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GENERALIZED HANKEL-CLIFFORD TRANSFORMATION OF CERTAIN SPACES OF DISTRIBUTIONS

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ABSTRACT

In this paper the generalized Hankel- Clifford transformation $F_{\alpha,\beta}\{f(x)\}=F(y)=\int_0^\infty (y/x)^{-(\alpha+\beta)/2} J_{\alpha-\beta}(2\sqrt{xy})f(x)dx$ $\quad (0<y<\infty)$ where $(\alpha-\beta)\geq -1/2$ and $J_\nu(x)$ denotes the Bessel function of the first kind and order ν, is extended to a class of generalized functions. A testing function space is constructed so as to contain the kernel, $(y/x)^{-(\alpha+\beta)/2} J_{\alpha-\beta}(2\sqrt{xy})$, of the transformation. Some properties of the function space and its dual are derived. An inversion theorem is established by interpreting the convergence in the weak distributional sense. The theory thus developed is applied to solve some partial differential equations involving the generalized Kepinski type operator $\Delta_{\alpha,\beta}=x^\beta D\, x^{\alpha-\beta+1} D\, x^{-\alpha}$ with distributional initial conditions.

KEY WORDS Hankel-Clifford transformation, generalized functions, countable union spaces, inversion theorem, adjoint method, kernel method, operational calculus, Cauchy problems.

1 INTRODUCTION

The conventional Hankel transformation defined by

$$h_\mu\{f(x)\}(y)=F(y)=\int_0^\infty \sqrt{xy}\; J_\mu(xy)f(x)dx \qquad (0<y<\infty) \tag{1.1}$$

was extended by Zemanian [13] to certain generalized functions of slow growth through a generalization of Parseval's equation. Later on Koh and Zemanian [3] extended (1.1) to a class of generalized functions by Kernel method, which is a more natural extension of (1.1) because the kernel appears explicitly as a testing function and is very well suited for specific computations.

The conventional Hankel-Clifford transformation defined by

$$cH_\mu\{f(x)\}=F(y)=\int_0^\infty (y/x)^{\mu/2} J_\mu(2\sqrt{xy})f(x)dx \qquad (0<y<\infty)$$
$$= y^\mu\int_0^\infty (xy)^{-\mu/2} J_\mu(2\sqrt{xy})f(x)dx \qquad (1.2)$$

was extended by Méndez and Socas [8] to certain generalized functions by the method of mixed Parseval's equation. The transformation (1.2) was also extended by Choudhary [1] to a class of generalized functions by kernel method in which the transformation (1.2) has not been precisely developed and studied in detail.

A simple generalization of (1.2), called generalized Hankel-Clifford transformation, is defined by

$$F_{\alpha,\beta}\{f(x)\}(y)=cH_{\alpha,\beta}\{f(x)\}(y)= \int_0^\infty (\frac{y}{x})^{-(\alpha+\beta)/2} J_{\alpha-\beta}(2\sqrt{xy})f(x)dx, (0<y<\infty)$$

$$=y^{-\alpha-\beta}\int_0^\infty (xy)^{(\alpha+\beta)/2}J_{\alpha-\beta}(2\sqrt{xy})f(x)dx$$

$$= F_1(y) \qquad (1.3)$$

where $(\alpha-\beta)\geq -1/2$ and $J_{\alpha-\beta}(z)$ is a Bessel function of the first kind of order $(\alpha-\beta)$. Note that when $\alpha=0$, $\beta=-\mu$ the transformation (1.3) reduces to (1.2) and when $\alpha=\mu/2, \beta=-\mu/2$ the transformation (1.3) reduces to the classical Hankel-transformation in Tricomi's form (by a suitable change of variables).

Following Méndez [7], Sneddon [10] and Watson [11], the convergence and the inversion for (1.3) and some of the important classical results which we require in subsequent work are given by the following theorems.

THEOREM 1 For $(\alpha-\beta)\geq -1/2$, if $f(x)$ is a locally integrable function on $0<x<\infty$ such that

$$f(x) = O(x^\eta), \quad x\to 0 \text{ and}$$
$$f(x) = O(x^\zeta), \quad x\to\infty,$$

then the integral $\int_0^\infty x^{(\alpha+\beta)/2} J_{\alpha-\beta}(2\sqrt{x})f(x)dx$ defining the transformation (1.3) is absolutely convergent when $\eta>-(\alpha+1)$ and $\zeta<-(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{3}{4})$.

THEOREM 2 (Inversion formula)

If f(x) is of bounded variation into a neighbourhood of the point $x_0>0$, $(\alpha-\beta)\geq -1/2$ and the integral $\int_0^\infty |f(x)|x^{-(\alpha+\beta)/2-(1/4)}dx$ exists, then

$$\lim_{R\to\infty} x_0^{-\alpha-\beta}\int_0^\infty f(x)\int_0^R y^{-\alpha-\beta}(xy)^{(\alpha+\beta)/2} J_{\alpha-\beta}(2\sqrt{(xy)})(x_0 y)^{(\alpha+\beta)/2} J_{\alpha-\beta}(2\sqrt{x_0 y})dydx$$

$$= \frac{1}{2}\{f(x_0+0)+f(x_0-0)\} \qquad (1.4)$$

or

$$\lim_{R\to\infty} \int_0^R (\frac{x_0}{y})^{-(\alpha+\beta)/2} J_{\alpha-\beta}(2\sqrt{xy})\ F_1(y)dy = \frac{1}{2}\{f(x_0+0)+f(x_0-0)\}$$

where $F_1(y)$ is as defined by (1.3).

THEOREM 3 : If $(\alpha-\beta)\geq -1/2$, then

$$\int_0^R J_{\alpha-\beta}(2\sqrt{ty})J_{\alpha-\beta}(2\sqrt{xy})dy$$

$$=\frac{\sqrt{R}}{(x-t)}[\sqrt{x}\ J_{\alpha-\beta+1}(2\sqrt{xR})J_{\alpha-\beta}(2\sqrt{tR})-\sqrt{t}\ J_{\alpha-\beta+1}(2\sqrt{tR})J_{\alpha-\beta}(2\sqrt{xR})] \qquad (1.5)$$

THEOREM 4: (Operational calculus)

(a) If $F_1(y)$ is as defined by (1.3) then

$$F_{\alpha,\beta}[f(ax)] = a^{-\alpha-\beta-1}F_{\alpha,\beta}[f(x)](y/a)$$

where a is a positive real constant.

(b) The transformation (1.3) satisfies the operational rule

$$F_{\alpha,\beta}\{\Delta^*_{\alpha,\beta}f(x)\}(y) = -y\ F_{\alpha,\beta}[f(x)](y) \qquad (1.6)$$

where

$$\Delta^*_{\alpha,\beta} = \Delta^*_{\alpha,\beta,x} = x^{-\alpha}\ D\ x^{\alpha-\beta+1}D\ x^{\beta} = [x\ D_x^2 + (\alpha+\beta+1)D_x + \alpha\beta x^{-1}] \qquad (1.7)$$

and f(x) is a suitable function .

