WALSH FUNCTIONS: DIGITAL ALTERNATIVE TO SINUSOIDAL FUNCTIONS

Mister President, Members of this Academy, Ladies and Gentlemen:
It is a real pleasure for me to celebrate with you my election as Corresponding Member of this Academy. Many thanks

The mathematical concept of sinusoidal functions and its application has a long tradition as well in science as in engineering. The modeling of planetary orbits by circles and epicycles as known from Ptolemy are early applications in astronomy. Even Copernicus made use of epicycles in his heliocentric model of the system of the sun. Arabic astronomy was familiar with the sinus function and the cosinus function which has been introduced to western astronomy by Georg von Peuberbach ("Purbachius"). They play an important part in such models. In physics the investigations of the vibrating string led such famous scientists as Daniel Bernoulli, Leonhard Euler and Joseph Luis Lagrange to the representation of vibrations by series of sinusoidal functions. By his work "The Analytical Theory of Heat" Joseph Fourier founded 1822a part of the theory of trigonometric series which is known today as the theory of "Fourier series". The final mathematical precision of the theory of Fourier series was provided by the work of Dirichlet and Riemann. Fourier series expansions of functions are an important tool of mathematical physics up to the present.

A further kind of application of sinusoidal functions supported the development of electrical engineering in the 19th century. Alternating currents can be described by sinusoidal functions. The same is true for the description of the characteristics of electrical networks in respect of alternating currents of different frequency. It was introduced by the German mathematician (later famous scientist in the US), Karl Rudolf Steinmetz (1865-1923), by his famous "symbolic method" (Komplexe Wechselstromrechnung) in dealing with alternating current phenomena. An extension of the symbolic method of Steinmetz is the "operational calculus" as developed by Oliver Heaviside for its application in the electromagnetic theory of electrical phenomena in a space-time continuum. Until now the Laplace transform, and as a special case the associated Fourier transform provide a solid mathematical basis for this type of problems in electrical engineering, control engineering and communication engineering. They allow the mathematical treatment of problems which are related to functions and operations that deal with electrical currents, control variables and signals and their transformation or processing in engineering systems.

Fourier-series expansions and Fourier-integral representations are conducive for the spectral representation of functions and operations with regard to their frequency-behavior (i.e.) with respect to their characteristics regarding sinusoidal functions. In all this kind of applications and mathematical models the system of real numbers (as well as the complex number system) play an important part. Signals and systems process "continuous" variables. In engineering this is known as an "Analog Technology". Today, in "Digital Technology" the leading technology in the field of electrical engineering system variables have only discrete values of finite number. Therefore, integers and finite sets are the main mathematical objects for modeling digital systems.

Although digital systems look back on a long history in electrical engineering (examples are given by switching circuits of high voltage power lines and by telephone switching circuits realized by electromagnetic relais, selectors and crossbar-switches), their actual importance is a result of the rapid progress of semiconductor-electronics and of computer technology. Microelectronics and microprocessors are the reason that Digital Technology has dramatically changed the existing systems. Analog systems have been replaced by digital systems. In this context the natural question arises, if there exists a system of functions which can perform similarly for digital systems as sinusoidal functions for analog systems.

The system of Walsh-Functions, which is known in mathematics since 1923, can claim to be partly useful for digital systems. This paper will try to show this. Special attention will be given to the pioneering work of H. Harmuth. His contributions to the application of Walsh-functions in the field of communication engineering (in that early times looking mainly for realizations by analog technology) are considered of big importance until today. In addition to such historical considerations we will also point to current modern applications of Walsh functions in mobile telecommunication and in secure information processing.

From a mathematical point of view the fact that Walsh functions have a similar applicability in modeling signals and systems as sinusoidal function is not so surprising.: Both systems of functions are special cases of "character functions" of a (topological) abelian group. The mathematical field of "Abstract Harmonic Analysis" covers many properties of both function systems in a more general way and provides a source for additional specific (orthogonal) function systems of similar use for applications in science and engineering.

