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A NEW APPROACH TO NONSTANDARD ANALYSIS

Manuel Suárez Fernández

Profesor de Matemáticas de la U.C.L.M.

Abstract.

The aim of this paper is to present an new and simple approach to nonstandard analysis.

Preface.

The nonstandard analysis is the modern version of Newton and Leibniz's old mathematical analysis in which unlimitedly big or unlimited and unlimitedly small or infinitesimal numbers are used. The most significant rediscoverer of such numbers is Abraham Robinson [3] and, later on, there are relevant people such as Edward Nelson [2] and Georges Reeb.

In my opinion, apart from its theoretical value that, doubtless, nonstandard ideas have, such ideas have a paramount practical interest.

But for it to be shown and so, first, the above mentioned ideas, come to be known, then accepted and, finally, normally used not only by mathematicians but also by other professionals that as it is the case with physicists, chemists, engineers, economists, etc., use mathematics and, even, nonstandard analysis come to be used in a systematic way in University mathematical analysis programs, it is essential to simplify more and more its approach. To present an new, simple and didactic approach to nonstandard analysis, so that the nonstandard ideas are easy and attractive to reader, it is the main objetive of this paper, which contains three parts.

The first part aim is to fundament, in the theory of sets, the non standard theorems of the second part, which, on its part, fundaments the non standard principles of the third in which there appear non standard definitions of some basic concepts of infinitesimal calculus.

Posing and reasoning of everyone of the above mentioned parts, are original, as well as the non standard principles stated in the third of them (though this principles could be proved also in [2]) and, in this paper (as in [2]), the non standard universe of sets is the same as the standard universe of sets.

And if what the reader wishes is just to know with the slightest possible effort a non standard theory to, for example, use it in infinitesimal calculus, can pay attention only to the third part (without reading either the first or the second) notwithstanding he must know ZF (Zermelo-Fraenkel set theory). And, to understand the fist part, he must know ZFC (Zermelo-Fraenkel set theory with axiom of choice[1]).

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[1]The axioms of ZFC are (according to [1]) the extensionality axiom, the power set axiom, the union set axiom, the class of axioms of subtitution, the axiom of infinity, the statement «there exists some set» and the axiom of choice. Specific sings that we use (in a simple way, as much as possible) are, for T_N model, and, in order to symplify the denotations, also are for T_N model, the usual signs,
$\wedge \; \neg \; \vee \; \Rightarrow \Leftrightarrow \exists \; \forall \; \in \; \notin \; = \; \neq \; \subset \; \not\subset \; \cup \; \cap \; (\;) \; \setminus \; \{ \; \} \; | \; \leq \; < \; + \; - \; . \; /$

First part: A model of ZFC valid for nonstandard analysis.[2]

In this Firs part we present an original proof that if T_W is a consistent model of ZFC then there exists a consistent model T_N of ZFC valid for the nonstandard analysis because the classic set of natural numbers (of T_N model), besides the classic (or standard) "limited" natural numbers, has "unlimited" natural numbers[3], so that «zero is a limited natural number», «if n is a limited natural number then n+1 is a limited natural number» and «there exist unlimited natural numbers» (by virtue of which and of the classic recurrence principle , the limited natural numbers do not constitute a set of T_N model).

The T_N theory.

We suppose that T_W is a model of ZFC for which **W** is the set of natural number, F_W is a Fréchet's ultrafilter[4] of model T_W and, from the before mentioned T_W model, we define a "theory of sets", that we call T_N, in the following way:

- We suppose that if $\alpha_n(\zeta_1,..,\zeta_j)$ is an expression whichever in which there are j expressions $\zeta_1,.., \zeta_j$ (bearing in mind that $j\in W^+$, $W^+=\{x|(x\in W)\wedge(x\neq 0_W)\}$ and 0_W is the zero of T_W model) and $\eta_1,.., \eta_j$ are j expressions, then $\alpha_n(\eta_1,..,\eta_j)$ is the expression with $\eta_1,.., \eta_j$ where in $\alpha(\zeta_1,..,\zeta_j)$, respectively,

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[2]Even this first part, which is the least simple of the three parts, it is easy to understand because its basic idea is nepeated from beginning to the end.

[3]Consequently, the classic sets of integer, rational, real and complex numbers (of T_N model), also have unlimited elements.

[4]See Appendix of First part.

there are $\zeta_1, .., \zeta_j$.

