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ON PARTIAL QUASI-BILATERAL GENERATING FUNCTIONS INVOLVING GEGENBAUER POLYNOMIALS

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Abstract : In this article, we have obtained a generalization of a known result on quasi-bilateral generating relation involving Gegenbauer polynomials from the existence of partial quasi-bilateral generating relation of the polynomial under consideration. Some particular cases of interest are also discussed here.

1. Introduction:

In **[1]** the term "partial quasi-bilateral generating function" is defined as follows:

$$G(x,z,w)=\sum_{n=0}^{\infty}a_n\ w^n P_{m+n}^{(\alpha)}(x)\,Q_r^{(m+n)}(z),$$

where the coefficients, a_n are quite arbitrary and $P_{m+n}^{(\alpha)}(x), Q_r^{(m+n)}(z)$ are two particular special functions of orders $m+n$, r and of parameters α and $m+n$ respectively. In particular, when $Q_r^{(m+n)}(z)=P_r^{(m+n)}(z)$, then it is " partial quasi-bilinear ".

In this note, we would like to show that the existence of a partial quasi-bilinear generating function implies the existence of a more general generating function from the group- theoretic view-point.

In [2] Chongdar proved the following theorem on bilateral generating function involving Gegenbauer polynomials by group theoretic method.

Theorem 1: If

$$G(x,t)=\sum_{n=0}^{\infty}a_n\ C_n^{\lambda}(x)\,t^n\ ,$$

then

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$$\left[1+4xz+4\left(x^2-1\right)z^2\right]^{\lambda-\frac{1}{2}} \quad G\left(x+2\left(x^2-1\right)z\,,\,zt\right) = \sum_{n=0}^{\infty} f_n\left(x,t\right)z^n,$$

where

$$f_n\left(x,t\right)=\sum_{m=0}^{\infty}\binom{n}{m}a_m\frac{\left(1-m-2\lambda\right)_{n-m}}{\left(1-\lambda\right)_{n-m}}C_n^{\lambda-(n-m)}\left(x\right)t^m,$$

for $\lambda \neq 1,2,3,...$

In [3] Sen and Chongdar obtained an extension of the above theorem in the following form :

Theorem 2 : If

$$G\left(x,u,w\right)=\sum_{n=0}^{\infty}a_n\,C_n^{\lambda}\left(x\right)\,C_m^{n}\left(u\right)\,w^n,$$

then

$$\left(1-2w\right)^{\frac{-m}{2}}\left\{1+2wx+\left(x^2-1\right)w^2\right\}^{\left(\lambda-\frac{1}{2}\right)}$$

$$\times G\left(x+w\left(x^2-1\right),\frac{u}{\sqrt{1-2w}},\frac{wv}{1-2w}\right)$$

$$=\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}a_n\,\frac{w^{p+q+n}}{p!\,q!}\,v^n\,\frac{\left(n+1\right)_p}{2^p}\,\frac{\left(-n-2r+1\right)_p}{\left(-\lambda+1\right)_p}$$

$$\times 2^q\,\left(n\right)_q\,C_{n+p}^{\lambda-p}\left(x\right)\,C_m^{n+q}\left(u\right),$$

for $\lambda \neq 1,2,3,...$

The object of the present article is to further generalize the above theorem from the concept of partial quasi-bilateral (or partial quasi-bilinear) generating functions. In fact, we have obtained the following theorem as the main result of our investigation.

Theorem 3: If

$$G\left(x,u,w\right) = \sum_{n=0}^{\infty}a_n\,C_{n+r}^{\lambda}\left(x\right)\,C_m^{n+r}\left(u\right)\,w^n$$

then

$$(1-2w)^{-\frac{m}{2}-r}\ \left\{1+2wx+\left(x^2-1\right)w^2\ \right\}^{\lambda-\frac{1}{2}}$$

$$\times G\left(x+w\left(x^2-1\right),\frac{u}{\sqrt{1-2w}},\frac{wv}{1-2w}\right)$$

$$=\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}a_n\ \frac{w^{p+q+n}\ v^n}{p!\ q!}\ \frac{(1+r+n)_p\ (-n-r-2\lambda+1)_p\ 2^q(n+r)_q}{2^p\ (-\lambda+1)_p}$$

$$\times\ C_{n+r+p}^{\lambda-p}\ (\ x\ )\ \ C_{m}^{n+r+q}\ (\ u\ )\ ,$$

for $\lambda\neq 1,2,3,...$

Proof of the theorem

Let

$$G\ (x,u,w)\ \ =\ \sum_{n=0}^{\infty}a_n\ \ w^n\ \ C_{n+r}^{\lambda}\ (x)\ C_m^{n+r}\ (u). \qquad (1)$$