2. Mathematical Overview

2.1 Walsh-Functions

We begin with the definition of the system of Rademacher-functions (Rademacher 1922).

With $\phi : [0,1) \to \mathbf{R}$ we denote a function which is given by $\phi(x) = 1$ for $x \in [0,1/2)$ and $\phi(x) = -1$ for $x \in [1/2,1)$. On the basis of the function ϕ we are able to define the system $\Phi = \{\phi_n : n = 0,1,2,\ldots\}$ of Rademacher-functions by $\phi_n : [0,1) \to \mathbf{R}$ with

$$\phi_n(x) := \phi(2^n x) \qquad (1)$$

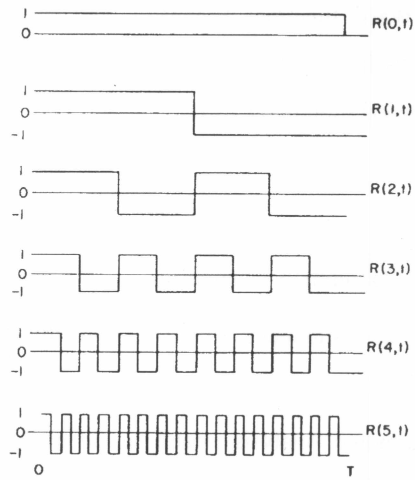


Figure 1: Rademacher-functions ϕ_0 to ϕ_5

With the help of the system Φ of Rademacher-functions the system $\Psi = \{\psi_n : n = 0,1,2,...\}$ of Walsh-functions $\psi_n : [0,1) \rightarrow \mathbf{R}$ is defined by

$$\psi_n(x) := [\phi_{n_0}(x)]^{n_0} [\phi_{n_1}(x)]^{n_1} ... [\phi_{n_k}(x)]^{n_k} \qquad (2)$$

where the integer n is represented by $n = n_0 2^0 + n_1 2^1 + ... + n_k 2^k$ (Walsh 1923).

Walsh-functions are, as we see, finite products of Rademacher-functions. To give an example, for $n = 2^0 + 2^2$ (n=5) the Walsh-function ψ_5 is given by $\psi_5 = \phi_0 \phi_2$.

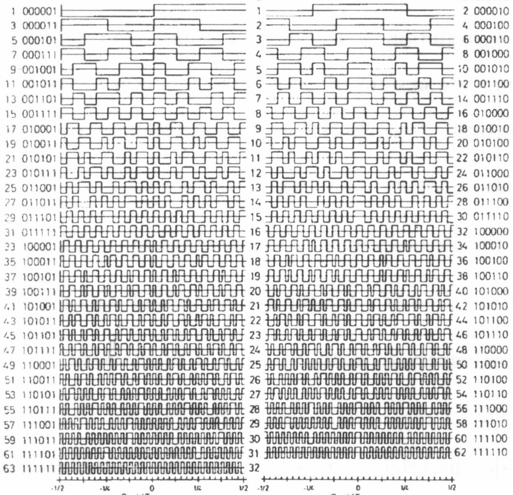


Figure 2: Walsh-functions $\psi_n(0 \leq n < 63)$ in sequency-order and divided into even functions cal(i,.) and odd functions sal(i,.) on the interval (-1/2, +1/2)

It is well known, that the Walsh-functions Ψ constitute a complete orthogonal system for the Hilbert-space $L_2[0,1)$. Consequently for each square-integrable function $f:[0,1)\rightarrow\mathbf{R}$ we have a Walsh-Fourier representation in the form of

$$f=\sum_{n=0}^{\infty}\hat{f}(n)\psi_n \qquad (3)$$

The coefficients $\hat{f}(n)$ of (3) define a discrete function $\hat{f}:\mathbf{N}_0\rightarrow\mathbf{R}$. $\hat{f}$ is called the Walsh-Fourier transform of f. The assignment $f\mapsto\hat{f}$ defines the Walsh-Fourier transformation WT; $WT(f)=\hat{f}$.