(c) If $y = x^{(\alpha+\beta)/2}J_{\alpha-\beta}(2\sqrt{x})$ then it satisfies the differential equation

$$x\,y''-(\alpha+\beta-1)y'+(\alpha\beta x^{-1}+1)y=0 \qquad (1.8)$$

(d) If $y = x^{-(\alpha+\beta)/2}J_{\alpha-\beta}(2\sqrt{x})$ then it satisfies the differential equation

$$x\,y''+(\alpha+\beta+1)y'+(\alpha\beta x^{-1}+1)y=0$$

(e) $x^{\alpha+\beta+1}\dfrac{d}{dx}\left[x^{-(\alpha-\beta)/2}J_{\alpha-\beta}(2\sqrt{x})\right]=-x^{(\alpha-\beta+1)/2}J_{\alpha-\beta+1}(2\sqrt{x})$

(f) $\dfrac{d}{dx}\left[x^{(\alpha-\beta)/2}J_{\alpha-\beta}(2\sqrt{x})\right]=x^{(\alpha-\beta-1)/2}J_{\alpha-\beta-1}(2\sqrt{x})$

(g) $\dfrac{d}{dx}\left[x^{(\alpha+\beta)/2}J_{\alpha-\beta}(2\sqrt{x})\right]=\sum_{i=0}^{1}C_i(-1)^i x^{\beta-1}\left[x^{(\alpha-\beta+i)/2}J_{\alpha-\beta+i}(2\sqrt{x})\right]$

where $C_0=\alpha$ and $C_1=1$.

(h) If $\Delta_{\alpha,\beta}=\Delta_{\alpha,\beta,x}=x^{\beta}D_x x^{\alpha-\beta+1}D_x x^{-\alpha}=[x D_x^2-(\alpha+\beta-1)D_x+\alpha\beta x^{-1}]$

(1.9)

then

$$\Delta^k_{\alpha,\beta}\left[\left(\frac{y}{x}\right)^{-(\alpha+\beta)/2}J_{\alpha-\beta}(2\sqrt{xy})\right]=(-1)^k y^k\left[\left(\frac{y}{x}\right)^{-(\alpha+\beta)/2}J_{\alpha-\beta}(2\sqrt{xy})\right]$$

(1.10)

for $k = 0,1,2,3,\ldots\ldots.$

In accordance with (1.7) and (d), one has

$$(\Delta^*_{\alpha,\beta})^k\left[\left(\frac{x}{y}\right)^{-(\alpha+\beta)/2}J_{\alpha-\beta}(2\sqrt{xy})\right]=(-1)^k y^k\left[\left(\frac{x}{y}\right)^{-(\alpha+\beta)/2}J_{\alpha-\beta}(2\sqrt{xy})\right]$$

for $k = 0,1,2,3,\ldots\ldots\ldots$ (1.11)

(j) $\dfrac{d^m}{dy^m}\left[(2\sqrt{xy})^{\nu}J_{\nu}(2\sqrt{xy})\right]=2^m x^m(2\sqrt{xy})^{\nu-m}J_{\nu-m}(2\sqrt{xy})$ (1.12)

THEOREM 5 (Parseval's relation):

If $f(x)\,x^{\alpha}$ and $G_1(y)y^{\alpha}$ are in $L_1(0,\infty)$ and $(\alpha-\beta)\geq-1/2$, $F_1(y)=F_{\alpha,\beta}[f(x)](y)$ and $g(x)=F^{-1}_{\alpha,\beta}[G_1(y)](x)$, then

$$\int_0^\infty x^{\alpha+\beta}f(x)g(x)dx=\int^\infty y^{\alpha+\beta}F_1(y)G_1(y)dy. \qquad (1.13)$$

By invoking Fubini's theorem we also can prove

<u>THEOREM 6</u> (Mixed Parseval's equation)

If $f(x)x^{\alpha}$ and $G_2(y)y^{\beta}$ are in $L_1(0,\infty)$,

$F_1(y) = F_{\alpha,\beta}[f(x)](y)$ and $G_2(y)= F_{\beta,\alpha}[g(x)](y)$

$$=\int_0^{\infty}(x/y)^{-(\alpha+\beta)/2} J_{\alpha-\beta}(2\sqrt{xy})g(x)dx \quad (1.14)$$

then for $(\alpha-\beta)\geq -1/2$

$$\int_0^{\infty} f(x)g(x)dx = \int_0^{\infty} F_1(y)G_2(y)dy \quad (1.15)$$

According to Méndez [8]the equality (1.15) is called the mixed Parseval equation for the $F_{\alpha,\beta}$-transformation (1.3) and $F_{\beta,\alpha}$-transformation (1.14).The $F_{\alpha,\beta}$-transformation (1.3) is recently extended by Malgonde and Bandewar [6] to certain generalized functions of slow growth through a generalization of mixed Parseval's equation (1.15) for $(\alpha-\beta)\geq-1/2$, as follows:

The Fréchet spaces $H_{\beta}(I)$ and $S_{\alpha}(I)$ of testing function defined below by (1.17) and (1.18) respectively are constructed such that ordinary $F_{\alpha,\beta}$-transformation and the $F_{\beta,\alpha}$-transformation given by (1.3) and (1.14) respectively are automorphisms on them respectively for $(\alpha-\beta)\geq -1/2$ .As suggested by the mixed Parseval's equation (1.15), the distributional $F'_{\alpha,\beta}$-transformation $F'_{\alpha,\beta}f$ of any $f\in S'_{\alpha}(I)$ is defined by

$$\langle F'_{\alpha,\beta}f , \phi \rangle = \langle f , F_{\beta,\alpha}\phi \rangle \quad (1.16)$$

for every $\phi \in S_{\alpha}(I)$.Analogously,we can define the gereralized transformation $F'_{\beta,\alpha}$ on $H'_{\beta}(I)$. It follows that $F'_{\alpha,\beta}$ and $F'_{\beta,\alpha}$ are automorphisms on $S'_{\alpha}(I)$ and $H'_{\beta}(I)$,respectively, for $(\alpha-\beta)\geq -1/2$.

The object of the present paper is to extend the transformation (1.3) to other class of generalized functions following a different procedure called the kernel method and prove the inversion formula by interpreting the convergence in the weak distributional sense as well as other important properties of the distributional generalized Hankel- Clifford transformation. In the end, we develop an operational calculus that is applied to solve certain partial differential equations involving generalized differential operator of Kepinski type.

The notation and terminology used here are those of Zemanian [14]. Throughout this work I denotes the open interval $(0,\infty)$. $D(I)$ denotes the space of functions whose supports are compact subsets of I. We assign to $D(I)$ the topology that makes its dual $D'(I)$ the space of Schwartz's distribution on I [9]. $E(I)$ and $E'(I)$ are, respectively, the space of smooth functions on I and the space of distributions having compact support with respect to I. These spaces have their customary topologies [14].

Let β be any real number. $H_\beta(I)$ denotes the linear space consisting of all smooth complex-valued functions $\phi(x)$ on I such that, for every pair of of non-negative integers (m,k) the number

$$\rho^{\beta}_{m,k}(\phi(x)) = \sup_{x\in I} |x^m D^k x^\beta \phi(x)| \qquad (1.17)$$

exist.

The set of seminorms $\{\rho^{\beta}_{m,k}\}^{\infty}_{m,k=0}$ generates the topology of $H_\beta(I)$. The dual of $H_\beta(I)$ is $H'_\beta(I)$. $D(I)\subset H_\beta(I)$ and $H'_\beta(I) \subset D'(I)$.

Let α be any real number. $S_\alpha(I)$ denotes the linear space consisting of all smooth complex valued functions $\phi(x)$ on I such that, for every pair of non-negative integers (m,k), the numbers

$$\zeta^{\alpha}_{m,k}(\phi)= \sup_{x\in I}\left| x^m D^k x^{-\alpha}\phi(x)\right| \qquad (1.18)$$

exist. The set of seminorms $\{\zeta^{\alpha}_{m,k}\}^{\infty}_{m,k=0}$ generates the topology of $S_\alpha(I)$. The dual of $S_\alpha(I)$ is $S'_\alpha(I)$. $D(I)$ is proper subspace of $S_\alpha(I)$. $S'_\alpha(I)$ can be considered a subspace of $D'(I)$.

