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B*-CONTINUITY AND OTHER GENERALISED CONTINUITY

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ABSTRACT The paper is devoted to study some properties of generalized types of continuity and B*-continuity and some examples are given in this connection.

INTRODUCTION AND PRELIMINARY EXAMPLES:

Continuous function is a source of studies for many of the research workers and it has been generalized in various ways, viz. quasi-continuity, cliquishness, simply -continuity, B-continuity, almost continuity, semi-continuity in the sense of Levine, functions having the WCIVP etc. It is seen that the notions of almost continuity, quasi-continuity and semi-continuity in the sense of Levine are same [4]. These functions involve the concept of generalized open sets, viz. semi-open, simply open etc. In recent years a good number of research has been done on many types of generalized continuity. Perhaps the notion of quasi-continuity has been focussed most intensively due to its close connection with other types of continuity and its various applications in Topology and Mathematical Analysis. The notion of B*-continuous function has been introduced by making some modification in the definition of quasi-continuous function and it has a closed interrelation with this function. B*-continuity is more general than quasi-continuity.

Our terminology is standard. Unless otherwise stated X and Y are topological spaces. We denote the closure and interior of a subset A of X by cl (A) and int (A). The following is a list of the definitions that are used in our paper.

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Definition 1:

- A function $f: X \to Y$ is quasi-continuous at a point $x \in X$ if, for each open neighborhood U of x and each open neighborhood V of f (x), there is a nonempty open set $G \subset U$ such that $f(G) \subset V$.

  If f is quasi-continuous at each point it is said to be quasi-continuous on X. [4]

- A set $A \subset X$ is said to be semi-open iff $\exists$ an open set O such that $O \subset A \subset cl(O)$.

- A function $f: X \to Y$ is semi- continuous in the sense of Levine if for any open set V in Y, $f^{-1}(V)$ is semi-open in X. [6]

- A function $f: X \to Y$ is simply continuous if for each open set V in Y, the set $f^{-1}(V)$ is the union of an open set and a nowhere dense set in X. [2]

- A function $f: X \to Y$ is called almost continuous at a point $p \in X$, if for any neighborhood V of the point f (p) in Y we have $p \in int(cl(f^{-1}(V)))$.

  If f is almost continuous at each point then it is said to be so on the whole of X. [8]

- A function $f: X \to Y$ (Y is a metric space with metric d) is cliquish at a point $x \in X$ if for each $\varepsilon > 0$ and each open neighborhood U of x, there is a nonempty open set $G \subset U$ such that, $d(f(x_1), f(x_2)) < \varepsilon \ \forall x_1, x_2 \in G$.

  If f is cliquish at each point, it is said to be so over the whole of X. [8]

We recall that a set A has the property of Baire if A can be expressed as $A = G \Delta P$, where G is open and P is of first category.

- A function $f: X \to Y$ is said to be B-continuous at $x \in X$, if for any open neighborhood U of x and V of f (x), there is a set B which is either open or second category having the Baire property such that $B \subset U \cap f^{-1}(V)$.

  If f is B-continuous at each point of X it is said to be B-continuous over X. [7]

The goal of this paper is to introduce a notion of B* continuity and present some results relating to various kinds of generalized continuous functions.

Definition 2: A set $A \subset X$ is said to be B* set if it is not nowhere dense in X and have the property of Baire.

Definition 3: A function f: $X \to Y$ is said to be B* continuous at x if for each open set U containing x in X and each open set V containing f (x) in Y, there is a B* set such that $B \subset U \cap f^{-1}(V)$.

If f be B* continuous at each point of X, then f is said to be B* continuous on the whole of X.

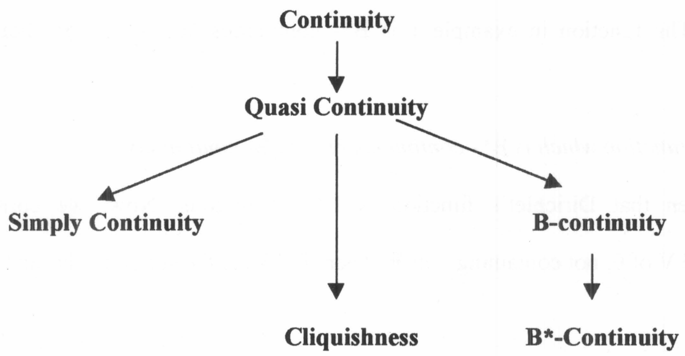
Definition 4: A function f: $[a, b] \to \mathbb{R}$ is said to have the WCIVP (Weak Cantor Intermediate Value Property), if for p and q in [a, b] with $p \neq q$ and $f(p) \neq f(q)$, there exists a Cantor set C between p and q such that f (C) lies between f (p) and f (q). [11]

Definition 5: A perfect road of a function f at a point x is a perfect set P such that,

i) x is a bi-lateral point of accumulation of P, and

ii) $f|_P$ is continuous at x. [3]

Among continuity, quasi continuity, simply continuity, cliquishness, B-continuity and B-continuity we have the following implication:*



The following discussion shows that there is no other implication among the functions:

Example 1: A function which is B*- continuous but not quasi continuous.