The field of Abstract Harmonic Analysis establishes in mathematics a general theory for functions defined on topological groups. In this context Walsh-functions can be identified as character-functions of the dyadic group **D** which is defined by the set of 0-1 sequences $\{(x_1,x_2,x_3,\ldots):x_i\in\{0,1\}\}$ with the addition modulo 2 (component-wise taken) as group-operation (Fine 1949, Vilenkin 1947).

For the identification of the Walsh-functions with the character-functions of **D** we use the map $bin:[0,1)\rightarrow\mathbf{D}$ which maps a real number $x=x_1 2^{-1}+x_2 2^{-2}+x_3 2^{-3}+\ldots$ to the binary sequence $bin(x)=(x_1,x_2,x_3,\ldots)$ (dyadic rational numbers are represented by a finite sum).

2.2 Sinusoidal Functions

The system of sinusoidal functions sin $2\pi nx$, n=1,2,3,...; cos 2π nx, n=0,1,2,... has, as pointed out in chapter 1, a long tradition in science and engineering. Similarly to the system of Walsh-functions it is a complete system of orthogonal function to span the Hilbert space $L_2[0,1)$. Therefore each function $f\in L_2[0,1)$ can be represented by a series expansion (Fourier serie) with coefficients $\hat{f}_s(n), n=1,2,\ldots$ and $\hat{f}_c(n), n=0,1,2,\ldots$.

The tupel $(\hat{f}_s,\hat{f}_c)$ of discrete functions defined by the coefficients is called the (real) Fourier-transform of f. The assignment $f\mapsto(\hat{f}_s,\hat{f}_c)$ defines the (real) Fourier transformation FT; $FT(f)=(\hat{f}_s,\hat{f}_c)$. From a mathematical point of view it is desirable to use the complex sinusoidal functions $\exp(j2\pi nx)=\cos 2\pi nx+j\sin 2\pi nx$ ($j=\sqrt{-1}$) for Fourier analysis in the classical sense. In this case the relation between classical Fourier analysis and Abstract Harmonic Analysis is easy to establish: The complex sinusoidal functions $\exp(j2\pi nx)$ are the character-functions of the additive group $([0,\infty),+)$ of non negative real numbers.

2.3 Finite system of functions

In a large number of applications of orthogonal functions in science and engineering it is sufficient to use finite many functions. In the case of Walsh-functions we take as domain the set $\mathbf{N}(n)=\{0,1,2,\ldots,2^n-1\}$ of integers and define the discrete Walsh-functions $w(i,.):\mathbf{N}(n)\rightarrow\{+1,-1\}$ of order n for $i\in N(n)$ by

$$w(i,k):=(-1)^{\|\mathrm{bin}(i)\oplus\mathrm{bin}(k)\|} \qquad (4)$$

where $bin(i)$ and $bin(k)$ denote the binary representation of the integers i and k, respectively ($\oplus$ is the „addition modulo 2" of binary numbers, $\|b\|$ is the Hamming weight of a binary number b).

The set $W(3)$ of discrete Walsh functions; + := + 1, – := – 1.

	000	001	010	011	100	101	110	111
$w(0,.)$	+	+	+	+	+	+	+	+
$w(1,.)$	+	+	+	+	–	–	–	–
$w(2,.)$	+	+	–	–	+	+	–	–
$w(3,.)$	+	+	–	–	–	–	+	+
$w(4,.)$	+	–	+	–	+	–	+	–
$w(5,.)$	+	–	+	–	–	+	–	+
$w(6,.)$	+	–	–	+	+	–	–	+
$w(7,.)$	+	–	–	+	–	+	+	–

Figure 3: Discrete Walsh-functions $w(i,.)$ for n=3

The system $W(n)$ of discrete Walsh-functions of order n can be identified with the character-functions of the dyadic group $\mathbf{D}(n)=(\mathbf{B}^n,\oplus)$, where $\mathbf{B}(n)$ denotes the set of binary numbers of length n. $\mathbf{D}(n)$ is isomorphic to the n-fold direct product $Z_2\otimes Z_2\otimes\cdots\otimes Z_2$ of the cyclic group $Z_2=(\{0,1\},\oplus)$.

