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OPTIMIZATION OF A FUNCTIONAL ON AN OPEN MANIFOLD WITH DIFFERENTIABLE VARIETY AS ITS BOUNDARY

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Abstract. The paper discusses control theoretic optimization of a functional with restrictions on the domain of definition of the functional, in the sense that, the domain is an open manifold with boundary, where the boundary is a differentiable variety. The paper also shows a realistic application of such an optimal control problem.

Keywords and Phrases: Differentiable manifold with boundary, Differentiable variety, Constrained optimal control problems with restricted domain.

Subject Classification Code [2005]: Primary 54C40; Secondary 46E25

1. Introduction

Optimal control problems are of two types - (i) when the restrictions are only in the parameter domain and (ii) when the restrictions are in the stable domain and also in the parameter domain. The necessary condition of optimality type (i) is known as Pontryagin's maximum principle [14], similar conditions of optimality in type (ii) is given in [1, 2, 13, 17]. So far as type (i) optimal control problem are concerned, their applications in real world problems and the corresponding analysis are found in [6, 7, 8, 9]. But it is very difficult to have realistic applications of type (ii) optimal control problems owing to arbitrary restrictions on the domain of the functional expressed by inequalities of differentiable functions. In realistic applications, it is found that the domain of restriction of the functional is a manifold with boundary [10, 15], where the manifold is an open set of R^n $(n > 1)$ and the boundary is a differentiability variety in R^n [10]. Naturally it remains open to formulate type (ii) optimal control problem with the aforesaid type of restrictions on the state domain and apply it in realistic optimal control problem. The present chapter deals with such control problems and their applications.

The whole matter of the paper is divided into four main sections, where section 1 gives the introduction. Section 2 gives the idea of differentiable manifold and differentiable variety and contains some results in this connection. In section 3, some constrained optimal control problem in parameter domain are given and discussing some real world problems on the domain of definition of the functional, in the sense that, the domain is a manifold with boundary, where the boundary is a differentiable variety. Section 4 discusses the stability analysis and the control-theoretic optimization of a functional of replicator dynamics on the same domain of definitions.

2. Some known Definitions and Results [10]

Definition 2.1. [10] A **differentiable variety** in R^{n+1} is defined as $\{f^{-1}(0)\}$, where $f : R^{n+1} \to R$ is a differentiable function such that at each $z \in M$, the matrix $[f_{,j}(z)]$ has rank one, $j = 1, 2, ..., n$
It can be shown that it is a differentiable manifold of dimension n.

Example 2.2. [10] *A 2-sphere* $S^2 = \{(z_1, z_2, z_3) \in R^3 : z_1^2 + z_2^2 + z_3^2 - 1 = 0\}$ *is a differentiable variety in* R^3 *and it is a manifold of dimension* 2.

3. Constrained optimal control problem with restrictions in the state space

Statement of the problem and its solution

$$Maximize\ J_x(u) = \phi(t, x_{t_1}) + \int_0^{t_1} F(x, t, u)dt\ \forall (x, u) \in X \tag{3.1}$$

where $X = \{(x, u)$: corresponding to each u, $x(t)$ is an integral curve of $x' = f(t, x, u)$ and $G(x, u) \leq 0\}$; $G(x, u) \leq 0$ denotes a manifold with boundary in R^{n+m} whose interior is open sub-manifold of R^{n+m} and whose boundary is a differentiable variety given by $G(x, u) = 0$; $u \in R^m$, $t \in [0, t_1]$, $x = x(t) \in R^n$ is C^1, $x = x_0$ when $t = 0$; further $f : R \times R^n \times R^m \to R^n$, $\phi : R \times R^n \to R$, $F : R \times R^n \times R^m \to R$ are all C^1-maps.

Proceeding as in [2, 17], we get the following

Theorem 3.1. *A necessary condition that* (x^*, u^*) *minimizes the control problem* (3.1) *is that there are costate vectors* $\lambda(t)$ *and* $\mu(t)$ *such that the following holds:*
(i) $\lambda(t_1) = (\frac{\partial \phi}{\partial x})_{t=t_1}$, *(ii)* $F_x + \lambda f_x + \mu G_x + \dot{\lambda} = 0$ *and*
(iii) $(F_u + \lambda f_u + \mu G_u)_{u=u^*} = 0$.

Remark 3.2. *It is noted that the necessary conditions of optimality as given in Theorem* 3.1 *reduce to the solution of a system of ordinary differential equations in co-state variables. Naturally it becomes almost impossible to find the solution analytically. This is why, steady state optimal solution is needed. Hence prior to finding out the optimal solution, at least local asymptotic stability of the system is to be assured.*

Example 3.3. *Suppose* x, y *are two non-interacting fishes and* z *is their predator moving in a part of an ocean, that is given by* $x^2 + y^2 + z^2 - a \leq 0$, $x > 0, y > 0, z > d > 0$, *for some real* a *and* d. *Let the natural dynamics of motion be given by [with standard meanings]*

$$\begin{aligned}\dot{x} &= rx(1 - \frac{x}{K}) - \alpha xz \\ \dot{y} &= sy(1 - \frac{y}{L}) - \beta yz \\ \dot{z} &= z(-f + \gamma x + \delta y)\end{aligned} \tag{3.2}$$

Let the dynamics of exploited motion under control parameter u *per unit biomass be given by the following differential equations [with standard meanings]*

$$\dot{x} = rx(1 - \frac{x}{K}) - \alpha xz - q_1 ux$$

(3.3)
$$\dot{y} = sy(1 - \frac{y}{L}) - \beta yz - q_2uy$$
$$\dot{z} = z(-f + \gamma x + \delta y) - q_3uz$$

$u = u(t)$ is the effort, q_i $(i = 1, 2, 3)$ are the catchability coefficients of x, y, z species. In this case, the state space of the natural dynamical system (3.2) *is a manifold with boundary; the interior is the three dimensional open sub-manifold of R^3: $x^2 + y^2 + z^2 - a < 0$ and the boundary is a differentiable variety of R^3, which is the surface of a two dimensional sphere given by $x^2 + y^2 + z^2 = a$. The state space of the exploited system* (3.3) *is $x^2 + y^2 + z^2 + bu - a \leq 0$ which is also a manifold with boundary; the interior is the four dimensional open sub-manifold of R^4: $x^2 + y^2 + z^2 + bu - a < 0$ and the boundary is a differentiable variety of R^4, which is a paraboloid (of dimension 3) given by $x^2 + y^2 + z^2 + bu = a$. Moreover, the state space of the exploited system is bounded since $(x, y, z, u) \in R^4_+$.*

Let c be the cost per unit effort u. Let p_1, p_2, p_3 be the prices of the species x, y, z respectively. Then the profit function is taken as

(3.4)
$$\pi(x, y, z, u) = (p_1q_1x + p_2q_2y + p_3q_3z - c)u(t)$$

where (x, y, z, u) belonging to the state space.

