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WEYL FRACTIONAL q-INTEGRAL OPERATOR INVOLVING A GENERALIZED BASIC HYPERGEOMETRIC FUNCTION

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ABSTRACT

This paper deals with a new fractional q-integral operator type Weyl involving a generalized basic hypergeometric function ${}_r\phi_s(.)$. For the new operator some composition formulae, fractional q-integration by parts and the q-Mellin transform are presented. Furthermore, various special cases are mentioned.
Keywords: Fractional q-integral operators; composition formulae; generalized basic hypergeometric function; fractional q-integration by parts; q-Mellin transform.

1. INTRODUCTION

The most widely used definition of an integral of fractional order is via an integral transform, called the Riemann-Liouville operator of fractional integration: [17, p. 146]

$$\begin{aligned} {}_aI_x^{\alpha}\varphi(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\varphi(t)\,dt, \quad \mathrm{Re}(\alpha)>0, \\ &= \frac{d^n}{dx^n}\,{}_aI_x^{\alpha+n}\varphi(x), \qquad -n<\mathrm{Re}(\alpha)\leq 0,\ n\in\mathbb{N}. \end{aligned} \tag{1}$$

Many authors, among them, Agarwal [1], Al-Salam [2], Galué [5-7], Garg [8], Kalia [10], Kalla *et al.* [11], Kalla and Kiryakova [12], Kiryakova [13], McBride and Roach [14], Ross [17], Saigo [18], Samko *et al.* [19], Saxena [20], Saxena *et al.* [21], have defined and studied operators of fractional integration with their applications. Some of these operators are:
1.1 Erdélyi-Kober Operator: [10, p. 4, Eq. (20)]

$$\begin{aligned} I_{\eta,\alpha}f(x) &= \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}t^{\eta}f(t)dt, \ \mathrm{Re}(\alpha)>0, \\ &= x^{-\alpha-\eta}\frac{d^n}{dx^n}x^{\eta+\alpha+n}I_{\eta,\alpha+n}f(x), \ -n<\mathrm{Re}(\alpha)\leq 0. \end{aligned} \tag{2}$$

1.2 Basic analogue of Kober fractional q-integral operator

A basic analogue of Kober fractional q-integral operator has been defined by Agarwal [1] in the following form:

$$I_q^{\eta,\mu} f(x) = \frac{x^{-\eta-\mu}}{\Gamma_q(\mu)} \int_0^x t^\eta \, (x-tq)_{\mu-1} \; f(t) \; d_q t, \quad \eta, \mu \in \mathbb{C}, \; Re(\mu) > 0, \tag{3}$$

where order of integration μ is arbitrary real or complex number, and

$$(x-y)_\nu = x^\nu \prod_{n=0}^{\infty} \left[\frac{1-(y/x)\, q^n}{1-(y/x)\, q^{n+\nu}} \right]. \tag{4}$$

1.3 Basic analogue of Weyl fractional q-integral operator

A basic analogue of Weyl fractional q-integral operator has been defined by Al-Salam [2] as follows:

$$K_q^{\eta,\mu} f(x) = \frac{q^{-\eta} x^\eta}{\Gamma_q(\mu)} \int_x^\infty (t-x)_{\mu-1} t^{-(\eta+\mu)} f(q^{(1-\mu)} t) d_q t, \quad \eta \in \mathbb{C}, \, Re(\mu) > 0. \tag{5}$$

$$K_q^{\eta,\mu} f(x) = \frac{(1-q)}{\Gamma_q(\mu)} \sum_{k=0}^{\infty} q^{k\eta} \left(1-q^{k+1}\right)_{\mu-1} f(xq^{-\mu-k}). \tag{6}$$

1.4 The operator $L_q^n(.)$

A fractional q-integral operator, denoted by $L_q^n(.)$, was introduced by the author [6] in the following form:

$$L_{\boldsymbol{q}}^n\{M, b_1, b_2, ..., b_r, \gamma, m_1, m_2, ..., m_r; f(x)\} =$$

$$\frac{x^{-\gamma-1}}{\Gamma_q(M+1)} \int_0^x t^\gamma {}_{r+1}\phi_r \left[\begin{matrix} q^{-M}, q^{m_1+b_1}, ..., q^{m_r+b_r} \\ q^{b_1}, ..., q^{b_r} \end{matrix} ; q, q^n \frac{t}{x} \right] f(t) \; d_q t, \tag{7}$$

$$M, m_1, ..., m_r \text{ non-negative integers}, \; n \in \mathbb{N}, \gamma \in \mathbb{C}, \; b_1, ..., b_r \neq 0, -1, -2, ..., \left| \frac{t}{x} \right| < 1.$$

1.5 The generalized fractional q-integral operator $I_q\left[\begin{smallmatrix}\delta,\nu,a_1,\cdots,a_{r-1}\\ b_1,\cdots,b_s\end{smallmatrix}\right] f(x)$

It is defined as follows: [7]

$$I_q \left[\begin{matrix} \delta, \nu, a_1, \cdots, a_{r-1} \\ b_1, \cdots, b_s \end{matrix} \right] f(x) = \frac{x^{-\delta-\nu}}{\Gamma_q(\nu)} \quad \text{x}$$

$$\int_0^x t^\delta \;\; (x-tq)_{\nu-1} \;\; {}_r\phi_s \left[\begin{matrix} a_1, \cdots, a_{r-1}, q^{\nu+k} \\ b_1, \cdots, b_s \end{matrix} ; q, q \right] f(t) \; d_q t \tag{8}$$

$$= \frac{(1-q)}{\Gamma_q(\nu)} \sum_{k=0}^{\infty} q^k \, q^{\delta k} \;\; \left(1-q^{k+1}\right)_{\nu-1} \; {}_r\phi_s \left[\begin{matrix} a_1, \cdots, a_{r-1}, q^{\nu+k} \\ b_1, \cdots, b_s \end{matrix} ; q, q \right] f(xq^k) \tag{9}$$

where $\delta \in \mathbb{C}, \; \mathrm{Re}(\nu) > 0, \; b_1, ..., b_s \neq q^{-m}$ for $m = 0, 1, \cdots$.

