}(b)$. The open sets U_b and the homeomorphisms γ_{U_b} will be called decomposing neighborhoods and decomposing functions respectively.

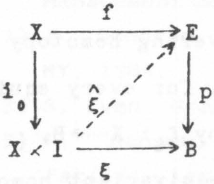
2. THE RESULT

We now state the result of this paper.

<u>Proposition 2.1</u> $p: E \to B$ has the G-BP $\Longrightarrow$ $p: E \to B$ has the G-CHP for the family of finite G-complexes.

In other words, a G-locally trivial G-space over a finite G-complex is an equivariant Serre-fibration.

<u>Proof</u>. Let X be a finite G-complex and let us consider the following commutative diagramm



where $i_0(x) = (x,0)$, f is an equivariant map and $\xi_t(x) = \xi(x,t)$, $0 \leq t \leq 1$, any equivariant homotopy of the map $p \cdot f: X \to B$. To prove the Proposition we have to construct an equivariant homotopy $\hat{\xi}$ of f which covers ξ.

Let $U = \{U_x / x \in B\}$ be a collection of decomposing neighborhoods. We have decomposing functions

$\gamma^{-1}_{U_x}: p^{-1}U_x \longrightarrow U_x \times p^{-1}(x)$

for all $x \in B$. By II.5.5 in [1] we can assume that

$\forall\ y \in U_x \Longrightarrow G_y < G_x$, (2.2)

that is, in the neighborhood U_x, x is have the biggest isotropy group. This fact is essentially used in the proof.

The collection $\xi^{-1}U$ of open sets of $X \times I$ forms an open covering of

$X \times I$. Since $X \times I$ is compact $\xi^{-1}(U)$ has a refinement of the form $\{W_\lambda \times I_\mu\}$, where $\{W_\lambda\}$ is a finite covering of X and $I_\mu = \{I_1, \ldots, I_r\}$ is a sequence of open subintervals of I which cover I. We may assume that I_μ meets only $I_{\mu-1}$ and $I_{\mu+1}$ for each $\mu = 2, \ldots, r-1$. Choose numbers $0 = t_0 < t_1 < \ldots < t_r = 1$ such that t_μ is in the intersection $I_\mu \cap I_{\mu+1}$. On the other side, taking a sufficiently fine triangulation of X, we may assume that X is a simplicial complex such that every simplex σ in X is contained in some W_λ of the finite open covering $\{W_\lambda\}$ constructed above.

We are now ready to construct the equivariant homotopy $\hat{\xi}$ inductively : For $t_0 = 0$ define $\hat{\xi}(x,0) := f(x)$. Suppose now that there exists an equivariant homotopy $\hat{\xi}: X \times [0,t_\mu] \longrightarrow E$ such that $\hat{\xi} \circ i_0 = f$ and $p \circ \hat{\xi} = \xi$. We want to extend ξ over $X \times [0,t_{\mu+1}]$.
For each simplex σ in X choose a decomposing neighborhood U_x, denoted by U_σ, such that $\xi(y,t) \in U_\sigma$ for all $y \in \sigma$ and $t \in [t_\mu, t_{\mu+1}]$. For σ a 0-simplex in X define

$$\hat{\xi}(g\sigma,t) := g \cdot \eta_{U_\sigma}(\xi_t\sigma, \psi_{U_\sigma} \circ \eta^{-1}_{U_\sigma} \circ \hat{\xi}_{t_\mu}(\sigma)),$$

where $t \in [t_\mu, t_{\mu+1}]$, $g \in G$ and
$\psi_{U_\sigma} : U_\sigma \times p^{-1}(\sigma) \longrightarrow p^{-1}(\sigma)$
is the proyection onto the second factor.
ξ is well-defined : Because $\xi_t : X \longrightarrow B$ is an equivariant map, we have $G_\sigma < G_{\xi_t(\sigma)}$ with $\xi_t\sigma \in U_\sigma = U_x$ for some $x \in B$. From (2.2) follows $G_{\xi_t(\sigma)} < G_x$ and we have $G_\sigma < G_x$. But $\eta^{-1}_{U_\sigma}$ is a G_x-equivariant map (= equivariant for the subgroup G_x of G), therefore from $g\sigma = g_1\sigma$ follows that $\hat{\xi}(g\sigma,t) = \hat{\xi}(g_1\sigma,t)$ and $\hat{\xi}$ is an equivariant well-defined extension of ξ.

Suppose now that $\hat{\xi}$ is already been defined on $X \times [0,t_\mu] \cup X^{n-1} \times [t_\mu, t_{\mu+1}]$. We proceed to extend $\hat{\xi}$ over the n-dimensional skeleton X^n of X. Let σ be any n-simplex of X and put

$P := \sigma \times \{t_\mu\} \cup \dot{\sigma} \times [t_\mu \; t_{\mu+1}]$

$Q := \sigma \times [t_\mu, t_{\mu+1}]$.

P is a retract of Q. Let r: Q $\longrightarrow$ P be a retraction and define

$\hat{\xi}(gy,t) := g\cdot\eta_{U_\sigma} \; (\xi_t y, \psi_{U_\sigma}\cdot\eta_{U_\sigma}^{-1}\hat{\xi}\cdot r(y,t))$

for $y \in \sigma$, $t \in [t_\mu, t_{\mu+1}]$ and $g \in G$.

Finally, by the same argument used above for the 0-simplexes, we have that $\hat{\xi}$ is a well-defined equivariant extension of ξ. This completes the construction of $\hat{\xi}$ and the proof is finished.

Remark 2.3. An application of Proposition 2.1 to equivariant fixed point theory is described in [2] .

REFERENCES

[1] Bredon, G. , Introduction to Compact Transformation Groups. Acad. Press (1972).

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