The control problem is as follows:

(3.5)
$$Maximize \int_0^T (p_1q_1x + p_2q_2y + p_3q_3z - c)udt, u \in U \text{ [the control set]}.$$

Remark 3.4. *As mentioned in Remark* 3.2*, that first of all local stability analysis of the system* (3.3) *is to be done and then corresponding optimal analysis is to be performed. For this problem local stability analysis is already known in connection with fishery management problems, but the optimal analysis is definitely much harder and completely new.*

Proposition 3.5. *The dynamical system* (3.2) *possesses equilibrium* $(x_1, y_1, z_1) > (0, 0, 0)$*, if*

(3.6)
$$r > \frac{\alpha(\gamma K + \delta L - f)}{P}, s > \frac{\beta(\gamma K + \delta L - f)}{P}, \gamma K + \delta L > f$$

where $P = \frac{\gamma\alpha K}{r} + \frac{\beta\delta L}{s}$.
Similarly (3.3) *possesses equilibrium* $(x_2, y_2, z_2) > (0, 0, 0)$ *if* (3.6), (3.7) *and* (3.8) *hold,*

(3.7)
$$q_1 < \frac{\alpha Q}{P}, q_2 < \frac{\beta Q}{P}$$

(3.8)
$$u < \frac{\gamma K + \delta L - f}{Q}$$

where $Q = \frac{\gamma q_1 K}{r} + \frac{\delta q_2 L}{s} + q_3$

Proof. It follows that
$x_1 = K[1 - \frac{\alpha}{rP}(\gamma K + \delta L - f)]$, $y_1 = L[1 - \frac{\beta}{sP}(\gamma K + \delta L - f)]$, $z_1 = \frac{\gamma K + \delta L - f}{P}$ where $P = \frac{\gamma\alpha K}{r} + \frac{\beta\delta L}{s}$.
So, $(x_1, y_1, z_1) > (0, 0, 0)$, if (3.6) holds. Similarly, it follows that

$$x_2 = x_1 - \frac{Ku}{r}(q_1 - \frac{\alpha Q}{P})$$

$$y_2 = y_1 - \frac{Lu}{s}(q_2 - \frac{\beta Q}{P})$$

$$z_2 = z_1 - \frac{Qu}{P}$$

where $Q = \frac{\gamma q_1 K}{r} + \frac{\delta q_2 L}{s} + q_3$.
Clearly $(x_2, y_2, z_2) > (0, 0, 0)$, if (3.6), (3.7) and (3.8) hold. □

Proposition 3.6. *The vanishing equilibrium* $(0, 0, 0)$ *is always an unstable saddle point and the interior equilibrium* (x_1, y_1, z_1) *is asymptotically stable for the system* (3.2).

Proof. The variational matrix for the equation in model (3.2) at origin is

$$J_0 = \begin{pmatrix} r & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & -f \end{pmatrix}$$

which clearly shows that $(0, 0, 0)$ is an unstable saddle point.
To show the interior equilibrium (x_1, y_1, z_1) is asymptotically stable, we first rewrite the model (3.2) as

$$\begin{aligned} \dot{x} &= x[-\frac{r}{K}(x - x_1) - \alpha(z - z_1)] \\ \dot{y} &= y[-\frac{s}{L}(y - y_1) - \beta(z - z_1)] \qquad (3.9) \\ \dot{z} &= z[\gamma(x - x_1) + \delta(y - y_1)] \end{aligned}$$

Consider the Liapunov function as

$$V(x, y, z) = x - x_1 - x_1 \log(\tfrac{x}{x_1}) + c_1[y - y_1 - y_1 \log(\tfrac{y}{y_1})] + c_2[z - z_1 - z_1 \log(\tfrac{z}{z_1})]$$

where $c_1, c_2 > 0$ are constants to be determined suitably. It is obvious that $V(x, y, z)$ is positive definite.

$$\dot{V}(x, y, z) = (\frac{x - x_1}{x})\dot{x} + c_1(\frac{y - y_1}{y})\dot{y} + c_2(\frac{z - z_1}{z})\dot{z}$$

$= (x - x_1)[-\frac{r}{K}(x - x_1) - \alpha(z - z_1)] + c_1(y - y_1)[-\frac{s}{L}(y - y_1) - \beta(z - z_1)] + c_2(z - z_1)[\gamma(x - x_1) + \delta(y - y_1)]$
$= -\frac{r}{K}(x - x_1)^2 - \frac{s}{L}(y - y_1)^2 + (c_2\gamma - \alpha)(x - x_1)(z - z_1) + (c_2\delta - c_1\beta)(y - y_1)(z - z_1)$

Let us choose $c_2 = \frac{\alpha}{\gamma}$, then $c_1 = \frac{\alpha\delta}{\beta\gamma}$ which implies that $\dot{V}(x, y, z)$ is negative and consequently the interior equilibrium (x_1, y_1, z_1) is asymptotically stable. □

Bionomic Equilibrium and its feasibility[3]

Let L denote the locus of dynamic equilibrium of the three species system (3.3) and let $\pi = 0$ denote the zero profit function. A feasible equilibrium is the point of intersection of $L = 0$ and $\pi = 0$, provided all the coordinates of this point are positive and also the value of the control parameter $u(t)$ is positive at this point. It is usually denoted by $(x_\infty, y_\infty, z_\infty)$.

The optimal steady state analysis is taken around the bionomic equilibrium of the model, so its existence is to be assured. In this connection we prove the following theorem.