The system S(n) consisting of discrete complex sinusoidal functions s(i,.) of order n are defined by

$$s(i,k):=\exp(j(2\pi/n)ik) \qquad (5)$$

It is known that the discrete complex sinusoidal functions of order n can be identified as character functions of the cyclic group $\mathbf{Z}_n=(\{0,1,2,\ldots,n-1\},+\bmod n)$.

Besides of the Walsh-functions ψ_n originally introduced into mathematics by Walsh (1923) there exist also "generalized" Walsh-functions as considered by Levy (1944) and Vilenkin (1947).

Generalized discrete Walsh-functions in this sense can be identified with the character-functions of finite abelian groups $G = Z_{k(1)} \otimes Z_{k(2)} \otimes \ldots \otimes Z_{k(n)}$ (here $Z_{k(i)}$ denotes the cyclic group of order n). For the special cases $G = Z_n$ and $G = Z_2 \otimes Z_2 \otimes \ldots \otimes Z_2$ (n-fold) the generalized discrete Walsh-functions become discrete sinusoidal functions of order n and respectively discrete Walsh-functions of order n. By this interpretation, generalized discrete Walsh-functions can also be considered as n-dimensional discrete sinusoidal functions of order k(1), k(2),..., respectively k(n). Consequently, discrete Walsh-functions w(i,.) of order n can then be interpreted as n-dimensional discrete sinusoidal functions $s(i,.) = \exp(j\pi i(.)$ of order 2.

This viewpoint helps to consider generalized discrete Walsh functions and (ordinary) discrete Walsh functions not as exotic mathematical constructions but to be closely related to the (classical) discrete sinusoidal functions.

2.5 Fast Transform Algorithms

For the case of discrete sinusoidal functions s(i,.) of order n the development of a fast algorithm, the "Fast Fourier Transform" algorithm FFT, to implement the Fourier transformation FT is credited to Good (1958) and Cooley-Tukey (1965).

A similar effective algorithm to implement the discrete Walsh-transformation WT of order n has been found by Whelchel (1968) ("Fast Walsh Transform" algorithm; FWT).

For the case of the generalized discrete Walsh functions the development of the corresponding "Fast Generalized Walsh Transform" algorithm ($FGWT$) to implement effectively the generalized Walsh transformation GWT has been discovered by Nicholson (1971).

The implementation of the $FGWT$ is covered by,Kunz (1977) and also by Fellner (1982) . Hellwagner (1988) investigated problems of testability of systolic hardware-implementation of the $FGWT$; the integration of the $FGWT$ into common tools for digital image processing has been made by Hellwagner (1990) and also by Scharinger (1995).

3. Application in Coding and Signal Processing

Discrete Walsh-function have a rather long tradition with regard to their application in the fields of coding and signal processing. We will give few examples:

(1) Let **W**(n) denote the matrix which is defined by the system W(n) of discrete Walsh functions of order n in the following way:

For i∈**N**(n) the i'th row of **W**(n) is given by $w(i,.) = (w(i,0), w(i,1), \ldots, w(i,2^n - 1))$. **W**(n) is a special kind of a Hadamard-matrix. The rows define an orthogonal code (Walsh-code) which has been applied in early space mission projects (Viterbi 1964). A reason for a successful application was the fact that this code allows the development of effective digital circuits for coding and decoding (Green 1958).

(2) The cancellation of the first row and the first column of the matrix **W**(n) results in a matrix **C** which defines an orthogonal code of length 2^n-1. It can be shown that there exist permutations P of the columns of **C** such that the permuted matrix **M**=P(**C**) constitutes a cyclic orthogonal code which can be generated by a maximum length linear feedback shiftregister MLFSR. This indicates that a relationship between "Walsh-codes" and "Pseudo Noise Codes" (which are cyclic codes generated by MLFSR's) can be established.