2. THE TESTING FUNCTION SPACES $\mathbb{H}_{\alpha,\beta,a}$ AND $\mathbb{H}_{\alpha,\beta}(\sigma)$ AND THEIR DUALS:

Let a denote a positive real number and α and β are any real numbers. Then we define $\mathbb{H}_{\alpha,\beta,a}$ as the space of testing functions $\phi(x)$ which are defined and smooth on $0<x<\infty$ and for which

$$\gamma_k^{\alpha,\beta,a}(\phi) = \sup_{0<x<\infty} \left| e^{-ax} x^{-\alpha} \Delta^k_{\alpha,\beta} \phi(x) \right| < \infty \qquad (2.1)$$

for k = 0,1,2,..., where $\Delta_{\alpha,\beta}$ is as defined in (1.9). We assign to $\mathbb{H}_{\alpha,\beta,a}$ the topologygenerated by the countable multinorm $\{\gamma_k^{\alpha,\beta,a}\}_{k=o}^{\infty}$. $\mathbb{H}_{\alpha,\beta,a}$ is a Hausdorff space, since $\gamma_o^{\alpha,\beta,a}$ is a norm on $\mathbb{H}_{\alpha,\beta,a}$ Moreover, $\mathbb{H}_{\alpha,\beta,a}$ is a locally convex linear space that satisfies the first axiom of countability. The dual space $\mathbb{H}'_{\alpha,\beta,a}$ consists of all continuous linear functionals on $\mathbb{H}_{\alpha,\beta,a}$. The dual is a linear space to which we assign the weak topology generated by the multinorm $\{\xi_\phi(f)\}_\phi$, where $\xi_\phi(f) = |\langle f,\phi\rangle|$ and ϕ varies through $\mathbb{H}_{\alpha,\beta,a}$.

We now list some properties of these spaces:

(i) Let $(\alpha-\beta) \geq -1/2$ and $a>0$. For a fixed positive real number y,

$$\frac{\partial^m}{\partial y^m}\Big[K(x,y) \Big] \in \mathbb{H}_{\alpha,\beta,a} \quad \text{for } m = 0,1,2,....$$

where $K(x,y)= (y/x)^{-(\alpha+\beta)/2} J_{\alpha-\beta}(2\sqrt{xy})$.

Indeed, it is easily verified that

$$\frac{\partial^m}{\partial y^m}\Big[K(x,y)\Big] = \frac{\partial^m}{\partial y^m}\Big[2^{-(\alpha-\beta)} x^{\beta} y^{-\alpha} (2\sqrt{xy})^{(\alpha-\beta)} J_{\alpha-\beta}(2\sqrt{xy})\Big]$$

$$=2^{-(\alpha-\beta)} x^{\beta} \sum_{j=o}^{m} \binom{m}{j} \frac{d^{m-j}}{dy^{m-j}} (y^{-\alpha}) \frac{d^j}{dy^j}\{(2\sqrt{xy})^{(\alpha-\beta)} J_{\alpha-\beta}(2\sqrt{xy})\}$$

$$= \sum_{j=o}^{m} C_j(\alpha)\, y^{j-m}[\, (y/x)^{-(\alpha+\beta+j)/2} J_{\alpha-\beta-j}(2\sqrt{xy}\,)]$$

using (1.12), where the $C_j(\alpha)$ are constants depending only on α. By the series and asymptotic expansion of $J_{\alpha-\beta-j}(2\sqrt{xy}\,)$, it follows that the quantities

$$\gamma_k^{\alpha,\beta,a}\left[(y/x)^{-(\alpha+\beta+j)/2} J_{\alpha-\beta-j}(2\sqrt{xy}\,)\right]$$ exist for all k=0,1,2,.. and $(\alpha-\beta)\geq -1/2$. Hence

$$\gamma_k^{\alpha,\beta,a}\left\{ \frac{\partial^m}{\partial y^m}\left[K(x,y)\right]\right\}$$

$$\leq \sum_{j=0}^{m} C_j(\alpha) y^{j-m}\, \gamma_k^{\alpha,\beta,a}\{ (y/x)^{-(\alpha+\beta+j)/2}\, J_{\alpha-\beta-j}(2\sqrt{xy}\,)\} < \infty$$

for any fixed y>0.

(ii) $\mathbb{H}_{\alpha,\beta,a}$ is sequentially complete and therefore a Fréchet space. Hence, $\mathbb{H}'_{\alpha,\beta,a}$ is also sequentially complete.

To prove the first part, we use an inductive argument based on

$$\Delta^k_{\alpha,\beta}(\phi(x)) = \sum_{i=0}^{2k} C_{i.k}\, x^{k-i} D^{2k-i}\, \phi(x) \qquad (2.2)$$

where, the $C_{i.k}$'s are constants depending upon α and β and k is a nonnegative integer.

If $\{\phi_\nu\}_{\nu=1}^{\infty}$ is a Cauchy sequence in $\mathbb{H}_{\alpha,\beta,a}$ then by seminorms $\gamma_k^{\alpha,\beta,a}$ and (2.2) there exists a smooth function ϕ on I such that for every $n\in\mathbb{N}$ $\lim_{\nu\to\infty} D^n\phi_\nu(x)=D^n\phi(x)$ uniformely on every compact subset contained in I.

Since $\gamma_k^{\alpha,\beta,a}(\phi_\nu)$ is finite for every $\nu\in\mathbb{N}$, it follows readily that $\phi_\nu\to\phi$,as $\nu\to\infty$ in the sense of convergence in $\mathbb{H}_{\alpha,\beta,a}$.

(iii) If a>b>0, then $\mathbb{H}_{\alpha,\beta,b}\subset\mathbb{H}_{\alpha,\beta,a}$,and the topology of $\mathbb{H}_{\alpha,\beta,b}$ is stronger than that induced on it by $\mathbb{H}_{\alpha,\beta,a}$ .This follows immediately from the inequality $\gamma_k^{\alpha,\beta,a}(\phi)\leq\gamma_k^{\alpha,\beta,b}(\phi)$ for $\phi\in\mathbb{H}_{\alpha,\beta,b}$. Hence the restriction of $f\in\mathbb{H}'_{\alpha,\beta,a}$ to $\mathbb{H}_{\alpha,\beta,b}$ is in $\mathbb{H}'_{\alpha,\beta,b}$, and the convergence in $\mathbb{H}'_{\alpha,\beta,a}$ implies convergence in $\mathbb{H}'_{\alpha,\beta,b}$.

(iv)Let $(\alpha-\beta)\geq -1/2$. $S_\alpha(I)$ is a subspace of $\mathbb{H}_{\alpha,\beta,a}$ for every choice of a>0 and the topology of $S_\alpha(I)$ being stronger than that induced on it by

$\mathbb{H}_{\alpha,\beta,a}$.