Theorem 3.7. *Let the dynamic model be given by* (3.3) *under the restrictions* (3.6), (3.7) *and* (3.8). *Let the objective function be given by* (3.4). *Then there exists a feasible bionomic equilibrium if* (3.10) *and* (3.11) *holds separately where*

$$p_1q_1P < p_2q_2Q + p_3q_3R \tag{3.10}$$

$$c < p_1q_1x_1 + p_2q_2y_1 + p_3q_3z_1 \tag{3.11}$$

Proof. The locus of dynamic equilibrium (x_2, y_2, z_2) is given by

$$L : \frac{x - x_1}{P} = \frac{y - y_1}{-Q} = \frac{z - z_1}{-R} = u \tag{3.12}$$

The zero profit function is given by

$$\pi = p_1q_1x + p_2q_2y + p_3q_3z - c = 0 \tag{3.13}$$

If (3.12) intersects (3.13) at (x^*, y^*, z^*) where $u = u_1$, then it follows that
$$x = x^* + Pu_1,\ y^* = y_1 - Qu_1,\ z^* = z_1 - Ru_1$$
Again from $\pi = 0$, it follows that
$p_1q_1x_1 + p_2q_2y_1 + p_3q_3z_1 - c = (p_2q_2Q + p_3q_3R - p_1q_1P)u_1.$
Hence

$$u_1 = \frac{p_1q_1x_1 + p_2q_2y_1 + p_3q_3z_1 - c}{p_2q_2Q + p_3q_3R - p_1q_1P} \tag{3.14}$$

Obviously, we get $u_1 > 0$ if (3.10) and (3.11) hold. □

Statement of the optimal control problem and its solution

Let the state space of the exploited system (3.3) be given by
$X = \{(x, y, z, u)$: corresponding to each u, $(x(t), y(t), z(t))$ is an integral curve of the exploited system (3.3) and $G(x, y, z, u) \le 0\}$ where $G(x, y, z, u) = x^2+y^2+z^2+bu-a$ then $G(x, y, z, u) \le 0$ denotes a manifold with boundary in R^4 whose interior is an open sub-manifold of R^4 and whose boundary is a differentiable variety given by $G(x, y, z, u) = 0$; $u \in R$, $t \in [0, T]$, $(x(t), y(t), z(t)) \in R^3$ is C^1, $(x, y, z) = (x_0, y_0, z_0)$ when $t = 0$; further let $f : R^3 \times R \to R^3$, $\pi : R^3 \times R \to R$ be all C^1-maps where $f = (f_1, f_2, f_3)$

$$f_1(x, y, z, u) = x[r(1 - \frac{x}{K}) - \alpha z - q_1u]$$

$$f_2(x, y, z, u) = y[(1 - \frac{y}{L}) - \beta z - q_2u]$$

$$f_3(x, y, z, u) = z[-f + \gamma x + \delta y - q_3u]$$

We assume that the total time taken to control the biomass is T. Then the control problem is to maximize

$$J = \int_0^T \pi(x, y, z, u)dt \ \ \forall(x, y, z, u) \in X \tag{3.15}$$

over the control parameter u, where $u \in (0, u_{max})$ and to find a suitable $u = u^*$ in $(0, u_{max})$ for which J is maximum where

$$\pi(x, y, z, u) = (p_1q_1x + p_2q_2y + p_3q_3z - c)u(t)$$

Before going to the main theorem we want to find out the particular solution of a 3- system of ordinary differential equation with constant coefficients. Such solution of differential equation in co-state variable will be necessary in our subsequent realistic example.

Particular Solution of a 3- system of ordinary differential equation

Let us consider the 3-system of ordinary differential equations

$(I)\qquad \frac{d\lambda_1}{dt} = a_1\lambda_1 + b_1\lambda_2 + c_1\lambda_3 + d_1$

$(II)\qquad \frac{d\lambda_2}{dt} = a_2\lambda_1 + b_2\lambda_2 + c_2\lambda_3 + d_2$

$(III)\qquad \frac{d\lambda_3}{dt} = a_3\lambda_1 + b_3\lambda_2 + c_3\lambda_3 + d_3$

Differentiating (I) with respect to t and using (II) and (III), we get,

$(IV)\qquad X = A\lambda_2 + B\lambda_3$

where $X = \frac{d^2\lambda_1}{dt^2} - a_1\frac{d\lambda_1}{dt} - (b_1a_2 + c_1a_3)\lambda_1 - (b_1d_2 + c_1d_3)$,
$A = b_1b_2 + b_3c_1$, $B = b_1c_2 + c_1c_3$.

Again differentiating (IV) with respect to t and using (II), (III), we get,

$(V)\qquad Y = C\lambda_2 + D\lambda_3$

where $Y = \frac{d^3\lambda_1}{dt^3} - a_1\frac{d^2\lambda_1}{dt^2} - (b_1a_2 + c_1a_3)\frac{d\lambda_1}{dt} - (a_2A + a_3B)\lambda_1 - (d_2A + d_3B)$,
$C = b_2A + b_3B$, $D = c_2A + c_3B$.

Solving (IV) and (V) we get,

$$\lambda_2(t) = \frac{DX - BY}{AD - BC}$$

$$\lambda_3(t) = \frac{AY - CX}{AD - BC}$$

provided $AD - BC \neq 0$. Putting the values of $\lambda_2(t)$ and $\lambda_3(t)$ in (I), we get,

$$\lambda_1(t) = -\frac{S}{R}$$

where $S = \frac{d_1(c_1A - b_1B)}{AD - BC} - \frac{(b_1d_2 + c_1d_3)(b_1D - c_1C)}{c_1A - b_1B} - (d_2A + d_3B)$

$R = \frac{a_1(AD - BC) - (b_1a_2 + c_1a_3)(b_1D - c_1C)}{c_1A - b_1B} - (a_2A + a_3B)$.

Thus the values of $\lambda_2(t)$ and $\lambda_3(t)$ are given as follows

$\lambda_2(t) = \frac{1}{AD-BC}[b_1D(\frac{a_2S}{R} - d_2) + c_1D(\frac{a_3S}{R} - d_3) + AB(d_2 - \frac{a_2S}{R}) + B^2(d_3 - \frac{a_3S}{R})]$

$\lambda_3(t) = \frac{1}{AD-BC}[A^2(\frac{a_2S}{R} - d_2) + AB(\frac{a_3S}{R} - d_3) + b_1C(d_2 - \frac{a_2S}{R}) + c_1C(d_3 - \frac{a_3S}{R})]$.