(3) The existence of the Fast Walsh transformation algorithm *FWT* was essential for the applications of Walsh-function in signal processing. Pioneering work on that topic is reported in Ahmed-Rao (1975). The concept of a wave-filter which operates in the domain of the Walsh-Fouriertransform was firstly introduced by Harmuth (1964). The accompanying description of such filters by dyadic convolution was established by Pichler (1968). The theory of optimization of such filters (Pichler 1970) needed the introduction of the concept of the dyadic autocorrelation function (DAKF) and the formulation of the Wiener-Chinchin theorem for the case of Walsh-Fourier analysis. In the context of dyadic filtering it was necessary to introduce the "sampling theorem" of Walsh-Fourier analysis, as a true analogon to the famous sampling theorem of Shannon (Pichler 1970).

(4) The possibility to relate different spectral representations to each other is of special interest in signal processing. A transformation for discrete-time processes which relates Fourier power-spectra into Walsh power-spectra has been investigated and implemented by Gibbs-Gebbie (1969). Besides of the use of the existing transformations between power spectra and the appropriate related type of autocorrelation function (Wiener-Chinchin theorems) the transformation of such autocorrelation functions into each other had to be developed (Gibbs-Pichler 1971).

(5) The comparison of Fourier spectra and Walsh spectra for certain signals is of great significance in applied sciences. As an example we mention here (Trappl-Horn-Rappelsberger 1982).

Although the mentioned examples of signal processing based on concepts of Walsh Harmonic

Analysis deal with applications to analog signals, they are also true for digital signals and indicate the applicability of Walsh functions in digital signal processing. Other specific applications of Walsh functions in digital technology, specially, for the design of boolean functions (switching functions) and their characterization with respect to fault detection, has been known for a long time (Karpovsky 1976, 1985).

4. Transmission of Information by Orthogonal Functions: The contribution of H. Harmuth

The successful promotion for the application of Walsh functions in communications engineering is due to Henning F. Harmuth, as well as the contribution of important papers during the years from 1960 to 1970 and of his book (Harmuth 1971). In the following we want to refer to some of the treatises as experienced by the author during his cooperation with H. Harmuth.

Harmuth started with discovering the Walsh functions as a useful orthogonal code and their application in wireless communication (Harmuth 1960). This was followed by a patent of the concept of a multi-channel transmission system with Walsh functions as carriers (Harmuth 1964). Harmuth considered mainly – with respect to the state of the art at this time – analog transmission systems with time-continuous signals to represent information.

To develop a valid mathematical basis for such systems Harmuth contacted the Institute of Mathematics of the University of Innsbruck, Austria, where mathematical research in Walsh function was under way (R. Liedl, P. Weiss). The contribution of the author (Pichler 1967, 1968, 1970) helped to give a mathematical and systemstheoretical basis for the work of Harmuth. The symposia which were organized by Harmuth in Washington D.C. (from 1969 on) roused international interest. They reported about applications in information technology and inspired further research. As a result Walsh functions and their applications are today well known in communication engineering and in signal processing. They have found important applications in modern digital information technology. As a conclusion we may say, that Henning F. Harmuth established a milestone in the development of information technology by his work on the application of Walsh functions as an alternative to sinusoidal functions.