If $a_i = 1$, for any $i = 1, 2, \cdots, r-1$ the operator (8) reduces to the following special form:

$$I_q\left[\begin{matrix}\delta,\nu,a_1,\cdots,a_{i-1},1,a_{i+1},\cdots,a_{r-1}\\ b_1,\cdots,b_s\end{matrix}\right]f(x) = \frac{x^{-\delta-\nu}}{\Gamma_q(\nu)}\int_0^x t^{\delta}\ (x-tq)_{\nu-1}\, f(t)\ d_qt$$
$$\equiv I_q^{\delta,\nu} f(x), \quad \delta\in\mathbb{C},\ \mathrm{Re}(\nu)>0. \quad (10)$$

The fractional calculus theory is applied in almost all the areas of science and engineering. Operators of fractional calculus and their q-analogues have many applications, for example, they can used to solve dual integral and series equations which arise in crack problems in elasticity [22]. They find applications also in chaos theory ([25]-[26]), control systems, signal processing, bio-medical engineering, radars, sonars, etc ([3], [23]-[24]). Further one more important property of fractional integrals is that certain multiples of them map some orthogonal polynomials to orthogonal polynomials [15].

In this paper we define a new fractional q-integral operator type Weyl, using a generalized basic hypergeometric function ${}_r\phi_s(.)$. For the new operator some composition formulae, fractional q-integration by parts and the q-Mellin transform are presented. Furthermore, various special cases are mentioned.

2. BASIC HYPERGEOMETRIC SERIES

In this section, we present some basic definitions, which are useful in our analysis.

2.1 The q-shifted factorial

The q-shifted factorial is defined as [9]

$$(a;q)_n = \begin{cases} 1, & n=0 \\ (1-a)(1-aq)(1-aq^2)...(1-aq^{n-1}), & n=1,2... \\ [(1-aq^{-1})(1-aq^{-2})...(1-aq^{-n})]^{-1}, & n=-1,-2,... \end{cases} \quad (11)$$

and

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n = \prod_{k=0}^{\infty}(1-aq^k), \quad (12)$$

which converges for $|q|<1$ and diverges for $a\neq 0$ and $|q|\geqslant 1$, and

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \quad n\in Z, |q|<1. \quad (13)$$

2.2 Identities

We recall here following q-identities [9, p. 233, Eq. (I11); p. 235, Eq. (I.35); p. 21, exercise 1.5]:

$$\frac{(a;q)_{n-k}}{(b;q)_{n-k}} = \frac{(a;q)_n}{(b;q)_n}\frac{(q^{1-n}/b;q)_k}{(q^{1-n}/a;q)_k}\left(\frac{b}{a}\right)^k. \quad (14)$$

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \quad 0<q<1. \quad (15)$$

$$\frac{(c,bq^n;q)_m}{(b;q)_m} = \frac{(b/c;q)_n}{(b;q)_n}\sum_{k=0}^{n}\frac{(q^{-n},c;q)_k\ q^k}{(q,cq^{1-n}/b;q)_k}\left(cq^k;q\right)_m. \quad (16)$$

From (4), (12), (13) and (15) we obtenemos the following identity

$$\left(1-q^{k+1}\right)_{\nu-1}=\frac{\left(q^{k+1};q\right)_\infty}{\left(q^{k+\nu};q\right)_\infty}=\frac{(q;q)_\infty}{(q^\nu;q)_\infty}\frac{(q^\nu;q)_k}{(q;q)_k}=\frac{\Gamma_q(\nu)}{(1-q)^{1-\nu}}\frac{(q^\nu;q)_k}{(q;q)_k}. \tag{17}$$

2.3 Basic hypergeometric series

This series is due to Heine, [9]

$$_2\phi_1(a,b;c;q,z)=\sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n}{(q;q)_n(c;q)_n}z^n, \tag{18}$$

where it is assumed that $c\neq q^{-m}$ for $m=0,1,...,$ and $(a;q)_n$ is the q-shifted factorial defined in (11).

2.4 Generalized basic hypergeometric series

A generalization of the basic hypergeometric series $_2\phi_1$, is as follows [9]

$$_r\phi_s(a_1,a_2,...,a_r;b_1,b_2,...,b_s;q,z)\equiv{}_r\phi_s\left[\begin{array}{c}a_1,a_2,...,a_r\\ b_1,b_2,...,b_s\end{array};q,z\right]=$$

$$\sum_{n=0}^{\infty}\frac{(a_1;q)_n(a_2;q)_n...(a_r;q)_n}{(q;q)_n(b_1;q)_n...(b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}z^n, \tag{19}$$

where $b_1,...,b_s\neq q^{-m}$ for $m=0,1,...;\binom{n}{2}=\frac{n(n-1)}{2}$; $q\neq 0$ when $r>s+1$ and $\lim\limits_{q\to 1^-}{}_r\phi_s={}_rF_s$.