To see this, we note that

$$x^{-\alpha}\Delta^{k}_{\alpha,\beta}\phi(x) = \sum_{j=0}^{2k} C_{jk}\, x^{k-j} D^{2k-j} x^{-\alpha}\phi(x)$$

where the constants C_{jk} depend on α and β. Since, for any choice of a>0, $e^{-ax}\leq 1$ on $0<x<\infty$,we have

$$\gamma_k^{\alpha,\beta,a}(\phi) \leq \sum_{j=0}^{2k}|C_{jk}|\ \zeta_{k-j,2k-j}(\phi) \quad \text{for any } \phi\in S_\alpha(I).$$

This implies our assertion and shows that $S_\alpha(I)$ is continuously imbedded on $\mathbb{H}_{\alpha,\beta,a}$. It now follows that the restriction of any $f\in \mathbb{H}'_{\alpha,\beta,a}$ to $S_\alpha(I)$ in $S'_\alpha(I)$ and that convergence in $\mathbb{H}'_{\alpha,\beta,a}$ implies convergence in $S'_\alpha(I)$. We note that $S_\alpha(I)$ is the proper subset of $\mathbb{H}_{\alpha,\beta,a}$. Indeed $K(x,y)\in\mathbb{H}_{\alpha,\beta,a}$ for all y fixed as was already shown in note(1), but not in $S_\alpha(I)$, because this function is not of rapid descent at infinity [6].

(v) $D(I)\subset\mathbb{H}_{\alpha,\beta,a}$, and the topology of D(I) is stronger than that induced on it by $\mathbb{H}_{\alpha,\beta,a}$. Hence, the restriction of any $f\in \mathbb{H}'_{\alpha,\beta,a}$ to D(I) is in D´(I), and convergence in $\mathbb{H}'_{\alpha,\beta,a}$ implies weak convergence in D´(I)

(vi) For every choice of a, α and β, $\mathbb{H}_{\alpha,\beta,a}\subset E(I)$. Moreover, it is dense in E(I) because $D(I)\subset\mathbb{H}_{\alpha,\beta,a}$ and D(I) is dense in E(I). The topology of $\mathbb{H}_{\alpha,\beta,a}$ is stronger than that induced on it by E(I). Hence, E´(I) can be identified with a subspace of $\mathbb{H}'_{\alpha,\beta,a}$.

(vii) The generalized Kepinski operator $\Delta_{\alpha,\beta}$ is a continuous linear mapping from the space S(I) into itself. Analogously $\Delta^{*}_{\alpha,\beta}$ defined in (1.7) is a continuous linear mapping from the space S (I) into itself.

(viii) The operation $\phi\rightarrow\Delta_{\alpha,\beta}\phi$ is a continuous linear mapping of $\mathbb{H}_{\alpha,\beta,a}$ into itself since

$$\gamma_k^{\alpha,\beta,a}(\Delta_{\alpha,\beta}\phi) = \gamma_{k+1}^{\alpha,\beta,a}(\phi) \qquad \text{for } k = 0,1,2 \ldots.$$

We define the operator $\Delta^{*}_{\alpha,\beta}$ on $\mathbb{H}'_{\alpha,\beta,a}$ by

$$\langle \Delta^{*}_{\alpha,\beta} f , \phi \rangle = \langle f , \Delta_{\alpha,\beta}\phi \rangle , \ \phi\in\mathbb{H}_{\alpha,\beta,a} \text{ and } f\in \mathbb{H}'_{\alpha,\beta,a}.$$

This is consistent with the usual definitions of multiplication by a smooth function and differentiation of generalized functions ([12],pp.28, 47).Since $\phi\to\Delta_{\alpha,\beta}\phi$ is a continuous linear mapping of $\mathbb{H}_{\alpha,\beta,a}$ into itself, it follows that $f\to\Delta^{*}_{\alpha,\beta}f$ is a continuous mapping of $\mathbb{H}'_{\alpha,\beta,a}$ into itself.

(ix)For each $f\in\mathbb{H}'_{\alpha,\beta,a}$,there exist a nononegative integer r and a positive constant c such that, for all $\phi\in\mathbb{H}_{\alpha,\beta,a}$,

$$|\langle f,\phi\rangle| \le c \max_{0\le k\le r} \gamma_k^{\alpha,\beta,a}(\phi)$$

The proof of this statement is similar to that of Zemanian [12,Theorem 3.3-1].

(x) Let f(x) be a locally integrable function on $0<x<\infty$ and such that $\int_0^\infty |f(x)\, e^{ax} x^{\alpha}|dx<\infty$.Then f(x) generates a regular generalized function in $\mathbb{H}'_{\alpha,\beta,a}$ defined by

$$\langle\, f\ ,\ \phi\,\rangle\ =\ \int_0^\infty f(x)g(x)dx\quad ,\ \phi\in\mathbb{H}_{\alpha,\beta,a}.$$

That f(x) is truly in $\mathbb{H}'_{\alpha,\beta,a}$ follows from the inequality

$$\xi_{\phi}(f)\ =\ |\langle f,\phi\rangle| \le \gamma_0^{\alpha,\beta,a}(\phi)\ \int^{\infty} |f(x)\ e^{ax}x^{\alpha}|\,dx\ <\infty$$

(xi)$\mathbb{H}_{\alpha,\beta,a}$ is not in general closed with respect to differentiation.For example the function defined as $\phi(x)=x^{\alpha+\beta}$ is in $\mathbb{H}_{\alpha,\beta,a}$ and however the derivative $D\phi(x)=(\alpha+\beta)x^{\alpha+\beta-1}$is not in $\mathbb{H}_{\alpha,\beta,a}$ provided that $0<\beta<1-\alpha$ and $\alpha+\beta\neq0$.

Next,we give the structure formula for the restriction of an element in $\mathbb{H}'_{\alpha,\beta,a}$ to D(I).

(xii)Let f be an arbitrary element of $\mathbb{H}'_{\alpha,\beta,a}$ .Then there exist bounded measurable functions $g_i(x)$ defined for x>0 and i=0,1,2...r where r is some non-negative integer depending upon f, such that for an arbitrary $\phi\in D(I)$ we have

$$\langle f,\phi\rangle\ =\ \langle \sum_{i=0}^{r} (\Delta^{*}_{\alpha,\beta})^{i}\ \{e^{-ax}x^{-\alpha}\ (-D_x)\ g_i(x)\}\ ,\ \phi(x)\,\rangle.$$

We turn now to the definition of a certain countable-union space $\mathbb{H}_{\alpha,\beta}(\sigma)$ that arises from the $\mathbb{H}_{\alpha,\beta,a}$ spaces. Our subsequent discussion takes on a simpler form when the $\mathbb{H}_{\alpha,\beta}(\sigma)$ spaces are used in place of the $\mathbb{H}_{\alpha,\beta,a}$ spaces. Following Friedman [2,p.21] and Koh and Zemanian [3], $\mathbb{H}_{\alpha,\beta}(\sigma)=\bigcup_{\nu=1}^{\infty}\mathbb{H}_{\alpha,\beta,a_\nu}$ is the countable-union space where $\{a_\nu\}_{\nu=1}^{\infty}$ is a monotonic sequence of positive numbers such that $a_\nu\to\sigma$ ($\sigma=+\infty$ is allowed). A generalized function f is $F'_{\alpha,\beta}$-transformable if $f\in\mathbb{H}'_{\alpha,\beta}(\sigma)$ for some $\sigma>0$ where $\mathbb{H}'_{\alpha,\beta}(\sigma)$ is the dual of $\mathbb{H}_{\alpha,\beta}(\sigma)$.

In view of our definition of $\mathbb{H}_{\alpha,\beta}(\sigma)$ and its dual, the following Lemmas are immediate.

LEMMA 1: For any fixed $y>0$, $\dfrac{\partial^m}{\partial y^m}$ [$K(x,y)$] $\in\mathbb{H}_{\alpha,\beta}(\sigma)$, $m=0,1,2\ldots$ where $\sigma>0$.