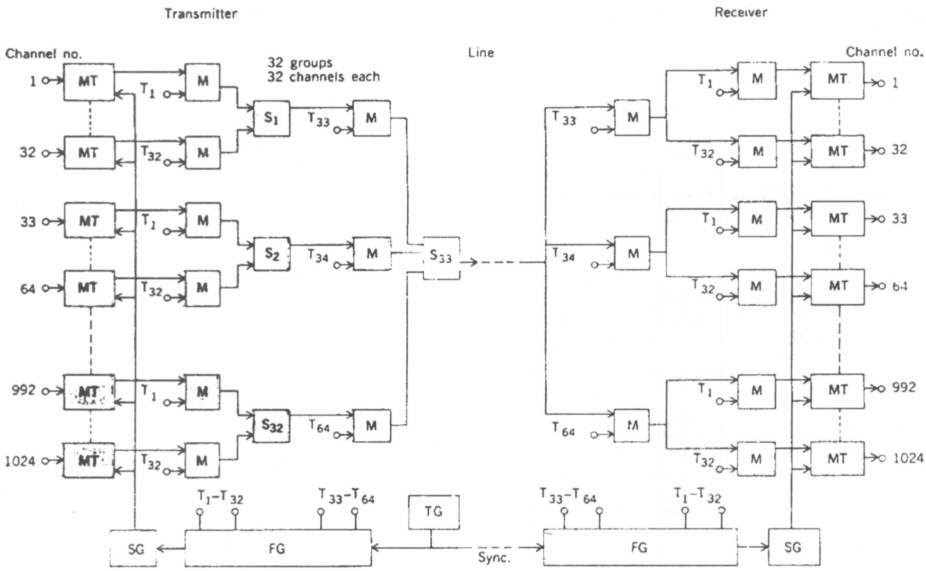


Figure 4: Block diagram of a sequency-multiplex system for 1024 analog telephony channels (after H.F. Harmuth)

5. Current Applications

In the following we want to discuss three examples of actual applications of Walsh-functions in the field of coding and signal processing. They should serve to show that this system of orthogonal functions which was introduced in mathematics more than seventy years ago is until today of practical importance.

5.1 Multiplexing in digital wireless communication

Cellular wireless communication systems with mobile stations are nowadays an important component for modern digital systems for telephony and data. The concentration of the traffic from the mobile stations to the base stations in the different cells needs a multiplexing system. Aside from the utilization of frequency division (FDMA systems) and time division (TDMA systems) orthogonal division, as introduced by Harmuth (1964) is used (CDMA systems; CDMA stands for “Code Division Multiple Access”). The specific CDMA system of Qualcomm Inc. uses for the forward link channels Walsh functions as digital carriers (Qualcomm Inc. 1992). The concentration of channel traffic by sequency-multiplexing for an analog telephone exchange was already suggested in Pichler (1967).

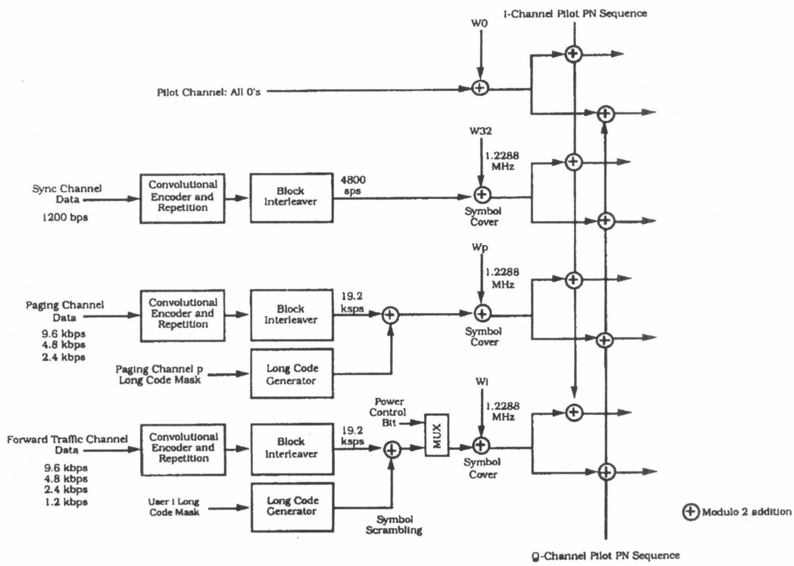
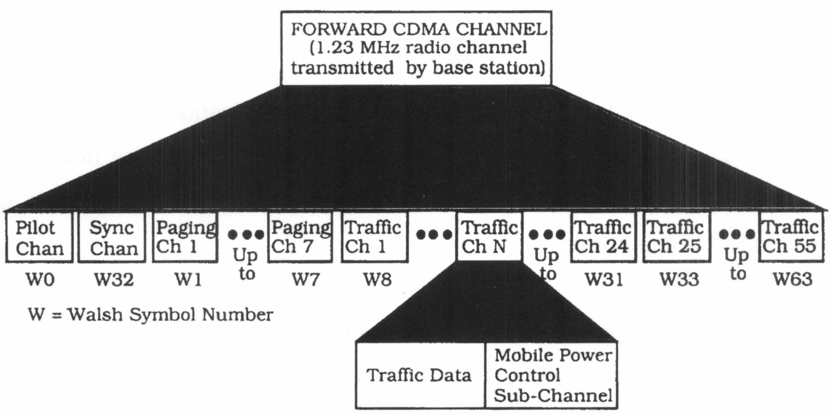