2.5 The q-binomial theorem

One of the important summation formulae for hypergeometric series is given by the following binomial theorem:

$$_2F_1(a,c;c;z)={}_1F_0(a;-;z)=\sum_{n=0}^{\infty}\frac{(a)_n}{n!}z^n=(1-z)^{-a},\quad |z|<1, \tag{20}$$

whose q-analogue was derived by Cauchy (1843), Heine (1847) and others [9]

$$_1\phi_0(a;-;q,z)=\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n=\frac{(az;q)_\infty}{(z;q)_\infty},\quad |z|<1,|q|<1. \tag{21}$$

2.6 The q-derivative operator

This is denoted by $\mathcal{D}_q$ and defined for fixed q as [9, p. 22]

$$\mathcal{D}_q f(z)=\frac{f(z)-f(qz)}{(1-q)\,z}. \tag{22}$$

2.7 The q-integral

Thomae (1869) and Jackson (1910) introduced the q-integral in the following form [9, p. 19, Eq. (1.11.1)]

$$\int_0^1 f(t)d_qt=(1-q)\sum_{n=0}^{\infty}f(q^n)q^n. \tag{23}$$

and also defined an integral on $(0,\infty)$ by [9, p. 19, Eq. (1.11.4)]

$$\int_0^\infty f(t)d_qt = (1-q)\sum_{n=-\infty}^{\infty} f(q^n)q^n, \tag{24}$$

therefore from (23) and (24) we have

$$\int_1^\infty f(t)d_qt = (1-q)\sum_{n=1}^{\infty} f(q^{-n})q^{-n}. \tag{25}$$

3. DEFINITION OF THE OPERATOR

We define the generalized fractional q-integral operator $K_q\left[\begin{smallmatrix}\delta,\nu,a_1,\cdots,a_{r-1}\\ b_1,\cdots,b_s\end{smallmatrix}\right]f(x)$ as follows:

$$K_q\begin{bmatrix}\delta,\nu,a_1,\cdots,a_{r-1}\\ b_1,\cdots,b_s\end{bmatrix}f(x) = \frac{q^{-\delta}x^{\delta}}{\Gamma_q(\nu)} \quad \mathrm{x}$$

$$\int_x^\infty t^{-\delta-\nu}\ (t-x)_{\nu-1}\ {}_r\phi_s\left[\begin{matrix}a_1,\cdots,a_{r-1},q^{\nu-1}\frac{x}{t}\\ b_1,\cdots,b_s\end{matrix};q,q\right]\ f(q^{1-\nu}t)\ d_qt \tag{26}$$

where $\delta\in\mathbb{C},\ \mathrm{Re}(\nu)>0,\ b_1,...,b_s\neq q^{-m}$ for $m=0,1,\cdots$.

Making a simple change of variable and using (22) we get

$$K_q\begin{bmatrix}\delta,\nu,a_1,\cdots,a_{r-1}\\ b_1,\cdots,b_s\end{bmatrix}f(x) = \frac{q^{-\delta}}{\Gamma_q(\nu)} \quad \mathrm{x}$$

$$\int_1^\infty w^{-\delta-1}\left(1-\frac{1}{w}\right)_{\nu-1}\ {}_r\phi_s\left[\begin{matrix}a_1,\cdots,a_{r-1},q^{\nu-1}/w\\ b_1,\cdots,b_s\end{matrix};q,q\right]\ f(q^{1-\nu}xw)\ d_qw \tag{27}$$

which, by virtue of (25) and after of a change of index, can be written as

$$K_q\begin{bmatrix}\delta,\nu,a_1,\cdots,a_{r-1}\\ b_1,\cdots,b_s\end{bmatrix}f(x) = \frac{(1-q)}{\Gamma_q(\nu)} \quad \mathrm{x}$$

$$\sum_{k=0}^{\infty} q^{\delta k}\ (1-q^{k+1})_{\nu-1}\ {}_r\phi_s\left[\begin{matrix}a_1,\cdots,a_{r-1},q^{\nu+k}\\ b_1,\cdots,b_s\end{matrix};q,q\right]\ f(q^{-\nu-k}x). \tag{28}$$

Now, using (17) we obtain

$$K_q\begin{bmatrix}\delta,\nu,a_1,\cdots,a_{r-1}\\ b_1,\cdots,b_s\end{bmatrix}f(x) = (1-q)^{\nu} \quad \mathrm{x}$$

$$\sum_{k=0}^{\infty} q^{\delta k}\ \frac{(q^\nu;q)_k}{(q;q)_k}\ {}_r\phi_s\left[\begin{matrix}a_1,\cdots,a_{r-1},q^{\nu+k}\\ b_1,\cdots,b_s\end{matrix};q,q\right]\ f(q^{-\nu-k}x). \tag{29}$$

If $a_i=1$ for any $i=1,2,\cdots,r-1$ the operator (26) reduces to the following special form:

$$K_q\begin{bmatrix}\delta,\nu,a_1,\cdots,a_{i-1},1,a_{i+1},\cdots,a_{r-1}\\ b_1,\cdots,b_s\end{bmatrix}f(x) = \frac{q^{-\delta}x^{\delta}}{\Gamma_q(\nu)}\int_x^{\infty} t^{-\delta-\nu}\ (t-x)_{\nu-1} f(q^{1-\nu}t)\ d_qt$$
$$\equiv K_q^{\delta,\nu}f(x),\quad \delta\in\mathbb{C},\ \mathrm{Re}(\nu)>0. \qquad (30)$$