Consider the operators [4,5]:

$$R_1=\left(x^2-1\right)y^{-1}\,z\frac{\partial}{\partial x}+2xz\frac{\partial}{\partial y}-xy^{-1}z$$

$$R_2=ut\frac{\partial}{\partial u}+2t^2\frac{\partial}{\partial t}+(m+2r)t$$

such that

$$R_1\ \left(\ C_{n+r}^{\lambda}\ \ (x\ )\ \ y^{\lambda}\ \ z^n\right)$$

$$=\frac{(n+r+2\lambda-1)(n+r+1)}{2(\lambda-1)}\ C_{n+r+1}^{\lambda-1}\ \ (x)\ \ y^{\lambda-1}\ \ z^{n+1}\ , \qquad (2)$$

$$R_2\ \left(C_m^{n+r}\ (u)\ t^n\right)\ =\ 2\ (n+r)\ \ C_m^{n+r+1}\ (u)\ \ t^{n+1} \qquad (3)$$

and

$$e^{wR_1}\ f(x,y,z) = \{1+2wxy^{-1}z+(x^2-1)\,w^2y^{-2}z^2\}^{-\frac{1}{2}}$$

$$\times f\left(x+w(x^2-1)y^{-1}z\,,\ y\{1+2wxy^{-1}z+(x^2-1)w^2y^{-2}z^2\},z\right) \qquad (4)$$

$$e^{wR_2}\ f(u,t) = (1-2wt)^{-\frac{m}{2}-r}\ f\left(\frac{u}{\sqrt{1-2wt}},\frac{t}{1-2wt}\right). \qquad (5)$$

Replacing w by $wztv$ and then multiplying both sides of (1) by y^{λ} we get

$$y^{\lambda}\ G\ (x,u,wztv)$$

$$= \sum_{n=0}^{\infty} a_n\,(wv)^n\left(C_{n+r}^{\lambda}(x)\,y^{\lambda}z^n\right)\left(C_m^{n+r}(u)\,t^n\right) \qquad (6)$$

Operating $e^{wR_1}e^{wR_2}$ on both sides of (6), we get

$$e^{wR_1}\ e^{wR_2}\left[y^{\lambda}\ G\,(x,u,wztv)\right]$$

$$= e^{wR_1}\ e^{wR_2}\left[\sum_{n=0}^{\infty} a_n\,(wv)^n\left(C_{n+r}^{\lambda}(x)\ y^{\lambda}\ z^n\right)\left(C_m^{n+r}(u)\,t^n\right)\right] \qquad (7)$$

The left member of (7) with the help of (4) and (5) is

$$e^{wR_1}\ e^{wR_2}\left[y^{\lambda}\ G\,(x,u,wztv)\right]$$

$$= e^{wR_1}\left[(1-2wt)^{-\frac{m}{2}-r}\ y^{\lambda}\ G\left(x,\frac{u}{\sqrt{1-2wt}},\frac{wztv}{1-2wt}\right)\right]$$

$$= (1-2wt)^{-\frac{m}{2}-r}\ y^{\lambda}\left\{1+2wxy^{-1}z+(x^2-1)\,w^2y^{-2}z^2\right\}^{\lambda-\frac{1}{2}}$$

$$\times G\left(x+w\left(x^2-1\right)y^{-1}z,\frac{u}{\sqrt{1-2wt}},\frac{wztv}{1-2wt}\right),$$

$$=(1-2wt)^{-\frac{m}{2}-r}\,y^{\lambda}\left\{1+2wxy^{-1}z+\left(x^2-1\right)w^2y^{-2}z^2\right\}^{\lambda-\frac{1}{2}}$$

$$\times G\left(x+w\left(x^2-1\right)y^{-1}z,\frac{u}{\sqrt{1-2wt}},\frac{wztv}{1-2wt}\right). \qquad (8)$$

The right member of (7), with the help of (2) and (3), becomes

$$e^{wR_1}\,e^{wR_2}\left[\sum_{n=0}^{\infty}a_n\,(wv)^n\left(C_{n+r}^{\lambda}(x)\,y^{\lambda}z^n\right)\left(C_m^{n+r}(u)\,t^n\right)\right]$$

$$=\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}a_n\,(wv)^n\,\frac{w^{p+q}}{p!\,q!}\,\frac{(1-n-r-2\lambda)_p\,(1+r+n)_p}{2^p\,(-\lambda+1)_p}$$

$$\times C_{n+r+p}^{\lambda-p}(x)\,y^{\lambda-p}\,z^{n+p}\,(x)\,2^q\,(n+r)_q\,C_m^{n+r+q}(u)\,t^{n+q}. \qquad (9)$$

Equating (8) and (9) we get

$$(1-2wt)^{-\frac{m}{2}-r}\,y^{\lambda}\left\{1+2wxy^{-1}z+\left(x^2-1\right)w^2y^{-2}z^2\right\}^{\lambda-\frac{1}{2}}$$

$$\times G\left(x+w\left(x^2-1\right)y^{-1}z,\frac{u}{\sqrt{1-2wt}},\frac{wztv}{1-2wt}\right).$$

$$=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{n=0}^{\infty}a_n\,\frac{w^{p+q+n}\,v^n}{p!\,q!}\,\frac{(1+r+n)_p\,(-n-r-2\lambda+1)_p}{2^p\,(-\lambda+1)_p}$$

$$\times 2^q\,(n+r)_q\,C_{n+r+p}^{\lambda-p}(x)\,y^{\lambda-p}\,z^{n+p}\,t^{n+q}\,C_m^{n+r+q}(u)\,t^{n+q}. \qquad (10)$$