Figure 5: Sequency-multiplex system for 62 digital mobile telephony channels

(after Qualcomm Inc. 1992)



Forward CDMA Channels Transmitted by a Base Station. *In addition to the pilot and sync channels, the forward link in each sector supports 62 channels that may be used for paging and traffic. Zero to seven channels are assigned to paging, the remainder are traffic channels.*

Figure 6: Organization of forward channels in the CDMA system of Qualcomm (1992)

5.2 Correlation-immune coupling of pseudo-noise sequences

Secure transmission of information by digital data can be achieved by mixing the data sequence with a random sequence (key sequence). Claude Shannon has shown that "stream ciphering" is cryptographically absolutely secure if the key sequence is purely random and is used only once ("one time key system"). Since the key sequence has also to be applied for demixing at the receiver station, an effective realization of stream ciphering has to be based on pseudo random noise generation by a finite state machine PNG (Pseudo-Noise Generator). To achieve an high degree of security the key sequences generated by a PNG has to approximate pure random noise in a "cryptographic best manner".

In practice, to get a cryptographic strong PNG, n sequences of $x_1, x_2, \ldots, x_n$ of weak quality are coupled by a certain operation C (the combiner) to result in a strong key sequence $y = C(x_1, x_2, \ldots, x_n)$. In the case of binary sequences a (static) combiner C can be represented by an Boolean function C:$\mathbf{B}^n \rightarrow \mathbf{B}$.

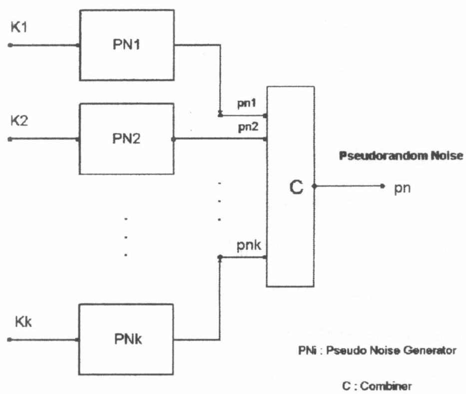


Figure 7: Architecture of a Pseudo Random Noise Generator PNG with Combiner C

The cryptological quality of a pseudo random noise sequence has to be determined by different statistical tests. Such tests relate to certain attacks to break the system. For PNG with an architecture according to Fig. 7 the so called correlation attack tries to identify the individual "weak" sequences $x_1, x_2, \ldots, x_n$ from observing the key sequence $y = C(x_1, x_2, \ldots, x_n)$ by a "divide and conquer" method using statistical tests and computer simulation. A PNG is robust with respect to the correlation attack if the combiner C possesses a certain degree of "correlation immunity". For boolean combiners C the following theorem was proved by Xiao-Massey (1985): A boolean combiner C:$\mathbf{B}^n \rightarrow \mathbf{B}$ is correlation immune of degree m if and only if the Walsh transform WT(C) of C has the following spectral property:

WT(C)(x)=0 for all x with $0 \leq \|x\| \leq m$ ($\|x\|$ denotes the Hamming weight of x; we identify $\mathbf{B}^n$ with N(n)).