Theorem 1 *If* $\delta\in\mathbb{C},\ \mathrm{Re}(\nu)>0,\ b_1,...,b_s\neq q^{-m}$ *for* $m=0,1,\cdots$, *then we have the following result for fractional q-integration by parts*

$$\int_0^{\infty} g\left(q^{-\nu}x\right) I_q\begin{bmatrix}\delta,\nu,a_1,\cdots,a_{r-1}\\ b_1,\cdots,b_s\end{bmatrix}f(x)\ d_qx = \int_0^{\infty} f\left(x\right) K_q\begin{bmatrix}\delta,\nu,a_1,\cdots,a_{r-1}\\ b_1,\cdots,b_s\end{bmatrix}g(x)\ d_qx \qquad (31)$$

provided that the q-integrals exist.

Proof Let be

$$I=\int_0^{\infty} g\left(q^{-\nu}x\right) I_q\begin{bmatrix}\delta,\nu,a_1,\cdots,a_{r-1}\\ b_1,\cdots,b_s\end{bmatrix}f(x)\ d_qx.$$

Using the definition of the operator $I_q\left[\begin{smallmatrix}\delta,\nu,a_1,\cdots,a_{r-1}\\ b_1,\cdots,b_s\end{smallmatrix}\right](\cdot)$ given in (9) and interchanging the order of integration and summation, in view of the absolute convergence, we get

$$I = \frac{(1-q)}{\Gamma_q(\nu)}\sum_{k=0}^{\infty} q^k\ q^{\delta k}\ \left(1-q^{k+1}\right)_{\nu-1}\ {}_r\phi_s\left[\begin{matrix}a_1,\cdots,a_{r-1},q^{\nu+k}\\ b_1,\cdots,b_s\end{matrix};q,q\right]\ \times$$
$$\int_0^{\infty} g\left(q^{-\nu}x\right) f(xq^k)\ d_qx$$

now, making a variable change and using (22), we obtain

$$I = \frac{(1-q)}{\Gamma_q(\nu)}\sum_{k=0}^{\infty} q^{\delta k}\ \left(1-q^{k+1}\right)_{\nu-1}\ {}_r\phi_s\left[\begin{matrix}a_1,\cdots,a_{r-1},q^{\nu+k}\\ b_1,\cdots,b_s\end{matrix};q,q\right]\ \times$$
$$\int_0^{\infty} g\left(q^{-\nu-k}x\right) f(x)\ d_qx$$

and interchanging the order of summation and integration we have

$$I = \frac{(1-q)}{\Gamma_q(\nu)}\int_0^{\infty} f(x)\left\{\sum_{k=0}^{\infty} q^{\delta k}\ \left(1-q^{k+1}\right)_{\nu-1}\ \times\right.$$
$$\left.{}_r\phi_s\left[\begin{matrix}a_1,\cdots,a_{r-1},q^{\nu+k}\\ b_1,\cdots,b_s\end{matrix};q,q\right]\ g\left(q^{-\nu-k}x\right)\right\}\ d_qx.$$

Finally, interpreting this expression in terms of the operator $K_q\left[\begin{smallmatrix}\delta,\nu,a_1,\cdots,a_{r-1}\\ b_1,\cdots,b_s\end{smallmatrix}\right](\cdot)$ we obtain (31).

From (10), (30) and Theorem 1 we obtain the following:

Corollary 1 *If* $\delta\in\mathbb{C},\ \mathrm{Re}(\nu)>0$, *then we have the following result for fractional q-integration by parts*

$$\int_0^{\infty} g\left(q^{-\nu}x\right) I_q^{\delta,\nu}f(x)\ d_qx = \int_0^{\infty} f\left(x\right) K_q^{\delta,\nu}g(x)\ d_qx \qquad (32)$$

provided that the q-integrals exist.

4. COMPOSITION FORMULAE

In this section we investigate some composition formulae for the fractional q-integral operator $K_q\left[{}^{\delta,\nu,a_1,\cdots,a_{r-1}}_{b_1,\cdots,b_s}\right]f(x)$.

Theorem 2 *Let $K_q^{\eta,\mu}f(x)$ and $K_q\left[{}^{\delta,\nu,a_1,\cdots,a_{r-1}}_{b_1,\cdots,b_s}\right]f(x)$ be fractional q-integral operators, as defined by (5) and (26) respectively. Then,*

$$K_q^{\delta-\mu,\mu}\left\{K_q\begin{bmatrix}\delta,\nu,a_1,\cdots,a_{r-1}\\ b_1,\cdots,b_s\end{bmatrix}f(x)\right\}=K_q\begin{bmatrix}\delta-\mu,\nu+\mu,a_1,\cdots,a_{r-1},q^{\nu}\\ b_1,\cdots,b_s,q^{\nu+\mu}\end{bmatrix}f(x) \quad (33)$$

where $\delta\in\mathbb{C},\ \mathrm{Re}(\mu)>0,\mathrm{Re}(\nu)>0,\ b_1,...,b_s\neq q^{-m}$ for $m=0,1,\cdots$.

