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A NEW PROOF OF THE UNIQUENESS OF THE SOLUTION OF A DIOPHANTINE EQUATION ASSOCIATED WITH THE NATURAL NUMBER 16

Isabel Fernández & José M. Pacheco*
Departamento de Matemáticas
Universidad de Las Palmas de Gran Canaria

Abstract.
This paper offers a new proof of the fact that a certain diophantine equation associated to the natural number 16 has a unique solution in the domain of natural numbers. The proof derives in a straightforward manner from the division of the naturals into odd and even numbers.

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Key words: "Book proof", Diophantine equation, Natural numbers, Uniqueness.

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1. Introduction.

The fourth power of 2 is $2^4 = 16$, and in turn, $16 = 4^2$ is the square -or second power- of 4. Therefore there exist at least two natural numbers $x = 2, y = 4$ satisfying the diophantine equation $x^y = y^x$. A rather natural problem is to decide whether the pair (2,4), found by simple inspection, is the only solution in natural numbers to the equation. Indeed we are aware that (4,2) is a solution as well, but both can be identified via the symmetry of the equation. Moreover, if we allow $x = y$, there exist infinitely many trivial solutions, so from now on $x \neq y$ will be supposed.

The study of the equation $x^y = y^x$ and its solutions has been addressed rather often and by many authors to a considerable degree of generality, including solutions in $\mathbf{Z}/p$ and in the complex domain. As an example, simple inspection determines as well the real number 4.81047... in the form $i^{-i} = (-i)^i$. See Euler 1748, Dickson 1966, Hausner 1961, Hurwitz 1967, Sved 1990 and references therein.

Our equation features uniqueness of solution *only when considered over the natural numbers*. Even the simplest extension of the domain, *viz.* to the whole numbers domain, has no unique solution, for $(-2,-4)$ is a solution as well. Here the common value is also related with 16: $x^y = y^x = 16^{-1}$.

The next extension, to the rational domain, is best studied by considering first the larger real field. In this case, under the hypothesis $x < y$, let us write $y = mx$, where $m > 1$ is an otherwise arbitrary real number. Both members in the equation become:

$$x^{mx} = x^{x(m-1+1)} = x^x x^{x(m-1)}$$

* Corresponding author. Address: Campus de Tafira Baja, 35017 LAS PALMAS, Spain.
pacheco@dma.ulpgc.es

and:

$$(mx)^x = m^x x^x .$$

From here it is straightforward to find that (observe: $m = 2$ yields the natural case!):

$$x = m^{\frac{1}{m-1}},$$

$$y = mx = m^{\frac{m}{m-1}}$$

Elementary Calculus shows that for $m \to \infty$ we have $x \to 1$ and $y \to \infty$. Therefore there exist an infinite number of solutions, and all of them belong to the part of the curve -whose parametric equations are the above expressions- lying in the plane strip $\{(x, y) : 1 < x < e\}$.

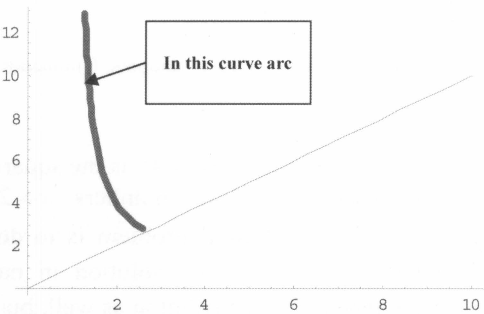

Fig. 1. Where real solutions live

In order to look for rational solutions we impose $m = \dfrac{p}{q}$, a fraction in lowest terms such that $p > q$, and conditions are found for the solutions to be rational. A known result of Euler (Euler 1748) -which, in a certain sense, is also an uniqueness result because all rational solutions are provided by these formulae (a proof is offered in an appendix)- is obtained when $p = n+1$ and $q = n$:

$$x = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n, \; y = \left(\frac{n+1}{n}\right)^{n+1} = \left(1+\frac{1}{n}\right)^{n+1}.$$

In this paper a new and elementary proof of the uniqueness of the solution *over the natural numbers* is presented. To establish it, only a remark on parity is needed, so quite possibly proof presented could be the "Book proof" in the sense of Erdös (see *e.g.* Babai and Spencer 1998, p. 65).