Let the function f: $[0, 1] \to [0, 1]$ be such that,

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

Let $x_0 \in [0, 1]$ be irrational point, and $0 < \varepsilon < 1$. Then for any open neighborhood U of x_0 and any open set $G \subset U$, there always exist rational points in G.

Then, $|f(x) - f(x_0)| = 1 - 0 = 1 > \varepsilon$, where $x \in G$, x rational. Thus f is not quasi continuous at x_0. But f is B*- continuous. In fact, x_0 being irrational, for any neighborhood V of f (x_0), $f^{-1}(V)$ contains all the irrational points in [0, 1], which is dense in [0, 1] and has the property of Baire. Similar arguments hold for the rational points.

Example 2: A function which is simply continuous but not B*- continuous.

Let f: $\mathbb{R} \to [0, 1]$ be such that,

$$f(x) = \begin{cases} 0 & \text{if } x = 1,2,\ldots,n \text{ (n is fixed and finite)} \\ 1 & \text{elsewhere} \end{cases}$$

Then, f is clearly simply continuous. But f is not B*- continuous at each of 1,2,…,n.

Remark 1: The above function in example 2 is also cliquish.

Remark 2: The function in example 1 is B*- continuous but is neither cliquish nor simply continuous.

Example 3: *Function which is* B*- *continuous but not* B- *continuous.*

We have seen that Dirichlet's function is B*- continuous. Now, we consider any open neighborhood V of 0, not containing 1 in $\mathbb{R}$. Then, $f^{-1}(V)$ is the set of all rational points of [0, 1].

Now, for any rational p in [0, 1] and any neighborhood U of p, $\nexists$ any second category set in $U \cap f^{-1}(V)$. Hence, f is not B-continuous.

MAIN RESULTS:

Theorem 1: *If* f *is* B**- continuous then,*

i) For each open set V *of* Y *with* $f(x) \in V$, *$\exists$ a semi open set* O *containing* x, *such that*

$O \subset cl(f^{-1}(V))$;

ii) For each neighborhood V *of* f (x), $x \in cl(int(cl(f^{-1}(V))))$.

Proof: i) For each open neighborhood U of x and each open set V in Y, with $f(x) \in V$, $int(U) \cap f^{-1}(V)$ is not nowhere dense. Hence we have, $(\varphi \neq)$ $int(cl(f^{-1}(V) \cap int(U))) \subset cl(int(U))$. Put $G = int(cl(f^{-1}(V) \cap int(U))) \cap int(U)$. Then G $(\neq\varphi)$ is an open set such that $G \subset U$ and $G \subset cl(f^{-1}(V))$.

Let V be any open set containing f (x). $\boldsymbol{U_x}$ be the family of open neighborhood of x. For each $U \in \boldsymbol{U_x}$, $\exists$ an open set G_U of X such that $G_U (\neq\varphi) \subset U$ and $G_U \subset cl(f^{-1}(V))$.

Set $W = \cup \{G_U\}$. Then W is open set of X, $x \in cl(W)$ and $W \subset cl(f^{-1}(V))$. Take $O = W \cup \{x\}$.

Then, $W \subset O \subset cl(W)$. So, O is a semi open set and $O \subset cl(f^{-1}(V))$.

ii) Let N be any neighborhood of f (x). Then, there exists semi open set U containing x such that $U \subset cl(f^{-1}(V))$. Therefore, $x \in cl(int(U)) \subset cl(int(cl(f^{-1}(V))))$.

Theorem 2: *Let* $f: X \rightarrow Y$, *then the following conditions are equivalent:*

i) For every dense set D, $f(cl(D)) \subset cl(f(D))$,

ii) f *is quasi continuous.*

Proof: i) ⇒ ii)

Let $p \in X$ and U is an open set in X containing p, and V be an open set in Y containing f (p). If f is not quasi continuous at p, then U would not contain any open set G such that $f(G) \subset V$.