Proof To prove the result (33), we make use of the relations (6), (17) and (29) in the left hand side, to obtain

$$K_q^{\delta-\mu,\mu}\left\{K_q\begin{bmatrix}\delta,\nu,a_1,\cdots,a_{r-1}\\ b_1,\cdots,b_s\end{bmatrix}f(x)\right\}=(1-q)^{\mu+\nu}\sum_{h=0}^{\infty}q^{h(\delta-\mu)}\frac{(q^{\mu};q)_h}{(q;q)_h} \quad \mathrm{x}$$

$$\sum_{k=0}^{\infty}q^{\delta k}\frac{(q^{\nu};q)_k}{(q;q)_k}\ {}_r\phi_s\left[\begin{matrix}a_1,\cdots,a_{r-1},q^{\nu+k}\\ b_1,\cdots,b_s\end{matrix};q,q\right]\ f(q^{-\mu-\nu-h-k}x).$$

Using the result [16, p. 56, Eq. 1]

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n-k)$$

we get

$$K_q^{\delta-\mu,\mu}\left\{K_q\begin{bmatrix}\delta,\nu,a_1,\cdots,a_{r-1}\\ b_1,\cdots,b_s\end{bmatrix}f(x)\right\}=(1-q)^{\mu+\nu}\sum_{h=0}^{\infty}f(q^{-\mu-\nu-h}x) \quad \mathrm{x}$$

$$\sum_{k=0}^{h}q^{(\delta-\mu)h}\frac{(q^{\mu};q)_{h-k}}{(q;q)_{h-k}}q^{\mu k}\frac{(q^{\nu};q)_k}{(q;q)_k}\ {}_r\phi_s\left[\begin{matrix}a_1,\cdots,a_{r-1},q^{\nu+k}\\ b_1,\cdots,b_s\end{matrix};q,q\right].$$

Now, the result (14) leads to

$$K_q^{\delta-\mu,\mu}\left\{K_q\begin{bmatrix}\delta,\nu,a_1,\cdots,a_{r-1}\\ b_1,\cdots,b_s\end{bmatrix}f(x)\right\}=(1-q)^{\mu+\nu}\sum_{h=0}^{\infty}q^{(\delta-\mu)h}\frac{(q^{\mu};q)_h}{(q;q)_h}f(q^{-\mu-\nu-h}x) \quad \mathrm{x}$$

$$\sum_{k=0}^{h}\frac{(q^{-h};q)_k}{(q^{1-h}/q^{\mu};q)_k}q^{k}\frac{(q^{\nu};q)_k}{(q;q)_k}\ {}_r\phi_s\left[\begin{matrix}a_1,\cdots,a_{r-1},q^{\nu+k}\\ b_1,\cdots,b_s\end{matrix};q,q\right].$$

Hence applying (19) we have

$$K_q^{\delta-\mu,\mu}\left\{K_q\begin{bmatrix}\delta,\nu,a_1,\cdots,a_{r-1}\\ b_1,\cdots,b_s\end{bmatrix}f(x)\right\}=(1-q)^{\mu+\nu}\sum_{h=0}^{\infty}q^{(\delta-\mu)h}\frac{(q^{\mu};q)_h}{(q;q)_h}f(q^{-\mu-\nu-h}x) \quad \mathrm{x}$$

$$\sum_{j=0}^{\infty} \frac{(a_1;q)_j (a_2;q)_j ... (a_{r-1};q)_j}{(q;q)_j (b_1;q)_j ... (b_s;q)_j} \left[(-1)^j q^{\binom{j}{2}}\right]^{1+s-r} q^j \sum_{k=0}^{h} \frac{(q^{-h};q)_k}{(q^{1-h}/q^{\mu};q)_k} q^k \frac{(q^{\nu};q)_k}{(q;q)_k} \left(q^{\nu+k};q\right)_j,$$

where we have interchanging the order of the sum on the basis of absolute convergence.

The use of (16), in order to evaluate the inner sum, yields

$$\sum_{k=0}^{h} \frac{(q^{-h};q)_k}{(q^{1-h}/q^{\mu};q)_k} q^k \frac{(q^{\nu};q)_k}{(q;q)_k} \left(q^{\nu+k};q\right)_j = \frac{(q^{\nu}, q^{\mu+\nu+h};q)_j}{(q^{\mu+\nu};q)_j} \frac{(q^{\mu+\nu};q)_h}{(q^{\mu};q)_h}$$

therefore,

$$K_q^{\delta-\mu,\mu}\left\{K_q\begin{bmatrix}\delta,\nu,a_1,\cdots,a_{r-1}\\ b_1,\cdots,b_s\end{bmatrix}f(x)\right\} = (1-q)^{\mu+\nu}\sum_{h=0}^{\infty} q^{(\delta-\mu)h}\frac{(q^{\mu+\nu};q)_h}{(q;q)_h} f(q^{-\mu-\nu-h}x) \quad \text{x}$$

$$\sum_{j=0}^{\infty} \frac{(a_1;q)_j (a_2;q)_j ... (a_{r-1};q)_j}{(q;q)_j (b_1;q)_j ... (b_s;q)_j} \frac{(q^{\nu}, q^{\mu+\nu+h};q)_j}{(q^{\mu+\nu};q)_j} \left[(-1)^j q^{\binom{j}{2}}\right]^{1+s-r} q^j,$$

that is,

$$K_q^{\delta-\mu,\mu}\left\{K_q\begin{bmatrix}\delta,\nu,a_1,\cdots,a_{r-1}\\ b_1,\cdots,b_s\end{bmatrix}f(x)\right\} = (1-q)^{\mu+\nu}\sum_{h=0}^{\infty} q^{(\delta-\mu)h}\frac{(q^{\mu+\nu};q)_h}{(q;q)_h} f(q^{-\mu-\nu-h}x) \quad \text{x}$$

$${}_{r+1}\phi_{s+1}\begin{bmatrix}a_1,\cdots,a_{r-1},q^{\nu},q^{\nu+\mu+h} & \\ b_1,\cdots,b_s,q^{\nu+\mu} & ;q,q\end{bmatrix}$$

on interpretation of the right hand side of the above expression in light of the definition (29), we finally obtain of the desired result.