Theorem 3.8. *Let the dynamic model be given by* (3.3) *with restrictions* (3.7) *and* (3.8) *and the profit function be given by* (3.4) *under restrictions* (3.10) *and* (3.11). *The problem is to maximize*

$$J = \int_0^T \pi(x, y, z, u)dt$$

where T is the total time. Then there exists $u = u^*$ satisfying (3.7) *and* (3.8) *for which J is maximum.*

Proof. Hamiltonian for our model (3.3) is given by

$$H = (p_1q_1x + p_2q_2y + p_3q_3z - c)u(t) + \lambda_1f_1 + \lambda_2f_2 + \lambda_3f_3 + \mu G$$

where $\lambda_i(t)$ for $i = 1, 2, 3$ and $\mu(t)$ are co-state variables to be determined suitably.

For steady state solution, we have

$$r(1-\frac{x}{K})-\alpha z-q_1u=0$$
$$s(1-\frac{y}{L})-\beta z-q_2u=0$$
$$-f+\gamma x+\delta y-q_3u=0$$

By applying Pontryagin's maximum principle we have (for a steady state solution)

$$\frac{d\lambda_1}{dt}=-\frac{\partial H}{\partial x}=-p_1q_1u+\frac{\lambda_1 r}{K}x-\lambda_3\gamma z-2\mu x$$

(3.16) $$\frac{d\lambda_2}{dt}=-\frac{\partial H}{\partial y}=-p_2q_2u+\frac{\lambda_2 s}{L}y-\lambda_3\delta z-2\mu y$$

$$\frac{d\lambda_3}{dt}=-\frac{\partial H}{\partial z}=-p_3q_3u+\alpha x\lambda_1+\beta y\lambda_2-2\mu z$$

and

$$\frac{\partial\pi}{\partial u}+\lambda_1\frac{\partial f_1}{\partial u}+\lambda_2\frac{\partial f_2}{\partial u}+\lambda_3\frac{\partial f_3}{\partial u}+\mu\frac{\partial G}{\partial u}=0$$

i.e.

(3.17) $$p_1q_1x+p_2q_2y+p_3q_3z-c-\lambda_1q_1x-\lambda_2q_2y-\lambda_3q_3z+b\mu=0$$

Equation (3.16) can be rewritten as

$$\frac{d\lambda_1}{dt}=a_1\lambda_1+b_1\lambda_2+c_1\lambda_3+d_1$$

(3.18) $$\frac{d\lambda_2}{dt}=a_2\lambda_1+b_2\lambda_2+c_2\lambda_3+d_2$$

$$\frac{d\lambda_3}{dt}=a_3\lambda_1+b_3\lambda_2+c_3\lambda_3+d_3$$

where $a_1=(\frac{r}{K}-\frac{2q_1x}{b})x$, $b_1=-\frac{2q_2xy}{b}$, $c_1=-(\gamma+\frac{2q_3x}{b})z$, $d_1=\frac{2rx}{b}-p_1q_1u$.
$a_2=-\frac{2q_1xy}{b}$, $b_2=(\frac{s}{L}-\frac{2q_2y}{b})y$, $c_2=-(\delta+\frac{2q_3y}{b})z$, $d_2=\frac{2ry}{b}-p_2q_2u$.
$a_3=(\alpha-\frac{2q_1z}{b})x$, $b_3=(\beta-\frac{2q_2z}{b})y$, $c_3=-\frac{2q_3z^2}{b}$, $d_3=\frac{2rz}{b}-p_3q_3u$ and
$r=p_1q_1x+p_2q_2y+p_3q_3z-c$.

Using the particular solution of system of ordinary differential equations as given in the previous section, we get the particular solution of (3.18) given by

$$\lambda_1(t)=\frac{1}{R}(\chi u-V)$$

where $\frac{p_1q_1(c_1A-b_1B)}{AD-BC}-\frac{(b_1p_2q_2+c_1p_3q_3)(b_1D-c_1C)}{c_1A-b_1B}-(Ap_2q_2+Bp_3q_3)$
$V=\frac{2r}{b}[\frac{(c_1A-b_1B)x}{AD-BC}-\frac{(b_1y+c_1z)(b_1D-c_1C)}{c_1A-b_1B}-(Ay+Bz)]$

$$\lambda_2(t)=\tau u+K$$

where $\tau=\frac{1}{BC-AD}[(\frac{\chi a_2}{R}+p_2\epsilon_2)(b_1D-AB)+(\frac{\chi a_3}{R}+p_3\epsilon_3)(c_1D-B^2)]$
$K=\frac{1}{AD-BC}[(\frac{a_2V}{R}-r)(b_1D-AB)+(\frac{a_3V}{R}-r)(c_1D-B^2)]$

$$\lambda_3(t)=\eta u+J$$

where $\eta=\frac{1}{AD-BC}[(\frac{\chi a_2}{R}+p_2\epsilon_2)(b_1C-A^2)+(\frac{\chi a_3}{R}+p_3\epsilon_3)(c_1C-AB)]$
$J=\frac{1}{AD-BC}[(\frac{a_2V}{R}-r)(A^2-b_1C)+(\frac{a_3V}{R}-r)(AB-c_1C)]$.
A, B, C, D, S, R have their usual meanings as discussed while giving the particular

solutions of a 3- system of ordinary differential equations mentioned above. Finally we take $u \longrightarrow u_{max}$ as $t \rightarrow T$, in λ_1, λ_2 and λ_3, and have the limiting values given as

$$\lambda_1(t) = \frac{1}{R}(\chi u_{\max} - V) \tag{3.19}$$

$$\lambda_2(t) = \tau u_{\max} + K \tag{3.20}$$

$$\lambda_3(t) = \eta u_{\max} + J \tag{3.21}$$

Using (3.17) the co-state vector $\mu(t)$ is given by

$$\mu(t) = \frac{1}{b}[\lambda_1(t)q_1 x + \lambda_2(t)q_2 y + q_3(t)q_3 z - r] \tag{3.22}$$