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2. The proof.

The main observation is a very simple one: If there is a solution (x, y) of $x^y = y^x$ over the natural numbers, then both x and y must be of the same parity. Should this not occur, the left and right members would be of different parities and could not have the same value. Let us write x for the smaller of the two numbers, so for some $k \in \mathrm{N}$, we obtain $y = x + 2k$. By plugging this expression into the equation we obtain:

$$\left[\frac{y^x}{x^y} = 1\right] \Leftrightarrow \left[\frac{(x+2k)^x}{x^{x+2k}} = 1\right],$$

or

$$\sqrt[2k]{\left(1+\frac{2k}{x}\right)^x} = x \; .$$

Under the radical symbol we recognise the familiar expression leading to the exponential e^{2k} in the limit when $x \to \infty$. This expression is monotonically increasing and satisfies the estimate

$$\left(1+\frac{2k}{x}\right)^x < e^{2k} .$$

By remarking that the function "to obtain the $2k$-th root" is a monotonic one, the following estimate holds:

$$x = \sqrt[2k]{\left(1+\frac{2k}{x}\right)^x} < e = 2.718\ldots,$$

so the only natural candidates for x are 1 and 2. First, let us consider $x = 1$. For any k, both members of the equation become:

$$x^y = 1^{1+2k} = 1$$

and

$$y^x = (1+2k)^1 = 1+2k > 1 .$$

Therefore, $x = 1$ *does not yield a solution* of the equation. Now we turn our attention to $x = 2$. Both members become:

$$x^y = 2^{2+2k} = 2^{2(1+k)} = 4^{1+k}$$

$$y^x = [2(1+k)]^2 = 4(1+k)^2$$

and their are equal when $k = 1$, 16 being their common value. Thus we find again what we found by simple inspection in the Introduction. No more natural pairs (x, y) can be found, for the set of possible candidates for x is already exhausted. Nevertheless, the

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following argument will reinforce our conviction: For any natural $k \geq 2$, the inequality $4^{1+k} > 4(1+k)^2$ always holds -because the exponential grows faster than the second degree polynomial- as the reader can easily check by induction on *k*. Therefore, we have obtained the following theorem:

THEOREM: "16 *is the only natural number that can be written in two different ways* $x^y = y^x$, *with* $x \neq y$ *natural numbers. The solution to the diophantine equation is provided by* $x = 2, y = 4$ (*or, symmetrically,* 4 *and* 2)"

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Appendix: The Euler formulae provide all positive rational solutions.

Let $m = \frac{p}{q}$ in lowest terms such that $p > q$. We obtain that

$$x = m^{\frac{1}{m-1}} = \left(\frac{p}{q}\right)^{\frac{q}{p-q}}, \quad y = m^{\frac{m}{m-1}} = \left(\frac{p}{q}\right)^{\frac{p}{p-q}}$$

and for these numbers to be rational ones , p and q must be $(p-q)$-powers of some other natural numbers, say P and Q. This condition determines the rationality of solutions. For instance, for $p = n+1$ and $q = n$ the Euler formulae are obtained:

$$x = \left(\frac{n+1}{n}\right)^{n} = \left(1+\frac{1}{n}\right)^{n}, \; y = \left(\frac{n+1}{n}\right)^{n+1} = \left(1+\frac{1}{n}\right)^{n+1}.$$

Therefore, for $p - q = 1$ there exist rational solutions. Do more rational solutions occur for some pair p and q such that $p - q > 1$? The answer is in the negative: If we let $p - q > 1$, then the following contradiction appears:

$$p - q = P^{p-q} - Q^{p-q} \geq (Q+1)^{p-q} - Q^{p-q} \geq 1 + (p-q)Q \geq 1 + (p-q) > p - q$$

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