LEMMA 2: For every choice of $\sigma>0$, $S_\alpha(I)\subset\mathbb{H}_{\alpha,\beta}(\sigma)$ and convergence in $S_\alpha(I)$ implies convergence in $\mathbb{H}_{\alpha,\beta}(\sigma)$. The restriction of $f\in\mathbb{H}_{\alpha,\beta}(\sigma)$ to $S_\alpha(I)$ is in $S'_\alpha(I)$, and convergence in $\mathbb{H}'_{\alpha,\beta}(\sigma)$ implies convergence in $S'_\alpha(I)$.

LEMMA 3: The operation $\phi\to\Delta_{\alpha,\beta}\phi$ is a continuous linear mapping of $\mathbb{H}_{\alpha,\beta}(\sigma)$ into itself. Hence the operation $f\to\Delta^*_{\alpha,\beta}$ is a continuous mapping of $\mathbb{H}'_{\alpha,\beta}(\sigma)$ into itself by Zemanian [14, Theorem 1.9-1].

As was indicated in note(vi), $\mathbb{H}'_{\alpha,\beta}(\sigma)$ contains all distributions of compact support on $I=(0,\infty)$. Similarly, any conventional function f satisfying the conditions stated in note(x) for some $a<\sigma$ is a member of $\mathbb{H}'_{\alpha,\beta}(\sigma)$, as is every generalized derivative $\Delta^k f$, $k=1,2,3\ldots$, according to Lemma 3. Moreover, we may say that the member of $\mathbb{H}'_{\alpha,\beta}(\sigma)$ are "generalized functions of exponential descent", since the multinorm $\{\gamma_k^{\alpha,\beta,a}\}$ shows that the testing functions $\phi\in\mathbb{H}_{\alpha,\beta,a}$ are at most of exponential growth.

3 THE DISTRIBUTIONAL GENERALIZED HANKEL-CLIFFORD TRANSFORMATION:

Let α and β be real numbers restricted to $-1/2<(\alpha-\beta)<\infty$. In view of note (iii) of § 2, to every $f\in\mathbb{H}'_{\alpha,\beta,a}$ there exists a unique real number σ_f (possibly, $\sigma_f=+\infty$) such that $f\in\mathbb{H}'_{\alpha,\beta,b}$ if $b<\sigma_f$ and $f\notin\mathbb{H}'_{\alpha,\beta,b}$ if $b>\sigma_f$. Therefore, $f\in\mathbb{H}'_{\alpha,\beta}(\sigma_f)$. We define the distributional generalized

Hankel-Clifford transform $F'_{\alpha,\beta}f$ of f as the application of f to the kernel K(x,y); i.e,

$$F_1(y) = (F'_{\alpha,\beta}f)(y) = \langle f(x) , K(x,y)\rangle$$
$$= \langle f(x), (y/x)^{-(\alpha+\beta)/2} J_{\alpha-\beta}(2\sqrt{xy})\rangle \quad (3.1)$$

where $0<y<\infty$ and $\sigma_f>0$. The right hand side of (3.1) has a sense because, by Lemma 1, $K(x,y)\in \mathbb{H}_{\alpha,\beta}(\sigma_f)$ for each $y>0$.

If f(x) satisfies the conditions of the note (x) in §2 for every $a<\sigma_f$,then we may write

$$F_1(y) = (F'_{\alpha,\beta}f)(y) = \int_0^\infty f(x)K(x,y)dx \quad (3.2)$$

for $0<y<\infty$.

<u>LEMMA 4</u>:Let a and σ_f be fixed real numbers such that $0<a<\sigma_f$. For all fixed $y>0$, for $(\alpha-\beta)\geq-1/2$ and for $0<x<\infty$

$$|e^{-ax}(2\sqrt{xy})^{-(\alpha-\beta)} J_{\alpha-\beta} (2\sqrt{xy})| < A_{\alpha,\beta} \quad (3.3)$$

where $A_{\alpha,\beta}$ is a constant with respect to x and y.

Proof:The proof is simple and can be easily verified following Koh and Zemanian [3].

We will now show that the transform $F_1(y)= F'_{\alpha,\beta}f$ is analytic. Namely,

<u>THEOREM</u> 7: For y > 0, let $F_1(y)$ be defined by (3.1). Then

$$\frac{d}{dy} F_1(y) = \langle f(x) , \frac{\partial}{\partial y} K(x,y) \rangle$$

Proof:Let y be an arbitrary but fixed positive real number. Let us choose Δy as a nonzero increment such that $0<|\Delta y| <y/2$.

For $\Delta y \neq 0$,

$$\frac{F_1(y+\Delta y)-F_1(y)}{\Delta y} - \langle f(x) , \frac{\partial}{\partial y} K(x,y)\rangle = \langle f(x) , \phi_{\Delta y} (x)\rangle \quad (3.4)$$

where

$$\phi_{\Delta y}(x) = \frac{K(x,y+\Delta y) - K(x,y)}{\Delta y} - \frac{\partial}{\partial y} K(x,y)$$

Our theorem will be proved when we show that (3.4) converges to zero as $|\Delta y|\to 0$ .This can be done by showing that $\phi_{\Delta y}(x)$ converges to zero as $|\Delta y|\to 0$ in $\mathbb{H}_{\alpha,\beta}(\sigma_f)$.By using the fact from (1.10)

$$\Delta^k_{\alpha,\beta}K(x,y) = (-1)^k\ y^k\ K(x,y) \tag{3.5}$$

and by interchanging $\frac{\partial}{\partial y}$ with $\Delta^k_{\alpha,\beta}$ we can write

$$e^{-ax}x^{-\alpha}\ \Delta^k_{\alpha,\beta}\phi_{\Delta y}(x) = e^{-ax}x^{-\alpha}\Big[\ \frac{(-1)\ \ (y+\Delta y)^k\ K(x,y+\Delta y)}{\Delta y}$$

$$- \frac{(-1)^k y^k\ \ K(x,y)}{\Delta y} - \frac{\partial}{\partial y}(-1)^k y^k\ K(x,y)\Big]$$

$$= e^{-ax}x^{-\alpha}(-1)^k\Big[\frac{1}{\Delta y}\int_y^{y+\Delta y}\frac{\partial}{\partial t}(t^k\ K(x,t))dt - \{\frac{\partial}{\partial t}(t^k\ \ K(x,t))\}_{t=y}\Big]$$

$$=\frac{e^{-ax}\ x^{-\alpha}\ \ (-1)^k}{\Delta y}\int_y^{y+\Delta y}dt\int_y^t\frac{\partial^2}{\partial u^2}\{u^k\ K(x,u)\}du$$

By letting the constant M be a bound on of the expression

$\left|\frac{\partial^2}{\partial u^2}\{u^k\ K(x,u)\}\right|$,we obtain

$$|e^{-ax}x^{-\alpha}\ \Delta^k_{\alpha,\beta}\ \phi_{\Delta y}(x)| \le \frac{M}{|\Delta y|}\int_y^{y+\Delta y}\int_y^t dtdu = \frac{M|\Delta y|}{2}\to 0 \text{ as } |\Delta y|\to 0.$$

Thus,as $|\Delta y|\to 0$, $\gamma_k^{\alpha,\beta,a}\ [\phi_{\Delta y}(x)] \to 0$ for every $k \in$ N.
Consequently , (3.4) vanishes as $|\Delta y| \to 0$.

<u>THEOREM 8</u>:Let $F_1(y)$ be defined by (3.1). Then $F_1(y)$ is bounded according to

$$|F_1(y)| \le \begin{cases} c\ y^{-\beta} & \text{as } y\to 0^+ \\ c\ y^{r-\beta} & \text{as } y\to\infty \end{cases} \tag{3.6}$$

where c is a positive constant and r is some non-negative integer depending on f.