Theorem 3 *Let $K_q^{\eta,\mu}f(x)$ and $K_q\left[\begin{smallmatrix}\delta,\nu,a_1,\cdots,a_{r-1}\\ b_1,\cdots,b_s\end{smallmatrix}\right]f(x)$ be fractional q-integral operators, as defined by (5) and (26) respectively. Then,*

$$K_q^{\delta-\mu_1-\mu_2-\cdots-\mu_{n-1},\mu_{n-1}} K_q^{\delta-\mu_1-\mu_2-\cdots-\mu_{n-2},\mu_{n-2}} K_q^{\delta-\mu_1-\mu_2-\cdots-\mu_{n-3},\mu_{n-3}} \cdots$$

$$K_q^{\delta-\mu_1-\mu_2,\mu_2} K_q^{\delta-\mu_1,\mu_1}\left\{K_q\begin{bmatrix}\delta,\nu,a_1,\cdots,a_{r-1}\\ b_1,\cdots,b_s\end{bmatrix}f(x)\right\} =$$

$$K_q\begin{bmatrix}\delta-\mu_1-\mu_2-\cdots-\mu_{n-1},\nu+\mu_1+\mu_2+\cdots+\mu_{n-1},a_1,\cdots,a_{r-1},q^{\nu}\\ b_1,\cdots,b_s,q^{\nu+\mu_1+\mu_2+\cdots+\mu_{n-1}}\end{bmatrix}f(x) \quad (34)$$

where $\delta \in \mathbb{C}$, $\mathrm{Re}(\mu_i) > 0, \mathrm{Re}(\nu) > 0$, $i = 1,2,\cdots,n-1$, $b_1,...,b_s \neq q^{-m}$ for $m = 0,1,\cdots$.

Proof We employ here the principle of mathematicial induction. For $n = 2$, (33) reduces to,

$$K_q^{\delta-\mu_1,\mu_1}\left\{K_q\begin{bmatrix}\delta,\nu,a_1,\cdots,a_{r-1}\\ b_1,\cdots,b_s\end{bmatrix}f(x)\right\} = K_q\begin{bmatrix}\delta-\mu_1,\nu+\mu_1,a_1,\cdots,a_{r-1},q^{\nu}\\ b_1,\cdots,b_s,q^{\nu+\mu_1}\end{bmatrix}f(x). \quad (35)$$

We suppose that (33) is valid for $n = k-1$, that is,

$$K_q^{\delta-\mu_1-\mu_2-\cdots-\mu_{k-2},\mu_{k-2}} K_q^{\delta-\mu_1-\mu_2-\cdots-\mu_{k-3},\mu_{k-3}} K_q^{\delta-\mu_1-\mu_2-\cdots-\mu_{k-4},\mu_{k-4}} \cdots$$

$$K_q^{\delta-\mu_1-\mu_2,\mu_2} K_q^{\delta-\mu_1,\mu_1} \left\{ K_q \begin{bmatrix} \delta, \nu, a_1, \cdots, a_{r-1} \\ b_1, \cdots, b_s \end{bmatrix} f(x) \right\} =$$

$$K_q \begin{bmatrix} \delta-\mu_1-\mu_2-\cdots-\mu_{k-2}, \nu+\mu_1+\mu_2+\cdots+\mu_{k-2}, a_1, \cdots, a_{r-1}, q^\nu \\ b_1, \cdots, b_s, q^{\nu+\mu_1+\mu_2+\cdots+\mu_{k-2}} \end{bmatrix} f(x). \quad (36)$$

On operating both sides of the relation (36) by the operator $K_q^{\delta-\mu_1-\mu_2-\cdots-\mu_{k-1},\mu_{k-1}}(.)$, we obtain

$$K_q^{\delta-\mu_1-\mu_2-\cdots-\mu_{k-1},\mu_{k-1}} \left\{ K_q^{\delta-\mu_1-\mu_2-\cdots-\mu_{k-2},\mu_{k-2}} K_q^{\delta-\mu_1-\mu_2-\cdots-\mu_{k-3},\mu_{k-3}} K_q^{\delta-\mu_1-\mu_2-\cdots-\mu_{k-4},\mu_{k-4}} \right.$$

$$\left. \cdots K_q^{\delta-\mu_1-\mu_2,\mu_2} K_q^{\delta-\mu_1,\mu_1} K_q \begin{bmatrix} \delta, \nu, a_1, \cdots, a_{r-1} \\ b_1, \cdots, b_s \end{bmatrix} f(x) \right\} = K_q^{\delta-\mu_1-\mu_2-\cdots-\mu_{k-1},\mu_{k-1}} \times$$

$$\left\{ K_q \begin{bmatrix} \delta-\mu_1-\mu_2-\cdots-\mu_{k-2}, \nu+\mu_1+\mu_2+\cdots+\mu_{k-2}, a_1, \cdots, a_{r-1}, q^\nu \\ b_1, \cdots, b_s, q^{\nu+\mu_1+\mu_2+\cdots+\mu_{k-2}} \end{bmatrix} f(x) \right\}. \quad (37)$$