Siegenthaler (1986) showed by algebraic methods how a boolean combiner C of sufficient high degree of correlation immunity can be constructed. Pichler (1985) established such a construction by means of Walsh-Fourier analysis. An extension of these results for (dynamic) combiners C realized by state machines with finite memory can be found in Pichler (1988). As a further recent publication to the topic of Boolean combiner design uses the Walsh transform we refer to Dobertin (1995).

5.3 Sequency-Hopping

The fundamental idea of the method of frequency hopping to achieve a secure communication by a wireless system is, that the carrier frequency of the transmitter and the receiver changes in time using a pseudo random pattern. It "hops" within a certain (broad) frequency band. The Austro-american actress and movie star Hedy Lamarr (born Hedy Kiesler) known as the "most beautiful woman of the world", is considered together with George Antheil ("bad boy of music") as the inventor of frequency hopping (US patent 2,292.387 of 1942). Following Harmuth (1971) we can replace the parameter "frequency" by "sequency" if we use a communication system based on Walsh functions as carrier. In this case "sequency hopping" is a method for secure information transmission. As a further generalization "code-hopping" can be introduced, when we use arbitrary orthogonal code sequences as carriers. A research project for the development of a code-hopping system in combination with a direct sequence CDMA system is currently under investigation (Pichler-Scharinger-Schütt 2000).

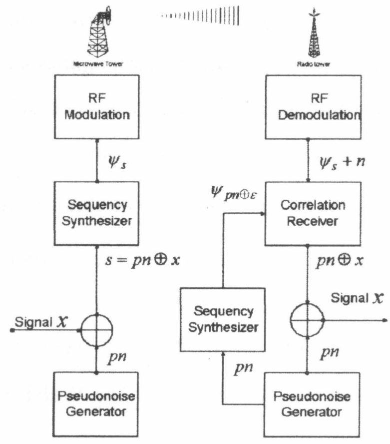


Figure 8: Block diagram of a digital system for secure information transmission by sequency hopping

6. Concluding Remarks

The paper tries to present a survey of the history of the development of the application of Walsh functions in signal processing and communication engineering. The goal was to show that Walsh functions and the method of Walsh-Fourier transformation can be used for digital systems in a similar way as the sinusoidal functions are used for analog systems. From a mathematical point of view this is not surprising: Both function systems are as we pointed out earlier, special cases of character functions of a topological group and the existing general method of Fourier transformation covers essential parts of the specific methods of the Walsh transformation and respectively the Fourier transformation. An example is given by the sampling theorem in signal processing. Kluvanec (1965) showed the validity of this theorem in abstract harmonic analysis, special cases of it for dyadic harmonic analysis and for real harmonic analysis are known from Pichler (1970) and Shannon (1948).

The application of Walsh functions and related concepts in modeling systems for signal processing and communication were – according to the state of art – originally developed for the case of analog information technology. Their realization used basic elements like resistors, coils, capacitors, operation amplifiers and sampling and hold circuits. The introduction of digital information technology, specifically the development of integrated microelectronic circuits, microprocessors and computers, required new approaches for modeling. For the processing of digital signals by digital systems new concepts and methods for dealing with hardware and software had to be developed. Examples for the case of hardware are the development of the theory of switching functions (boolean functions) and the theory of sequential switching circuits (finite automata). The different concepts and methods for analog signal processing and analog information transmission which exist for Walsh functions can be adapted to digital system by restriction to the set *W(n)* of discrete Walsh functions of order *n*. This is analogous to the approach of adapting concepts and methods of analog signal processing (for example by digital filters) by the use of the system *S(n)* of discrete sinusoidal functions of finite order.

The development of communication systems is in a steady competition between the requirements stated by the customers and the architectural and technical propositions made by the designers and engineers to realize such systems by the current existing basic technology. In this development mathematical and systemstheoretical models are of great importance. Walsh functions and related mathematical and systemstheoretical results can contribute to the construction of models for modern digital communication systems.

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