Proof :Since $f \in \mathbb{H}'_{\alpha,\beta,b}$ where $0 < a < b < \sigma_f$, we see from note (ix) of § 2 that there exist a constant $c'>0$ and a non-negative integer r such that

$$|F_1(y)| \le c' \max_{0\le k\le r} \sup_{0<x<\infty} \left| e^{-ax} x^{-\alpha} \Delta^k_{\alpha,\beta}[K(x,y)] \right| .$$

Here, c' and r depend in general on the choice of a. By (3.5), the right-hand side is equal to

$$c' \max_{0\le k\le r} \sup_{0<x<\infty} \left| e^{-ax} (-1)^k \; y^{k-\beta} 2^{\alpha-\beta} (2\sqrt{xy})^{-(\alpha-\beta)} J_{\alpha-\beta}(2\sqrt{xy}) \right| .$$

$$= c \max_{0\le k\le r} [y^{k-\beta}] ,$$

using Lemma 4, from which the theorem follows.

In view of note (iv) of §2 and Lemma 2 if f is in $\mathbb{H}'_{\alpha,\beta}(\sigma_f)$ then f belong to $S'_\alpha(I)$ provided that $(\alpha-\beta)\ge -1/2$. We now show that distributional generalized Hankel-Clifford transformation of $f\in \mathbb{H}'_{\alpha,\beta}(\sigma_f)$ given by (3.1) is equal (in the sense of equality in $S'_\alpha(I)$) to the distributional generalized Hankel-Clifford transformation of f as given by (1.16).

<u>THEOREM 9</u>: Let $f \in \mathbb{H}'_{\alpha,\beta}(\sigma_f)$, $\phi \in S_\alpha(I)$, and $(\alpha-\beta)\ge -1/2$. Then

$$\langle \langle f(x) , K(x,y)\rangle , \phi(y) \rangle = \langle f(x) , \int_0^\infty K(x,y)\phi(y)dy\rangle \qquad (3.7)$$

Proof: We now restrict y to the positive real line. Since $F_1(y)=\langle f(x),K(x,y)\rangle$ is of slow growth as $y\to\infty$ (Theorem 8) and is Lebesgue integrable on $0<y<Y$ for every Y on $(0,\infty)$, we may take into account that $F_1(y)$ generates a regular member in $S'_\alpha(I)$ with $(\alpha-\beta)\ge -1/2$ to write the left hand side of (3.7) as

$$\langle \langle f(x) , K(x,y)\rangle , \phi(y) \rangle = \int_0^\infty \langle f(x) , K(x,y)\rangle \phi(y)dy, \ \phi \in S_\alpha(I) \qquad (3.8)$$

Our theorem will be proved when we show that

$$\int_0^\infty \langle f(x) , K(x,y)\rangle \phi(y)dy = \langle f(x) , \int_0^\infty K(x,y)\phi(y)dy\rangle \qquad (3.9)$$

Following Zemanian [14,p.148] and Koh and Zemanian [3], the technique of Riemann sums can be used to express

$$\int_0^Y \langle f(x), K(x,y)\rangle \phi(y)dy = \langle f(x), \int_0^Y K(x,y)\phi(y)dy\rangle \tag{3.10}$$

Finally, the right hand side of (3.10) converges to the right hand side of (3.9) because of the following inequalities:

$$\left|e^{-ax}x^{-\alpha}\Delta^k_{\alpha,\beta}\int_Y^\infty \phi(y)K(x,y)dy\right|$$

$$= \left|\int_Y^\infty \phi(y)e^{-ax}x^{-\alpha}(-1)^k y^k K(x,y)dy\right|$$

$$\leq \int_Y^\infty |\phi(y)y^{k-\beta}2^{\alpha-\beta}e^{-ax}(2\sqrt{xy})^{-(\alpha-\beta)}J_{\alpha-\beta}(2\sqrt{xy})|dy$$

$$\leq A_{\alpha-\beta}\int_Y^\infty |\phi(y)y^{k-\beta}|dy.$$

The last inequality is due to Lemma 4.Since $\phi \in S_\alpha(I)$ is of rapid descent, the last integral, which is independant of x , vanishes as $Y\to\infty$.This completes the proof of the theorem.

We now state an inversion theorem for our distributional generalized Hankel-Clifford transformation ($F'_{\alpha,\beta}$-transformation).

<u>THEOREM 10</u>:Let $F_1(y) = (F'_{\alpha,\beta}f)(y)$, $f\in \mathbb{H}'_{\alpha,\beta}(\sigma_f)$ as in (3.1) where $y>0$.Let $(\alpha-\beta)\geq -1/2$.Then,in the sense of convergence in $D'(I)$

$$f(x) = \lim_{R\to\infty}\int_0^R F_1(y)K(y,x)dy \tag{3.11}$$

where $K(y,x) = (x/y)^{-(\alpha+\beta)/2}J_{\alpha-\beta}(xy)$

Proof:Let $\phi(x)\in D(I)$. We wish to show that

$$\langle \int_0^R F_1(y)K(y,x)dy , \phi(x)\rangle \tag{3.12}$$

tends to $\langle f(x), \phi(x)\rangle$ as $R\to\infty$. From the smoothness of $F_1(y)$ and the fact that support of $\phi(x)$ is a compact subset of I, we may write (3.12) as a repeated integral on (x,y) having a continuous integrand and a finite domain of integration. Hence we can change the order of the integration and obtain

$$\int_0^\infty \phi(x) \int_0^R F_1(y)K(y,x)dydx = \int_0^R \langle f(t) , K(t,y)\int_0^\infty \phi(x)K(y,x)dxdy \tag{3.13}$$

By an argument based on Riemann sums for the integral $\int_0^R \ldots dy$, the right side of (3.13) can be written as

$$\langle f(t) , \int_0^R K(t,y)\int_0^\infty \phi(x)K(y,x)dxdy \rangle \tag{3.14}$$

$$= \langle f(t), \int_0^R (\frac{y}{t})^{-(\alpha+\beta)/2} J_{\alpha-\beta}(2\sqrt{ty})\int_0^\infty \phi(x)(\frac{x}{y})^{-(\alpha+\beta)/2} J_{\alpha-\beta}(2\sqrt{xy})dxdy\rangle$$

From (1.5) of Theorem 3, and the asymptotic representation of the Bessel functions enable us to show that for any $a>0$, the testing function in (3.14) converges in $\mathbb{H}_{\alpha,\beta,a}$ to $\phi(t)$ as $R\to\infty$. Since $f\in\mathbb{H}'_{\alpha,\beta,a}$ where $0<a<\sigma_f$,it follows that (3.14) converges to $\langle f(t),\phi(t)\rangle$ as $R\to\infty$. This proves our theorem.

As a result of the inversion theorem, we have the following uniqueness theorem.

THEOREM 11 :Let $F_1(y)= F'_{\alpha,\beta}f$ for $y>0$ and $G_1(y)= F'_{\alpha,\beta}g$ for $y>0$, f and g being in $\mathbb{H}'_{\alpha,\beta}(\sigma)$. If $F_1(y)=G_1(y)$, for every $y>0$, then f=g in the sense of equality in $D'(I)$.

Proof : By theorem 10,

$$f-g = \lim_{R\to\infty} \int_0^R [F_1(y) - G_1(y)] K(y,x)dy = 0.$$

4. AN OPERATIONAL CALCULUS :

Our distributional generalized Hankel-Clifford transformation generates an operational calculus by means of which certain (partial) differential equations involving generalized functions can be solved.