The application of (33) to the right hand side of the above expression leads to

$$K_q^{\delta-\mu_1-\mu_2-\cdots-\mu_{k-1},\mu_{k-1}} \left\{ K_q^{\delta-\mu_1-\mu_2-\cdots-\mu_{k-2},\mu_{k-2}} K_q^{\delta-\mu_1-\mu_2-\cdots-\mu_{k-3},\mu_{k-3}} K_q^{\delta-\mu_1-\mu_2-\cdots-\mu_{k-4},\mu_{k-4}} \right.$$

$$\left. \cdots K_q^{\delta-\mu_1-\mu_2,\mu_2} K_q^{\delta-\mu_1,\mu_1} K_q \begin{bmatrix} \delta, \nu, a_1, \cdots, a_{r-1} \\ b_1, \cdots, b_s \end{bmatrix} f(x) \right\} =$$

$$K_q \begin{bmatrix} \delta-\mu_1-\mu_2-\cdots-\mu_{k-1}, \nu+\mu_1+\mu_2+\cdots+\mu_{k-1}, a_1, \cdots, a_{r-1}, q^\nu \\ b_1, \cdots, b_s, q^{\nu+\mu_1+\mu_2+\cdots+\mu_{k-1}} \end{bmatrix} f(x) \quad (38)$$

which is valid for $n = k$ $(k = 1, 2, ..., n)$. This completes the proof of (32).

The result (30) joint with Theorem 2 and Theorem 3 yield:

Corollary 2 *Let $K_q^{\eta,\mu} f(x)$ be a fractional q-integral operator as defined by (5), then*

$$K_q^{\delta-\mu,\mu} K_q^{\delta,\nu} f(x) = K_q^{\delta-\mu,\nu+\mu} f(x) \quad (39)$$

where $\delta \in \mathbb{C},\ \mathrm{Re}(\mu) > 0, \mathrm{Re}(\nu) > 0.$

Corollary 3 *Let $K_q^{\eta,\mu} f(x)$ be a fractional q-integral operator as defined by (5), then*

$$K_q^{\delta-\mu_1-\mu_2-\cdots-\mu_{n-1},\mu_{n-1}} K_q^{\delta-\mu_1-\mu_2-\cdots-\mu_{n-2},\mu_{n-2}} \cdots K_q^{\delta-\mu_1-\mu_2,\mu_2} K_q^{\delta-\mu_1,\mu_1} K_q^{\delta,\nu} f(x) =$$

$$K_q^{\delta-\mu_1-\mu_2-\cdots-\mu_{n-1},\nu+\mu_1+\mu_2+\cdots+\mu_{n-1}} f(x) \quad (40)$$

where $\delta \in \mathbb{C},\ \mathrm{Re}(\mu_i) > 0, \mathrm{Re}(\nu) > 0,\ i = 1, 2, \cdots, n-1.$

5. Q-MELLIN TRANSFORM

In this section we establish the q-Mellin transform of the $K_q\left[\begin{smallmatrix} \delta,\nu,a_1,\cdots,a_{r-1} \\ b_1,\cdots,b_s \end{smallmatrix}\right](.)$ fractional q-integral operator.

The q-Mellin transform of $f(x)$ is defined as [4]

$$M_q\{f(x);s\} = \frac{1}{(1-q)}\int_0^\infty x^{s-1}f(x)d_qx. \tag{41}$$

Theorem 4 *Let* $K_q\left[\begin{smallmatrix}\delta,\nu,a_1,\cdots,a_{r-1}\\ b_1,\cdots,b_s\end{smallmatrix}\right]f(x)$ *be a fractional q-integral operator, as defined by (26), then*

$$M_q\left\{x^\rho K_q\begin{bmatrix}\delta,\nu,a_1,\cdots,a_{r-1}\\ b_1,\cdots,b_s\end{bmatrix}f(x);s\right\} = \frac{\Gamma_q(\delta+\rho+s)q^{\nu(\rho+s)}}{\Gamma_q(\delta+\nu+\rho+s)} \times$$

$${}_{r+1}\phi_{s+1}\left[\begin{matrix}a_1,\cdots,a_{r-1},q^\nu,0\\ b_1,\cdots,b_s,q^{\delta+\nu+\rho+s}\end{matrix};q,q\right] M_q\{f(x);\rho+s\}, \tag{42}$$

where $M_q\{f(x);s\}$ *denote the q-Mellin transform of* $f(x)$ *and* $\delta,\rho\in\mathbb{C}$, $\mathrm{Re}(\nu)>0$, $b_1,...,b_s\neq$
q^{-m} *for* $m=0,1,\cdots$.

Proof From (29) and (41) we have,

$$M_q\left\{x^\rho K_q\begin{bmatrix}\delta,\nu,a_1,\cdots,a_{r-1}\\ b_1,\cdots,b_s\end{bmatrix}f(x);s\right\} = (1-q)^{\nu-1}\sum_{k=0}^\infty q^{\delta k}\frac{(q^\nu;q)_k}{(q;q)_k} \times$$

$${}_r\phi_s\left[\begin{matrix}a_1,\cdots,a_{r-1},q^{\nu+k}\\ b_1,\cdots,b_s\end{matrix};q,q\right]\int_0^\infty x^{\rho+s-1}f(xq^{-\nu-k})\,d_qx, \tag{43}$$

where we have interchanged the order of integral and summation.