Now, if H is maximum at $u = u^*$ (say), $0 < u < u_{\max}$, then $\frac{\partial H}{\partial u} = 0$ at $u = u^*$. Hence we have

$$r - \lambda_1(t)q_1 x - \lambda_2(t)q_2 y - \lambda_3(t)q_3 z + \mu(t)b = 0 \tag{3.23}$$

where $\lambda_i(t)$ $(i = 1, 2, 3)$ and $\mu(t)$ corresponds to $u = u^*$. Hence from (3.19) to (3.22), we get

$$\lambda_1(t) = \frac{1}{R}(\chi u^* - V)$$

$$\lambda_2(t) = \tau u^* + K$$

$$\lambda_3(t) = \eta u^* + J$$

$$\mu(t) = \frac{1}{b}[\lambda_1(t)q_1 x + \lambda_2(t)q_2 y + \lambda_3(t)q_3 z - r]$$

Again, as steady state optimal solution (x^*, y^*, z^*) is desired, u^* is given by

$$u^* = \frac{x - x_1}{P} = \frac{y - y_1}{-Q} = \frac{z - z_1}{-R} \tag{3.24}$$

Thus finally we have

$$\lambda_1(t) = \frac{1}{R}(\chi\frac{x - x_1}{P} - V) \tag{3.25}$$

$$\lambda_2(t) = \tau\frac{y - y_1}{-Q} + K \tag{3.26}$$

$$\lambda_3(t) = \eta\frac{z - z_1}{-R} + J \tag{3.27}$$

$$\mu(t) = \frac{1}{b}[\lambda_1(t)q_1 x + \lambda_2(t)q_2 y + \lambda_3(t)q_3 z - r] \tag{3.28}$$

where P, Q and R are given in proposition 3.5. Putting the values of x_1, y_1 and z_1, we obtain the equation of the optimal path. Solving (3.12) with the above optimal path, we obtain the optimal values x^*, y^*, z^* of x, y, z respectively and thus we find the optimal value of u^*. □

4. Stability analysis and the control-theoretic optimization of a functional of replicator dynamics

In this section we prefer to restrict our detailed analysis only to the following replicator system of dynamics, which is completely new in all respects.

Definition 4.1. [16] Let

$S_3^c = \{x = (x_1, x_2, x_3) \in R^n : \sum_{i=1}^{3} x_i = c, x_i \geq 0 \; for \; 1 \leq i \leq 3\}$. It is called the concentration simplex.

The dynamics on S_3^c is given by the differential equations

$$\dot{x}_i = x_i[q_i + \sum_{j=1}^{3} k_{ij}x_j - \frac{\phi}{c}] \tag{4.1}$$

where $c > 0$, q_i and $k_{ij} \in R$ and $\phi = \sum_{i=1}^{3} x_i(q_i + \sum_{j=1}^{3} k_{ij}x_j)$. x_i represents the concentration of the chemical or biological species i and $q_i \in R$ corresponds to the selfreproduction or decay of the species i and $k_{ij}x_j$ represents the effect of the species j on the reproduction of species i which is of mass action type, catalytic if $k_{ij} > 0$ and inhibiting if $k_{ij} < 0$. (4.1) is called a replicator system on S_3^c; if it keeps the boundaries and faces of S_3^c invariant.

Example 4.2. *Let us consider an inhomogeneous hypercycle defined on* 3- *concentration simplex* $S_3^c = \{(x, y, z) \in R^3 : x + y + z = c, x, y, z \geq 0\}$ *given by the system of differential equation*

$$\begin{aligned} \dot{x} &= x(q_1 + k_1 y - \frac{\phi}{c}) \\ \dot{y} &= y(q_2 + k_2 z - \frac{\phi}{c}) \\ \dot{z} &= z(q_2 + k_3 x - \frac{\phi}{c}) \end{aligned} \tag{4.2}$$

where $\phi = k_1xy + k_2yz + k_3zx$ *is called the dillution flux.* x, y, z *represents the concentrations of the chemical or biological species.* q_1, q_2, q_3 *denotes the self reproduction or decay of the species* x, y *and* z *respectively.* k_1, k_2, k_3 *represents the effect of the species* x *on* y, y *on* z *and* z *on* x *respectively.*

Let the dynamics of the exploited system of (4.2) *under control parameter* u *be given by*

$$\begin{aligned} \dot{x} &= x(q_1 + k_1 y - \frac{\phi}{c}) - \epsilon_1 ux \\ \dot{y} &= y(q_2 + k_2 z - \frac{\phi}{c}) - \epsilon_2 uy \\ \dot{z} &= z(q_2 + k_3 x - \frac{\phi}{c}) - -\epsilon_3 uz \end{aligned} \tag{4.3}$$

where u *is the effort of control per unit waste molecule.*

$\epsilon_1, \epsilon_2, \epsilon_3$: *coefficients of degradation product (waste) from evolution reaction vessels for the molecules* x, y *and* z *respectively. In this case, the state space of the dynamical system is a* 3*-simplex, which is actually a manifold with boundary, the manifold being the open submanifold in* R^3 *given by* $x + y + z < c$ *and the boundary* *being the differentiable variety given by $x+y+z=c$. The exploited system under the control parameter u is the 4 simplex $x+y+z+u-c \leq 0$, $u>0$, which is also a manifold with boundary; the interior is a 4- simplex which is an open submanifold of R^4 given by $x+y+z+u-c<0$ and boundary is a 3-simplex $x+y+z+u=c$.*

Let p_1, p_2, p_3 be the projected profit for degradation product (waste) of molecules x, y, z respectively from evolution reactor coming out of the vessel to avoid risks of breaking the walls of the reactor.

The total number of waste molecules x, y, z at time t taken by the control process are given by $\epsilon_1 ux, \epsilon_2 uy, \epsilon_3 uz$ respectively.

Therefore, the net projected profit for degradation (waste) of molecules x, y and z are respectively $p_1\epsilon_1 ux, p_2\epsilon_2 uy$ and $p_3\epsilon_3 uz$.

Let $\hat{c}$ be the cost per unit effort u at time t.

So total effort in the process is $\hat{c}u(t)$. Then the profit function is taken as

$$\pi(x,y,z,u) = (p_1\epsilon_1 x + p_2\epsilon_2 y + p_3\epsilon_3 z - \hat{c})u(t) \tag{4.4}$$

where (x,y,z,u) belonging to the state space.