Let $D = \{ x \in U: f(x) \notin V\}$. Then D is dense in U. Then $f(cl(D)) \subset cl(f(D))$. Now $p \in cl(D) \Rightarrow f(p) \in f(cl(D)) \subset cl(f(D))$. $\Rightarrow V \cap f(D) \neq \varphi$. But, this contradicts the construction of D. Hence f must be quasi continuous.

ii) ⇒ i) Let f be quasi continuous. If possible, let $f(cl(D)) \not\subset cl(f(D))$, where D is a dense set. Let $p \in cl(D)$. Then $f(p) \notin cl(f(D))$. Then for V (open in Y) $\ni f(p)$, $V \cap f(D) = \varphi$.

Now f being quasi continuous, for any open set $U \ni p$, $\exists$ a nonempty open set $G \subset U$ such that $f(G) \subset V$. Again D being dense, $D \cap G \neq \varphi$. Then, $V \cap f(D) \neq \varphi$.

This contradiction proves that $f(cl(D)) \subset cl(f(D))$.

Remark 3: H Blumberg showed that [1] for every real-valued function defined on $\mathbb{R}$, there exists a dense subset B of $\mathbb{R}$ such that $f|_B$ is continuous. If $f: X \to Y$ where X & Y are metric spaces and B is dense subset of X such that $f|_B$ is continuous, then we say that B is a Blumberg set of f. A set $D \subset X$ is called strongly Blumberg set for the function f if D is a Blumberg set for f and for every open set $G \subset X$, the set $f(G \cap D)$ is dense in f (G). [9]

J. C. Neugebauer established a close relation between quasi-continuity and Blumberg set [9]. The result states that: A function f: [0, 1] $\to \mathbb{R}$ possesses a strongly Blumberg set iff f is quasi-continuous. Applying these facts to our consideration we have the following corollaries of the theorem.

Corollary 1: *A function* f: [0, 1] $\to \mathbb{R}$ *possesses a strongly Blumberg set iff for every dense set* D, $f(cl(D)) \subset cl(f(D))$.

Corollary 2: *A function* f: X → Y, *having the Darboux property has a perfect road if for every dense set* D, f (cl (D)) ⊂ cl (f (D)).

This follows from the result [5] that a quasi-continuous function having the Darboux property has a perfect road.

Theorem 3: *If* Q (f) *is the set of points of quasi continuity of the function* f: X → Y *and* X \ Q (f) *is nowhere dense, then* f *is simply continuous.*

Proof: Let V be an open set in Y. Then, we know that [2], $Q(f) \cap (f^{-1}(V) \setminus \text{int } f^{-1}(V))$ is nowhere dense. Then the set $f^{-1}(V) \setminus \text{int } f^{-1}(V) \subset Q(f) \cap (f^{-1}(V) \setminus \text{int } f^{-1}(V)) \cup (X \setminus Q(f))$ is nowhere dense. Hence, f is simply continuous.

Theorem 4: *A simply continuous function* f: X → Y, *is* B*- *continuous if for each open set* V *in* Y, $f^{-1}(V)$ *contains a nonempty open set.*

Proof: Let f be simply continuous at a point $x_0 \in X$. Then for each neighborhood V of f (x_0) and for each neighborhood U of x_0 the set $f^{-1}(V)$ is the union of a nonempty open set G and a nowhere dense set. Then $U \cap f^{-1}(V)$ contains a nonempty open set and a nowhere dense set. Thus, $U \cap f^{-1}(V)$ contains a B*-set. Hence f is B*- continuous.

We know that a function is in *DB*1 class (Baire one functions having the Darboux property) iff it has a perfect road at each of its points [3]. We can construct a B*- continuous functions with a perfect road at each point such that the function fails to do have the Darboux property.

Example 4: Consider the function f: [0, 1] → [0, 1] such that,

$$f(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \in B \\ 0 & \text{if } x \in C \end{cases}$$

Where, A is the union of the closure of all the open intervals that are removed in the odd stage in the construction of the Cantor set,

B is the union of the closure of all the open intervals that are removed in the even stage in the construction of the Cantor set,

and C is the non end points points of the Cantor set.

Now, A and B are perfect sets. Clearly, f has a perfect road at each of its points. Again f is quasi continuous at every point of [0, 1], hence, f is B*- continuous over [0, 1].

But f ([0, 1]) = {0, 1}. Hence f is not Darboux.

We show that there are B*- continuous functions with perfect road at each point, but the functions do not possess the WCIVP.

Example 5: Let f: $[0, \pi] \to [0, 1]$ be such that, f (x) = sin x. Then f is continuous and so B*- continuous over $[0, \pi]$. Also f has a perfect road at each of its points. Now, consider the points $\pi/3$ and $3\pi/4$. Then $f(\pi/3) = \sqrt{3}/2 \neq 1/\sqrt{2} = f(3\pi/4)$. We construct a Cantor set C between $\pi/3$ and $3\pi/4$. But f (C) will not lie between $1/\sqrt{2}$ and $\sqrt{3}/2$.

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