Putting t = y = z =1 in (10), we get

$$(1-2w)^{-\frac{m}{2}-r}\left\{1+2wx+\left(x^2-1\right)w^2\right\}^{\lambda-\frac{1}{2}}$$

$$\times G\left(x+w\left(x^2-1\right),\frac{u}{\sqrt{1-2w}},\frac{wv}{1-2w}\right)$$

$$=\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}a_n\,\frac{w^{p+q+n}\,v^n}{p!\,q!}\,\frac{(1+r+n)_p\,(-n-r-2\lambda+1)_p\,2^q(n+r)_q}{2^p\,(-\lambda+1)_p}$$

$$\times C_{n+r+p}^{\lambda-p}\,(\,x\,)\;\;C_m^{n+r+q}\,(\,u\,)\,,$$

which is Theorem 3.

Corollary 1. Put r=0. Then we have

$$(1-2w)^{-\frac{m}{2}}\left\{1+2w\,x+\left(x^2-1\right)w^2\right\}^{\lambda-\frac{1}{2}}$$

$$\times G\left(x+w\left(x^2-1\right),\frac{u}{\sqrt{1-2w}},\frac{w\,v}{1-2w}\right)$$

$$=\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}a_n\,\frac{w^{p+q+n}}{p!\,q!}\,v^n\,\frac{(n+1)_p\,(-n-2\lambda+1)_p}{2^p\,(-\lambda+1)_p}$$

$$\times 2^q\,(\,n\,)_q\,C_{n+p}^{\lambda-p}\,(\,x\,)\,C_m^{n+q}\,(\,u\,),$$

for $\lambda\neq 1,2,3,\ldots,$ which is Theorem 2.

Corollary 2. Put m=0 and assume that $\lambda\neq 1,2,3,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$, then

$$(1-2w)^{-r}\left\{1+2wx+\left(x^2-1\right)w^2\right\}^{\lambda-\frac{1}{2}}\left(G\left(x+w\left(x^2-1\right),\frac{wv}{1-2w}\right)\right)$$

$$=\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}a_n\,\frac{w^{p+q+n}\,v^n}{p!\,q!}$$

$$\times \frac{(1+r+n)_p\ (-n-r-2\lambda+1)_p\ 2^q (n+r)_q}{2^p\ (-\lambda+1)_p}\ C_{n+r+p}^{\lambda-p}\ (\ x\ )$$

$$= \sum_{n=0}^{\infty}\ \sum_{p=0}^{\infty} \left(\frac{w}{2}\right)^{p+n} a_n\ \frac{(1+r+n)_p\ (-n-r-2\lambda+1)_p}{p!\,(-\lambda+1)_p}$$

$$\times C_{n+r+p}^{\lambda-p}\ (\ x\ )\ \sum_{q=0}^{\infty} \frac{(\ 2w\ )^q\ \ (n+r\ )_q}{q\ !}$$

$$= \sum_{n=0}^{\infty}\ \sum_{p=0}^{\infty} \left(\frac{w}{2}\right)^{p+n} a_n\ \frac{(1+r+n)_p\ (-n-r-2\lambda+1)_p}{p!\,(-\lambda+1)_p}$$

$$\times\ C_{n+r+p}^{\lambda-p}\ (\ x\ )(\ 2v\ )^n\ (1-2w)^{-(n+r)}\ .$$

Thus we see that

$$\left\{1+2w\,x+\left(x^2-1\right)\,w^2\right\}^{\lambda-\frac{1}{2}}\ G\left(\ x+w\ \left(x^2-1\right), \frac{w\,v}{1-2w}\right)$$

$$= \sum_{n=0}^{\infty} \left(\frac{w}{2}\right)^n \sigma_n\ (\ x,v\ ),$$

where

$$\sigma_n\ (x,\ v) = \sum_{p=0}^{n} a_p \binom{n+r}{p+r} \frac{(-p-r-2\lambda+1)_{n-p}}{(-\lambda+1)_{n-p}}\ C_{n+r}^{\lambda-\overline{n-p}}\ (x) \left(\frac{2\ v}{1-2w}\right)^p$$

.

Replacing v by $(1-2\ w)\ v$ on both sides, we get

$$\left\{1+2w\,x+\left(x^2-1\right)w^2\right\}^{-\lambda+\frac{1}{2}}\ G\left(x+w\left(x^2-1\right),\ w\,v\right) = \sum_{n=0}^{\infty}\left(\frac{w}{2}\right)^n \sigma_n\ (\ x,v\ ),$$

where

$$\sigma_n(x,v)=\sum_{p=0}^{n} a_p \binom{n+r}{p+r} \frac{(-p-r-2\lambda+1)_{n-p}}{(-\lambda+1)_{n-p}} C_{n+r}^{\overline{\lambda-n-p}}(x)(2v)^p ,$$

which is found derived in [4].

Corollary 3 : If we put r=0 in the above result, we get the Theorem 1.

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