The control problem is as follows:

$$Maximize\ \ J = \int_0^T \pi(x,y,z,u)dt\ \ \forall (x,y,z,u) \in X\ \ [\text{the control set}] \tag{4.5}$$

Proposition 4.3. *The replicator system (4.2) has equilibrium $(x_1, y_1, z_1) > (0,0,0)$ if c is large enough. Similarly (4.3) possesses equilibrium if c is large enough as well as (4.6) hold.*

$$\begin{aligned} &\frac{\epsilon_1-\epsilon_3}{k_1} - \frac{\epsilon_3-\epsilon_2}{k_2} + 1 < 0 \\ &\frac{\epsilon_2-\epsilon_1}{k_2} - \frac{\epsilon_1-\epsilon_3}{k_3} + 1 < 0 \\ &\frac{\epsilon_3-\epsilon_2}{k_3} - \frac{\epsilon_2-\epsilon_1}{k_1} + 1 < 0 \end{aligned} \tag{4.6}$$

Proof. Equilibrium point of (4.2) can be obtained by solving the system of linear equations given by

$$q_1 + k_1 y - \frac{\phi}{c} = 0$$

$$q_2 + k_2 z - \frac{\phi}{c} = 0$$

$$q_3 + k_3 x - \frac{\phi}{c} = 0$$

where $\phi = k_1xy + k_2yz + k_3zx$. It follows that
$x_1 = \frac{\frac{1}{k_3}}{N}[c - \frac{q_3-q_2}{k_2} + \frac{q_1-q_3}{k_1}]$, $y_1 = \frac{\frac{1}{k_1}}{N}[c - \frac{q_1-q_3}{k_3} + \frac{q_2-q_1}{k_2}]$ and $z_1 = \frac{\frac{1}{k_2}}{N}[c - \frac{q_2-q_1}{k_1} + \frac{q_3-q_2}{k_3}]$ where $N = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}$. Thus for inhomogeneous hypercycle (where q_i's are unequal) the inner equilibrium $(x_1, y_1, z_1) > (0,0,0)$ if c is large enough.

For the exploited system (4.3), it follows that

$$x_2 = x_1 - \frac{Fu}{N},\ y_2 = y_1 - \frac{Gu}{N},\ z_2 = z_1 - \frac{Hu}{N}$$

where

$F = \frac{1}{k_3}[1 - \frac{\epsilon_3-\epsilon_2}{k_2} + \frac{\epsilon_1-\epsilon_3}{k_1}]$
$G = \frac{1}{k_1}[1 - \frac{\epsilon_1-\epsilon_3}{k_3} + \frac{\epsilon_2-\epsilon_1}{k_2}]$
$H = \frac{1}{k_2}[1 - \frac{\epsilon_2-\epsilon_1}{k_1} + \frac{\epsilon_3-\epsilon_2}{k_3}]$

Clearly $(x_2, y_2, z_2) > (0, 0, 0)$ if c is large enough as well as (4.6) hold. □

Proposition 4.4. *The system* (4.2) *is globally asymptotically stable if* $(q_1+k_1y)(x-x_1)+(q_2+k_2z)(y-y_1)+(q_3+k_3x)(z-z_1) < 0\ \forall(x,y,z) > (0,0,0)$ *and* $x \neq x_1, y \neq y_1, z \neq z_1$.

Proof. To test the global stability analysis let us consider the following Lyapunov function
$V(x,y,z) = x - x_1 - x_1 log(\frac{x}{x_1}) + c_1[y - y_1 - y_1 \log(\frac{y}{y_1})] + c_2[z - z_1 - z_1 \log(\frac{z}{z_1})]$,
where c_1, c_2 are positive constants to be determined suitably.
We have $\dot{V}(x,y,z) = (\frac{x-x_1}{x})\dot{x} + c_1(\frac{y-y_1}{y})\dot{y} + c_2(\frac{z-z_1}{z})\dot{z}$. Using (4.2), we get
$\dot{V} = (q_1 + k_1y - \frac{\phi}{c})(x - x_1) + c_1(q_2 + k_2z - \frac{\phi}{c})(y - y_1) + c_2(q_3 + k_3x - \frac{\phi}{c})(z - z_1) = (q_1+k_1y)(x-x_1)+c_1(q_2+k_2z)(y-y_1)+c_2(q_3+k_3x)(z-z_1) - \frac{\phi}{c}[(x-x_1)+c_1(y-y_1) + c_2(z - z_1)]$.

Choosing $c_1 = c_2 = 1$, we get
$\dot{V} = (q_1 + k_1y)(x - x_1) + (q_2 + k_2z)(y - y_1) + (q_3 + k_3x)(z - z_1)$, since $\frac{\phi}{c}[(x - x_1) + c_1(y - y_1) + c_2(z - z_1)] = 0$ for $x + y + z = z_1 + y_1 + z_1 = c$.

Thus by LaSalle's theorem it follows that (x_1, y_1, z_1) for the system (4.2) is globally asymptotically stable if $(q_1 + k_1y)(x - x_1) + (q_2 + k_2z)(y - y_1) + (q_3 + k_3x)(z - z_1) < 0\ \forall(x,y,z) > (0,0,0)$ and $x \neq x_1, y \neq y_1, z \neq z_1$. □

Theorem 4.5. *Let the dynamic model be given by* (4.3) *under the restrictions* (4.6) *and c be large enough. Let the objective function be given by* (4.4), *then there exists a feasible bionomic equilibrium if* (4.7) *and* (4.8) *holds where*

(4.7) $$p_1\epsilon_1F < p_2\epsilon_2G + p_3\epsilon_3H$$

(4.8) $$\hat{c} < p_1\epsilon_1x_1 + p_2\epsilon_2y_1 + p_3\epsilon_3z_1$$

Proof. The locus of dynamic equilibrium (x_2, y_2, z_2) is given by

(4.9) $$L : \frac{x - x_1}{F} = \frac{y - y_1}{-G} = \frac{z - z_1}{-H} = u$$

The zero profit function is given by

(4.10) $$\pi = p_1\epsilon_1x + p_2\epsilon_2y + p_3\epsilon_3z - \hat{c} = 0$$

If (4.9) intersects (4.10) at (x^*, y^*, z^*) where $u = u_1$, then it follows that
$$x = x^* + Fu_1,\ y^* = y_1 - Gu_1,\ z^* = z_1 - Hu_1$$
Again from $\pi = 0$, it follows that
$p_1\epsilon_1x_1 + p_2\epsilon_2y_1 + p_3\epsilon_3z_1 - \hat{c} = (p_2\epsilon_2Q + p_3\epsilon_3R - p_1\epsilon_1P)u_1$.
Hence