We define the operator

$$\Delta^*_{\alpha,\beta} : \mathbb{H}'_{\alpha,\beta}(\sigma_f) \to \mathbb{H}'_{\alpha,\beta}(\sigma_f)$$

by the relation

$$< \Delta^*_{\alpha,\beta}\ f(x)\ ,\ \phi(x) > \ = \ < f(x)\ ,\ \Delta_{\alpha,\beta}\phi(x)>$$

for all $f\in \mathbb{H}'_{\alpha,\beta}(\sigma_f)$ and $\phi(x)\in \mathbb{H}_{\alpha,\beta}(\sigma_f)$, $(\alpha-\beta)\geq-1/2$. It can be readily seen that

$$< (\Delta^*_{\alpha,\beta})^k f(x)\ ,\ \phi(x) > \ = \ < f(x)\ ,\ \Delta^k_{\alpha,\beta}\phi(x)>$$

for each k=1,2,3,......In this case f is a regular distribution generated by an element of D(I),then

$\Delta^*_{\alpha,\beta}=\ [\ xD_x^2+(\alpha+\beta+1)D_x+\alpha\beta x^{-1}]$ as defined by (1.7).

The distributional generalized Hankel-Clifford transformation is useful in solving initial value problems. Indeed, we now establish a theorem that enables us to transform a differential equation of the form

$$P[\Delta^*_{\alpha,\beta}]\,u \ = \ g \tag{4.1}$$

where P is a polynomial and u and g possess $F'_{\alpha,\beta}$-transforms, into an algebraic equation of the form

$$P[\,-y\,]\,U(y) \ = \ G(y)$$

where $U(y) = (F'_{\alpha,\beta}u(x))(y)$ and $G(y) = (F'_{\alpha,\beta}g(x))(y)$.

<u>**THEOREM 12:**</u> For k = 0,1,2,.....

$$F'_{\alpha,\beta}[\Delta^{*k}_{\alpha,\beta}f] \ = \ (-1)^k y^k F'_{\alpha,\beta}f \tag{4.2}$$

for $f\in\mathbb{H}'_{\alpha,\beta}(\sigma_f)$.

Proof: From our definition of the operator $\Delta^*_{\alpha,\beta}$ and from (3.5), we have

$$\begin{aligned} F'_{\alpha,\beta}[\Delta^{*k}_{\alpha,\beta}f]\,(y) &= \langle(\Delta^*_{\alpha,\beta})^k f(x),\ K(x,y)\rangle \\ &= \langle f(x)\ ,\ \Delta^k_{\alpha,\beta}K(x,y)\rangle \\ &=(-1)^k y^k\ \langle f(x)\ ,\ K(x,y)\rangle \\ &=(-1)^k y^k\ (F'_{\alpha,\beta}f)(y) \end{aligned}$$

We now wish to find a generalized function $u\in\mathbb{H}'_{\alpha,\beta}(\sigma)$ for some $\sigma>0$ satisfying the distributional differential equation (4.1) where P is a polynomial having no roots on $-\infty<x\leq0$. By applying $F'_{\alpha,\beta}$

to (4.1) and invoking Theorem 12, we obtain

$$P[-y]\,U(y) = G(y)$$

where U and G are $F'_{\alpha,\beta}$ - transfomations of u and g respectively. Taking the inverse $F'_{\alpha,\beta}$-transformation, we get as our solution a generalized function u in $\mathbb{H}'_{\alpha,\beta}(\sigma)$ such that

$$\langle u,\phi\rangle = \lim_{R\to\infty} \left\langle \int_0^R \frac{G(y)}{P[-y]} K(y,x)dy\ ,\ \phi(x)\right\rangle \tag{4.3}$$

for every $\phi \in D(I)$ if $P[-y] \neq 0$.

We now wish to find $u \in \mathbb{H}'_{\alpha,\beta}(\sigma)$ such that $(F'_{\alpha,\beta}u)(y) = \dfrac{G(y)}{P[-y]}$.

By Theorem 8, we know that $|G(y)| \leq c\, y^{r-\beta}$ as $y \to \infty$ for some non-negative integer r depending upon g and so

$\dfrac{G(y)}{P[-y]}$ K(y,x) is an integrable function in $0<y<R$ and then

$\displaystyle\int_0^R \frac{G(y)}{P[-y]} K(y,x)dy$ is a locally integrable function in $0<x<\infty$

and it defines a regular distribution in $D'(I)$.Now let Q(x) be a polynomial of degree (r+1) having no zeros on the negative real axis. Then the convergence of the right hand side of (4.3) can be established as follows:

$$\left\langle \int_0^R \frac{G(y)}{P[-y]} K(y,x)dy\ ,\ \phi(x)\right\rangle$$

$$= \left\langle Q(\Delta^*_{\alpha,\beta})\int_0^R \frac{G(y)}{P[-y]Q[-y]} K(y,x)dy, \phi(x)\right\rangle$$

$$= \left\langle \int_0^R \frac{G(y)}{P[-y]Q[-y]} K(y,x)dy, Q(\Delta_{\alpha,\beta})\phi(x)\right\rangle$$

by integration by parts.

Let us suppose that the support of $\phi(x)$ is contained in [A,B]. Then, we can find a constant L such that for $R_1, R_2 > L$ we have

$$|J| = \left| \langle \int_{R_1}^{R_2} \frac{G(y)}{P[-y]} K(y,x)dy \ , \ \phi(x)\rangle \right|$$

$$\leq \int_{R_2}^{R_2} \frac{M}{|P[-y]|\,|Q[-y]|} y^r dy \to 0 \text{ as } R_1, R_2 \to \infty$$ **where M is a suitable positive number. Therefore**

$$\lim_{R\to\infty} \langle \int_0^R \frac{G(y)}{P[-y]} K(y,x)dy \ , \ \phi(x)\rangle$$

exists and by completeness of $D'(I)$ there exists $f \in D'(I)$ such that

$$\lim_{R\to\infty} \langle \int_0^R \frac{G(y)}{P[-y]} K(y,x)dy \ , \ \phi(x)\rangle = \langle f,\phi\rangle \tag{4.4}$$

Now for all $\phi \in D(I)$, we have

$$\lim_{R\to\infty} \langle P(\Delta^*_{\alpha,\beta}) \int_0^R \frac{G(y)}{P[-y]} K(y,x)dy \ , \ \phi(x)\rangle = \langle P(\Delta^*_{\alpha,\beta})f,\phi\rangle,$$

or

$$\lim_{R\to\infty} \langle \int_0^R G(y)K(y,x)dy \ , \ \phi(x)\rangle = \langle P(\Delta^*_{\alpha,\beta})f,\phi\rangle$$

Hence by our inversion Theorem 10 it follows that

$\langle g,\phi\rangle = \langle P(\Delta^*_{\alpha,\beta})f,\phi\rangle$.

This proves that f determined by (4.4), which belongs to $D'(I)$ and is the restriction of $u \in H'_{\alpha,\beta}(\sigma)$ to $D(I)$, satisfies the differential equation (4.1).