Making a change of variable and using (22) we obtain,

$$\int_0^\infty x^{\rho+s-1}f(xq^{-\nu-k})\,d_qx = q^{(\nu+k)(\rho+s)}\int_0^\infty w^{\rho+s-1}f(w)d_qw,$$

which can be written as:

$$\int_0^\infty x^{\rho+s-1}f(xq^{-\nu-k})\,d_qx = (1-q)q^{(\nu+k)(\rho+s)}M_q\{f(x);\rho+s\}.$$

Substitution of this expression in (43) leads to

$$M_q\left\{x^\rho K_q\begin{bmatrix}\delta,\nu,a_1,\cdots,a_{r-1}\\ b_1,\cdots,b_s\end{bmatrix}f(x);s\right\} = (1-q)^\nu q^{\nu(\rho+s)}M_q\{f(x);\rho+s\} \times$$

$$\sum_{k=0}^\infty q^{(\delta+\rho+s)k}\frac{(q^\nu;q)_k}{(q;q)_k}\,{}_r\phi_s\left[\begin{matrix}a_1,\cdots,a_{r-1},q^{\nu+k}\\ b_1,\cdots,b_s\end{matrix};q,q\right]. \tag{44}$$

Now applying (19), we get

$$\sum_{k=0}^\infty q^{(\delta+\rho+s)k}\frac{(q^\nu;q)_k}{(q;q)_k}\,{}_r\phi_s\left[\begin{matrix}a_1,\cdots,a_{r-1},q^{\nu+k}\\ b_1,\cdots,b_s\end{matrix};q,q\right] =$$

$$\sum_{j=0}^{\infty} \frac{(a_1;q)_j(a_2;q)_j...(a_{r-1};q)_j}{(q;q)_j(b_1;q)_j...(b_s;q)_j} \left[(-1)^j q^{\binom{j}{2}}\right]^{1+s-r} q^j \sum_{k=0}^{\infty} q^{(\delta+\rho+s)k} \frac{(q^\nu;q)_k}{(q;q)_k}(q^{\nu+k};q)_j. \quad (45)$$

On the other hand, from (13)

$$(q^{\nu+k};q)_j = \frac{(q^{\nu+k};q)_\infty}{(q^{\nu+k+j};q)_\infty} = \frac{(q^\nu;q)_\infty}{(q^{\nu+j};q)_\infty}\frac{(q^{\nu+j};q)_k}{(q^\nu;q)_k} = (q^\nu;q)_j \frac{(q^{\nu+j};q)_k}{(q^\nu;q)_k}. \quad (46)$$

Hence we have

$$S = \sum_{k=0}^{\infty} q^{(\delta+\rho+s)k} \frac{(q^\nu;q)_k}{(q;q)_k}(q^{\nu+k};q)_j = (q^\nu;q)_j \sum_{k=0}^{\infty} q^{(\delta+\rho+s)k} \frac{(q^{\nu+j};q)_k}{(q;q)_k}.$$

Using (21), (13) and (15) the above expression can be written as,

$$S = (q^\nu;q)_j \ {}_1\phi_0\left(q^{\nu+j};-;q,q^{\delta+\rho+s}\right) = \frac{(q^{\delta+\rho+s+\nu+j};q)_\infty}{(q^{\delta+\rho+s};q)_\infty}(q^\nu;q)_j$$

$$= \frac{(1-q)^{-\nu}\ \Gamma_q(\delta+\rho+s)}{\Gamma_q(\delta+\nu+\rho+s)} \frac{(q^\nu;q)_j}{(q^{\delta+\nu+\rho+s};q)_j}.$$

Therefore, from this result and (45),

$$\sum_{k=0}^{\infty} q^{(\delta+\rho+s)k} \frac{(q^\nu;q)_k}{(q;q)_k} \ {}_r\phi_s\left[\begin{matrix} a_1,\cdots,a_{r-1},q^{\nu+k} \\ b_1,\cdots,b_s \end{matrix};q,q\right] = \frac{(1-q)^{-\nu}\ \Gamma_q(\delta+\rho+s)}{\Gamma_q(\delta+\nu+\rho+s)} \times$$

$$\sum_{j=0}^{\infty} \frac{(a_1;q)_j(a_2;q)_j...(a_{r-1};q)_j(q^\nu;q)_j}{(q;q)_j(b_1;q)_j...(b_s;q)_j(q^{\delta+\nu+\rho+s};q)_j} \left[(-1)^j q^{\binom{j}{2}}\right]^{1+s-r} q^j,$$

$$= \frac{(1-q)^{-\nu}\ \Gamma_q(\delta+\rho+s)}{\Gamma_q(\delta+\nu+\rho+s)} \ {}_{r+1}\phi_{s+1}\left[\begin{matrix} a_1,\cdots,a_{r-1},q^\nu,0 \\ b_1,\cdots,b_s,q^{\delta+\nu+\rho+s} \end{matrix};q,q\right].$$

The substitution from this expression in (44) yields the desired result.

Finally, using (30) and (42) we get the following result, as a particular case:

Corollary 4 Let $K_q^{\delta,\nu}f(x)$ *be a fractional q-integral operator, as defined by (6), then*

$$M_q\left\{x^\rho K_q^{\delta,\nu}f(x)\right\} = \frac{\Gamma_q(\delta+\rho+s)\ q^{\nu(\rho+s)}}{\Gamma_q(\delta+\nu+\rho+s)} M_q\{f(x);\rho+s\}, \quad \delta,\rho\in\mathbb{C}, \mathrm{Re}(\nu)>0. \quad (47)$$

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