(4.11) $$u_1 = \frac{p_1\epsilon_1x_1 + p_2\epsilon_2y_1 + p_3\epsilon_3z_1 - \hat{c}}{p_2\epsilon_2G + p_3\epsilon_3H - p_1\epsilon_1F}$$

Obviously, we get $u_1 > 0$ if (4.7) and (4.8) hold. □

Statement of the optimal control problem and its solution

Let the state space of the exploited system (4.3) be given by $X = \{(x, y, z, u)$: corresponding to each u, $(x(t), y(t), z(t))$ is an integral curve of the exploited replicator system (4.3) and $G(x, y, z, u) \leq 0\}$ where $G(x, y, z, u) = x + y + z + u - c$ then $G(x, y, z, u) \leq 0$ denotes a manifold with boundary in R^4 whose interior is open sub-manifold of R^4 and whose boundary is a differentiable variety given by $G(x, y, z, u) = 0$; $u \in R$, $t \in [0, T]$, $(x(t), y(t), z(t)) \in R^3$ is C^1, $(x, y, z) = (x_0, y_0, z_0)$ when $t = 0$; further let $f : R^3 \times R \to R^3$, $\pi : R^3 \times R \to R$ be all C^1-maps where $f = (f_1, f_2, f_3)$

$$f_1(x, y, z, u) = x[q_1 + k_1 y - \frac{k_1 xy + k_2 yz + k_3 zx}{c} - \epsilon_1 u]$$

$$f_2(x, y, z, u) = y[q_2 + k_2 z - \frac{k_1 xy + k_2 yz + k_3 zx}{c} - \epsilon_2 u]$$

$$f_3(x, y, z, u) = z[q_3 + k_1 x - \frac{k_1 xy + k_2 yz + k_3 zx}{c} - \epsilon_3 u]$$

We assume that the total time taken to control waste molecules is T. Then the control problem is to maximize

$$J = \int_0^T \pi(x, y, z, u) dt \ \ \forall (x, y, z, u) \in X \tag{4.12}$$

over the control parameter u, where $u \in (0, u_{max})$ and to find a suitable $u = u^*$ in $(0, u_{max})$ for which J is maximum where

$$\pi(x, y, z, u) = (p_1 \epsilon_1 x + p_2 \epsilon_2 y + p_3 \epsilon_3 z - \hat{c}) u(t)$$

Theorem 4.6. *Let the dynamic model be given by* (4.3) *with restrictions* (4.6) *and the profit function be given by* (4.4) *under restrictions* (4.7) *and* (4.8). *The problem is to maximize*

$$J = \int_0^T \pi(x, y, z, u) dt$$

where T is the total time. Then there exists $u = u^*$ satisfying (4.6) *for which J is maximum. Further the optimal path is given by*

$(\epsilon_1 - q_1) x \frac{1}{R} (\chi \frac{x - x_1}{F} - V) + (\epsilon_2 - q_2) y [\tau \frac{y - y_1}{-G} + K] + (\epsilon_3 - q_3) z [\eta \frac{z - z_1}{-H} + J] = 0$

where F, G and H are given in proposition (4.3) *and (x_1, y_1, z_1) is the nontrivial equilibrium of the model* (4.2). *Lastly, the optimal values of x, y and z are obtained as the point of intersection of* (4.9) *with the above optimal path.*

Proof. Hamiltonian for our model (4.3) is given by

$$H = (p_1 \epsilon_1 x + p_2 \epsilon_2 y + p_3 \epsilon_3 z - \hat{c}) u(t) + \lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 + \mu G$$

where $\lambda_i(t)$ for $i = 1, 2, 3$ and $\mu(t)$ are co-state variables to be determined suitably.

For steady state solution, we have

$$q_1 + k_1 y - \frac{k_1 xy + k_2 yz + k_3 zx}{c} - \epsilon_1 u = 0$$

$$q_2 + k_2 z - \frac{k_1 xy + k_2 yz + k_3 zx}{c} - \epsilon_2 u = 0$$

$$q_3 + k_1 x - \frac{k_1 xy + k_2 yz + k_3 zx}{c} - \epsilon_3 u = 0$$

By applying Pontryagin's maximum principle we have (for a steady state solution)

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial x} = -[p_1\epsilon_1 u - \frac{\lambda_1}{c}(k_1xy + k_3zx) + \mu]$$

(4.13)
$$\frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial y} = -[p_2\epsilon_2 u - \frac{\lambda_2}{c}(k_1xy + k_2yz) + \mu]$$

$$\frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial z} = -[p_3\epsilon_3 u - \frac{\lambda_3}{c}(k_2yz + k_3zx) + \mu]$$

and

$$\frac{\partial \pi}{\partial u} + \lambda_1\frac{\partial f_1}{\partial u} + \lambda_2\frac{\partial f_2}{\partial u} + \lambda_3\frac{\partial f_3}{\partial u} + \mu\frac{\partial G}{\partial u} = 0$$

i.e.

(4.14)
$$p_1\epsilon_1 x + p_2\epsilon_2 y + p_3\epsilon_3 z - \hat{c} - \lambda_1\epsilon_1 x - \lambda_2\epsilon_2 y - \lambda_3\epsilon_3 z + \mu = 0$$

