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ON A THEOREM OF CASS AND TRAUTNER

Fernando Pérez González
Departamento de Análisis Matemático
Facultad de Matemáticas
Universidad de La Laguna
38271 La Laguna, Tenerife
Spain

Abstract. In this note a short and standard proof of a theorem of F.P. Cass and R. Trautner, which extend to Banach valued functions a familiar result by Wiener, is given.

Key Words: Banach algebra, vector valued function.

1. INTRODUCTION

F.P. CASS and R. TRAUTNER have given in [1] an extension of the well known 1/f-Wiener's lemma for functions defined on the closed unit disc or on an annulus of **C** containing the unit circle and valued in a Banach algebra. Most of the proof of the main result in [1] is based upon four lemmas, the last one being strictly topologic.

In this note we show a simple theorem, proven by standard methods, which leads inmediately to the lemmas 1 through 3 in [1]. This is done in section 2.

We will use the following notations: $D_r = \{z \in \mathbb{C} : |z| \leq r\}$ and $\partial D_r = \{z\in\mathbb{C} : |z| \leq r\}$; A denotes a Banach algebra with identity e and $\|e\| = 1$; G(A) is the group of invertible elements in A, and $sp_A(w)$ represents the spectrum of $w \in A$. Finally $\mathcal{A}_A(D_r)$ is the space of all functions $f : D_r \longrightarrow A$ continuous in D_r and holomorphic in the interior of D_r. An account of the elementary theory of vector valued holomorphic functions is given in ([3], p. 92 ff.) We consider the arithmetic operations defined on $\mathcal{A}_A(D_r)$ as usual, and for $f \in \mathcal{A}_A(D_r)$ we set

$$\|f\|_{\infty,r} = \sup_{z\in D_r} \|f(z)\|.$$

2. THE THEOREM

With the notations as above, the following result holds:

Theorem.- (i) $(\mathcal{A}_A(D_r), \|.\|_{\infty,r})$ is a Banach algebra with identity.

(ii) $G(\mathcal{A}_A(D_r)) = \{ f \in \mathcal{A}_A(D_r) : f(z) \in G(A), \text{ for every } z \in D_r \}$.

(iii) For every $f \in \mathcal{A}_A(D_r)$

$$sp_{\mathcal{A}_A(D_r)}(f) = \bigcup_{z\in D_r} sp_A(f(z)). \qquad (1)$$

Proof: As in the case $A = \mathbb{C}$, it is clear that the space $C_A(D_r)$ of all A-valued continuous functions in D_r is a Banach algebra with the norm $\|.\|_{\infty,r}$

and $\mathcal{A}_A(D_r)$ is a subalgebra of $C_A(D_r)$. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{A}_A(D_r)$ converging to $f \in C_A(D_r)$. This convergence is uniform on any compact subset of D_r and, in particular, on ∂D_r. Now, applying the theorem 3.11.6 in [2] we conclude that f is holomorphic in the interior of D_r. This shows that $\mathcal{A}_A(D_r)$ is a closed subalgebra of $C_A(D_r)$, and (i) follows.

Suppose now that f is a function of $\mathcal{A}_A(D_r)$ such that $f(z) \in G(A)$ for every $z \in D_r$, and let $f^{-1} : D_r \longrightarrow G(A)$ be defined as $f^{-1}(z) = (f(z))^{-1}$ $(z \in D_r)$. Since the inversion mapping $x \longrightarrow x^{-1}$ is continuous in G(A) and holomorphic in each component of G(A) ([2], th. 4.3.4), f^{-1} is continuous on D_r and holomorphic in the interior of D_r. On the other hand, if $f \in G(\mathcal{A}_A(D_r))$ then there exists a $g \in \mathcal{A}_A(D_r)$ such that $fg \equiv e$, i.e., $f(z)g(z) = e$ for $z \in D_r$. This means that $f(z) \in G(A)$ for every $z \in D_r$, and (ii) is proven. Finally it is an easy exercise to compute the identity in (iii), and the theorem is proven. ∎

As it is well known the spectrum of any element of a Banach algebra is a non empty compact subset of the plane $\mathbb{C}$. Hence $sp_{\mathcal{A}_A(D_r)}(f)$ is so, and this exactly the result of the lemma 1.

Lemma 2 in [1] asserts that if $f \in \mathcal{A}_A(D_1)$ and $\varepsilon > 0$ is given, we can find a $\delta > 0$ such that whenever $g \in \mathcal{A}_A(D_1)$ and $\|f - g\|_{\infty,1} < \delta$ then $sp_{\mathcal{A}_A(D_1)}(g) \subset \Omega = \{z \in \mathbb{C} : \text{dist}\,[z, sp_{\mathcal{A}_A(D_1)}(f)] < \varepsilon\}$. But since Ω is an open set, it follows directly from (i) in the theorem and [3, theorem 10.20]. Note that, obviously, $sp_{\mathcal{A}_A(D_1)}(g) = sp_{\mathcal{A}_A(D_1)}(f + (g-f))$.

Finally let f be an element in $\mathcal{A}_A(D_1)$ and let $K \subset \mathbb{C}$ be a compact set such that $\text{dist}\,[K, sp_{\mathcal{A}_A(D_1)}(f)] > 0$. In these conditions, we have that

$$\sup \{ \|(t - f(z))^{-1}\| : t \in K, z \in D_1\} < \infty, \tag{2}$$

as established in [1, lemma 3]. But, since $K \subset \mathbb{C} \setminus sp_{\mathcal{A}_A(D_1)}(f)$ and the mapping $t \in K \longrightarrow \|(t - f)^{-1}\|_{\infty,1}$ is continuous and bounded, the estimate in (2) also follows.

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