5 APPLICATIONS:

This section is devoted to an application of the preceding theory to two Cauchy type problems having generalized functions like initial conditions. The distributional generalized Hankel-Clifford transformation can be used in solving certain partial differential equations involving the generalized Kepinski operator. We point out that other equations, as the generalized Fokker-Plank equation

$$\lambda \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}[\,p(x)\frac{\partial u}{\partial x}\,] + \frac{\partial}{\partial x}[\,q(x)u] + r(x)u$$

when $p(x) = x$, $q(x) = \alpha+\beta$ and $r(x) = \alpha\beta x^{-1}$ (α and β are real numbers and constant), as well as the general linear equation of heat conduction

$$a(x,t)\frac{\partial u}{\partial t} - b(x,t)\frac{\partial^2 u}{\partial x^2} - c(x,t)\frac{\partial u}{\partial x} - d(x,t)u = f(x,t)$$

when $a(x,t) = \lambda$, $b(x,t) = x$, $c(x,t) = \alpha+\beta+1$, $d(x,t) = \alpha\beta x^{-1}$

$f(x,t) = 0$, reduce to equations of generalized Kepinski type.

An operational calculus generated by $F'_{\alpha,\beta}$-transformation is now used in solving two Cauchy type problems:

Let us determine a function $u(x,t)$ on the domain $\{(x,t): 0<x<\infty, 0<t<\infty\}$ which satisfies the generalized Kepinski equation

$$x\frac{\partial^2 u}{\partial x^2} + (\alpha+\beta+1)\frac{\partial u}{\partial x} + \alpha\beta x^{-1}u - \lambda\frac{\partial u}{\partial t} = 0 \tag{5.1}$$

for $(\alpha-\beta)\geq -1/2$, and $\lambda>0$ under the initial condition $u(x,t)$ converges in $\mathbb{H}'_{\alpha,\beta}(\sigma)$ to $f(x) \in \mathbb{H}'_{\alpha,\beta}(\sigma)$ for some $\sigma > 0$ as $t\to0^+$.

We will find the solution in the space $\mathbb{H}'_{\alpha,\beta}(\sigma)$ for some $\sigma > 0$ and $(\alpha-\beta)\geq-1/2$.

Let $U(y,t) = F'_{\alpha,\beta}(u(x,t)(y)$. Then applying (4.2) to (5.1) we obtain

$$-y\,U(y,t) - \lambda\frac{\partial}{\partial t}U(y,t) = 0$$

whose solution is

$$U(y,t) = F_1(y)\exp(-yt/\lambda)$$

where $F_1(y)= (F'_{\alpha,\beta}f(x))(y)$.

Using the inversion formula stated in Theorem 10 we get

$$u(x,t) = \lim_{R\to\infty}\int_0^R F_1(y)\exp(-yt/\lambda)K(y,x)dy \tag{5.2}$$

in $D'(I)$.

For each $\phi \in D(I)$ one can show that

$$\langle u(x,t),\phi(x)\rangle = \int_0^\infty \phi(x)dx\int_0^\infty F_1(y)\exp(-yt/\lambda)K(y,x)dy \tag{5.3}$$

Now one can observe from (5.3) that

$$u(x,t) = \int_0^\infty F_1(y)\exp(-yt/\lambda)K(y,x)dy \quad (x,t>0) \tag{5.4}$$

Using the asymptotic orders of $F_1(y)$ as established in

Theorem 8 we can justify that

$$(\Delta^*_{\alpha,\beta} - \lambda \frac{\partial}{\partial t})u = \int_0^\infty F_1(y)(\Delta^*_{\alpha,\beta} - \lambda \frac{\partial}{\partial t})\exp(-yt/\Lambda)K(y,x)dy$$

Therefore u(x,t) as defined in (5.2) satisfies the generalized Kepinski equation (5.1).To verify the initial condition, assume as in §4 that Q(x) is a polynomial of degree (r+1) with no roots on the negative real axis . Then, we have

$$<u(x,t) , \phi(x)> = \int_0^\infty \phi(x)dx\, Q(\Delta^*_{\alpha,\beta})\int_0^\infty \frac{F_1(y)}{Q(-y)} \exp(-yt/\Lambda)K(y,x)dy$$

$$= \int_0^\infty Q(\Delta_{\alpha,\beta})\phi(x)dx \int_0^\infty \frac{F_1(y)}{Q(-y)} \exp(-yt/\Lambda)K(y,x)dy$$

(by integration by parts)

That is,

$$<u(x,t) , \phi(x)> = \lim_{R\to\infty} \int_A^B Q(\Delta_{\alpha,\beta})\phi(x)dx \int_0^R \frac{F_1(y)}{Q(-y)} \exp(-yt/\Lambda)K(y,x)dy \qquad (5.5)$$

We assume that the support of $\phi(x)$ is contained in (A,B), B>A>0. The right hand side expression in (5.5) converges uniformly for all t>0 as $R\to\infty$. Therefore, letting $t\to 0^+$ in (5.5) and switching the limit operation with respect to R and t in the right hand side of (5.5) we get

$$\lim_{t\to 0+} <u(x,t),\phi(x)> = \lim_{R\to\infty} \int_A^B Q(\Delta_{\alpha,\beta})\phi(x)dx \int_0^R \frac{F_1(y)}{Q(-y)} K(y,x)dy$$

or

$$\lim_{t\to 0+} <u(x,t),\phi(x)> = \lim_{R\to\infty} \int_a^b \phi(x)dx \int_0^R F_1(y)\, K(y,x)dy$$

(by integration by parts)

$$= <f,\phi> \text{ (in view of Theorem 11)}$$

Thus the condition is verified.The solution obtained is unique in the sense of equality over D(I) in view of uniqueness Theorem 11.

Similarly we can find the solution

$$u(x,t) = \lim_{R\to\infty} \int_0^R F_1(y)\cos(t\sqrt{y/\Lambda}\,)K(y,x)dy \qquad (5.6)$$

of the partial differential equation involving the generalized Kepinski equation

$$x\frac{\partial^2 u}{\partial x^2}+(\alpha+\beta+1)\frac{\partial u}{\partial x}+\alpha\beta x^{-1}u-\lambda\frac{\partial^2 u}{\partial t^2}=0 \tag{5.7}$$

satisfying the initial conditions

(i) $u(x,t)$ converges in $\mathbb{H}'_{\alpha,\beta}(\sigma)$ to $f(x)\in\mathbb{H}'_{\alpha,\beta}(\sigma)$ as $t\to 0^+$

(ii) $\frac{\partial}{\partial t}u(x,t)$ converges in $\mathbb{H}'_{\alpha,\beta}(\sigma)$ to zero, as $t\to 0^+$

and verify that (5.6) is a solution of (5.7) [14,p.158].

Remark 1: It is proposed to develope the theory in distributional sense for the second generalized Hankel-Clifford transformation defined by (1.14) which is the generalization of the second Hankel-Clifford transformation

$$F_2(y)=cH_\mu f=\int_o^\infty x^\mu\ C_\mu(xy)\ f(x)dx \tag{5.8}$$

defined by Méndez and Socas in [8], where $C_\mu(x)=x^{-\mu/2}J_\mu(2\sqrt{x})$ is the Bessel-Clifford function of the first kind of order μ. Note that when $\alpha=0,\beta=-\mu$, (1.14) reduces to (5.8) and then to show that the study of these two transformations (1.14)and (1.3) on certain spaces of generalized functions allows one to solve a class of partial differential equations of the generalized Kepinski type for any real values of its parameter $(\alpha-\beta)$.

Remark 2: In future it is proposed to extend these two transformations (1.3) and (1.14) to a larger space of generalized functions similar to Koh and Li [4].

Remark 3: It is proposed to develope the distributional set up for the finite generalized Hankel-Clifford transformation.

Remark 4: Using the theory of one-sided generalized Laplace transformation of generalized functions by Malgonde and Choudhary [5] and the theory thus developed can be used to solve a distributional integral equation which will be the future investigation.

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