Equation (4.13) can be rewritten as

$$(D - \frac{k_1xy + k_3zx}{c})\lambda_1 + p_1\epsilon_1 u - \mu = 0$$

(4.15)
$$(D - \frac{k_1xy + k_2yz}{c})\lambda_2 + p_2\epsilon_2 u - \mu = 0$$

$$(D - \frac{k_2yz + k_3zx}{c})\lambda_3 + p_3\epsilon_3 u - \mu = 0$$

where $D = \frac{d}{dt}$. Again using (4.14), the system (4.15) becomes

$$\frac{d\lambda_1}{dt} = a_1\lambda_1 + b_1\lambda_2 + c_1\lambda_3 + d_1$$

(4.16)
$$\frac{d\lambda_2}{dt} = a_2\lambda_1 + b_2\lambda_2 + c_2\lambda_3 + d_2$$

$$\frac{d\lambda_3}{dt} = a_3\lambda_1 + b_3\lambda_2 + c_3\lambda_3 + d_3$$

where $a_1 = l = \frac{k_1xy+k_3zx}{c} + \epsilon_1 x$, $b_1 = b_3 = \epsilon_2 y$, $c_1 = c_2 = \epsilon_3 z$, $a_2 = a_3 = \epsilon_1 x$, $b_2 = m = \frac{k_1xy+k_2yz}{c} + \epsilon_2 y$, $c_3 = n = \frac{k_2yz+k_3zx}{c} + \epsilon_3 z$
$L = -d_1 = p_1\epsilon_1 u + r$, $M = -d_2 = p_2\epsilon_2 u + r$, $N = -d_3 = p_3\epsilon_3 u + r$
$r = p_1\epsilon_1 x + p_2\epsilon_2 y + p_3\epsilon_3 z - \hat{c}$.

Solving the system of differential equation (5.4.16) we get the particular solutions by using the particular solution of the system of ordinary differential equations as given earlier given by

$$\lambda_1(t) = \frac{1}{R}(\chi u - V)$$

where $\chi = (Ap_2\epsilon_2 + Bp_3\epsilon_3) + \frac{b_1p_2\epsilon_2+c_1p_3\epsilon_3}{c_1A-b_1B} - \frac{p_1\epsilon_1(c_1A-b_1B)}{AD-BC}$,
$V = r[(c_1A - b_1B) - \frac{(b_1+c_1)(b_1D-c_1C)}{c_1A-b_1B} - (A + B)]$

$$\lambda_2(t) = \tau u + K$$

where $\tau = \frac{1}{BC-AD}[(\frac{\chi a_2}{R} + p_2\epsilon_2)(b_1D - AB) + (\frac{\chi a_3}{R} + p_3\epsilon_3)(c_1D - B^2)]$
$K = \frac{1}{AD-BC}[(\frac{a_2V}{R} - r)(b_1D - AB) + (\frac{a_3V}{R} - r)(c_1D - B^2)]$

$$\lambda_3(t) = \eta u + J$$

where $\eta = \frac{1}{AD-BC}[(\frac{\chi a_2}{R} + p_2\epsilon_2)(b_1C - A^2) + (\frac{\chi a_3}{R} + p_3\epsilon_3)(c_1C - AB)]$
$J = \frac{1}{AD-BC}[(\frac{a_2V}{R} - r)(A^2 - b_1C) + (\frac{a_3V}{R} - r)(AB - c_1C)]$.
A, B, C, D, S, R have their usual meanings as deduced in previous sections.

Finally we take $u \to u_{max}$ as $t \to T$, in λ_1, λ_2 and λ_3, we have the limiting values as

$$\lambda_1(t) = \frac{1}{R}(\chi u_{\max} - V) \tag{4.17}$$

$$\lambda_2(t) = \tau u_{\max} + K \tag{4.18}$$

$$\lambda_3(t) = \eta u_{\max} + J \tag{4.19}$$

Using (4.14) the co-state vector $\mu(t)$ is given by

$$\mu(t) = \lambda_1(t)\epsilon_1 x + \lambda_2(t)\epsilon_2 y + \lambda_3(t)\epsilon_3 z - r \tag{4.20}$$

Now, if H is maximum at $u = u^*$ (say), $0 < u < u_{\max}$, then $\frac{\partial H}{\partial u} = 0$ at $u = u^*$. Hence we have

$$r - \lambda_1(t)q_1x - \lambda_2(t)q_2y - \lambda_3(t)q_3z + \mu(t) = 0 \tag{4.21}$$

where $\lambda_i(t)$ $(i = 1, 2, 3)$ and $\mu(t)$ corresponds to $u = u^*$. Hence from (4.17) to (4.20), we get

$$\lambda_1(t) = \frac{1}{R}(\chi u^* - V)$$
$$\lambda_2(t) = \tau u^* + K$$
$$\lambda_3(t) = \eta u^* + J$$
$$\mu(t) = \lambda_1(t)\epsilon_1 x + \lambda_2(t)\epsilon_2 y + \lambda_3(t)\epsilon_3 z - r$$

Again, as steady state optimal solution (x^*, y^*, z^*) is desired, u^* is given by

$$u^* = \frac{x - x_1}{F} = \frac{y - y_1}{-G} = \frac{z - z_1}{-H} \tag{4.22}$$

Thus finally we have

$$\lambda_1(t) = \frac{1}{R}(\chi\frac{x - x_1}{F} - T) \tag{4.23}$$

$$\lambda_2(t) = \tau\frac{y - y_1}{-G} + K \tag{4.24}$$

$$\lambda_3(t) = \eta\frac{z - z_1}{-H} + J \tag{4.25}$$

$$\mu(t) = \lambda_1(t)\epsilon_1 x + \lambda_2(t)\epsilon_2 y + \lambda_3(t)\epsilon_3 z - r \tag{4.26}$$

Under (4.23)–(4.26), (4.21) reduces to

$(\epsilon_1 - q_1)x\lambda_1(t) + (\epsilon_2 - q_2)y\lambda_2(t) + (\epsilon_3 - q_3)z\lambda_3(t) = 0.$

This implies that

$(\epsilon_1 - q_1)x\frac{1}{R}(\chi\frac{x-x_1}{F} - T) + (\epsilon_2 - q_2)y[\tau\frac{y-y_1}{-G} + K] + (\epsilon_3 - q_3)z[\eta\frac{z-z_1}{-H} + J] = 0$

where F, G and H are given in proposition 4.3. Putting the values of x_1, y_1 and z_1, we obtain the equation of the optimal path. Solving (4.9) with the above optimal path, we obtain the optimal values x^*, y^*, z^* of x, y, z respectively and thus obtain the optimal value of u^*. □

Discussion

Theory of constrained optimization of a functional on a subset of R^n was known earlier. But realistic application of this theory and performance of corresponding analysis to determine the optimal effort and optimal bio-masses was not attempted earlier. This is the first instance where such problem of reality has been tackled nicely.

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