- We suppose that the sets of T_N theory are the sequences of sets of T_W model.[5]
- If f is a set of T_N theory then, for each element n of W^+, we call "component n of f" to the set f(n) of T_W model.[6]
- We call "variables" of T_N theory, to the same signs that we call "variables" of T_W model.
- We suppose that the formulas of T_N theory are the expressions that result if we replace in the formulas of T_W model, the constants[7] (of T_W model) appearing in them, for constants of T_N theory (So, for instance, if A is constant of T_W model, f is a constant of T_N theory and **x** is a variable then $\exists x(x \in A)$ is a formula of T_W model with the constant A and $\exists x(x \in f)$ is a formula of T_N theory with the constant f where in $\exists x(x \in A)$ there is the constant A).
- If α is a formula of T_N theory and n is an element of W^+ then,
  - if α is with no constants then we say that "α_n is the component n of α" if and only if α_n is the expression identical to α.
  - if α is with only p constants $f_1, .., f_p$ (bearing in mind that $p \in W^+$), then we say that "α_n is the component n of α"

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[5]So, f is a set of T_N theory if and only if f is a sequence $A_1, A_2, A_3, ...$ of T_W model. That is, f is a set of T_N theory if and only if f is a map from W^+ into a set E of T_W model (So, for each element n of W^+ there exists an element A_n of E such that $f(n)=A_n$).

[6]That is, is f is the sequence $A_1, A_2, A_3, ...$ of T_W model and n is an element of W^+ then we call "component n of f" to A_n.

[7]We call "constants" to the signs which denote sets, "constants of T_W model" to the signs which denote sets of T_W model and "constant of T_N theory" to the signs which denote sets of T_N theory.

if and only if α_n is the expression such that in α_n there are the constants $f_1(n),..,\ f_p(n)$ of T_W model where in α, respectively, there are the constants $f_1,..,\ f_p$ of T_N theory (that is, if $\alpha(f_1,..,f_p)$ is a formula of T_N theory with only p constants $f_1,..,\ f_p$ and $n\in W^*$ then the expression $\alpha(f_1(n),..,f_p(n))$ is the component n of $\alpha(f_1,..,f_p)$).

- We suppose (as in T_W model) that the statements of T_N theory are the formulas of T_N theory with no free variables (So, if α is a formula of T_N theory then α is a statement if and only if for each element n of W^*, the component n of α, is a statement of T_W model).
- We suppose that if α is a statement of T_N theory then α is true if and only if the set of the elements n of W^* such that the component n of α is a true statement of T_W model, is an element of (ultrafilter) F_W.
- If α, β are statements of T_N theory then (in a similar way to T_W model) we say that "α is equivalent to β", if and only if either α is true and β is true or α is false and β is false. And if $\alpha(x)$, $\beta(x)$ are formulas of T_N theory with the same variables and (at least) there is a free variable x then we say that "$\alpha(x)$ is equivalent to $\beta(x)$" if and only if, for all constant f of T_N theory, $\alpha(f)$ is equivalent to $\beta(f)$.
  (So, for example, if f, g are constants of T_N theory then the statements $f\neq g$, $f\notin g$, $f\not\subset g$ are equivalent to, respectively, the statements $\neg(f=g)$, $\neg(f\in g)$, $\neg(f\subset g)$).

Theorem 1.1. If α is a statement of T_N theory then either α is true or α is false.

Proof. If α is a statement of T_N theory and for each element n of W^+, α_n is the component n of α, then for all element n of W^+, α_n is a statement of T_W model. So, since F_W is an ultrafilter, one and only one of the two following formulations is true:

- The set of elements n of W^+ such that α_n is a true statement of T_W model, is an element of F_W.
- The set of elements n of W^+ such that α_n is a false statement of T_W model, is an element of F_W.

Therefore, one and only one of the two following formulations is verified:

- α is a true statement of T_N theory.
- α is a false statement of T_N theory.

Theorem 1.2. If α is a statement of T_N theory then α is true if and only if $\neg\alpha$ is false.

Proof. Taking into account now that F_W is a filter (because F_W is an ultrafilter), similar proof to that of Theorem 1.1.

Theorem 1.3. If α, β are statements of T_N theory then,

- $\alpha\wedge\beta$ is true if and only if «α is true and β is true».
- $\alpha\vee\beta$ is true if and only if «α is true or β is true».

Proof. In similar way to proof of Theorem 1, it is proved that,

- $\alpha\wedge\beta$ is true if and only if α is true and β is true, taking into account that F_W is a filter.
- $\alpha\vee\beta$ is true if and only if α is true or β is true, taking into account that F_W is an ultrafilter.

Lemma 1.1. If, in T_w model, H is an element of F_w and A_1, A_2, A_3,... is a sequence of sets such that for each element n of H, $A_n \neq \Phi_w$ (Φ_w being the empty set of T_w model) then there exists a set f of T_N theory such that for each element n of H, the component n of f is an element of A_n.

Proof. If $A=\bigcup_{n\in W+} A_n$ (that is, if A is the union of the A_n of sequence) then, by virtue of the axiom of choice (that satisfies the T_w model), there exists a map Ψ from $P(A)\backslash\{\Phi_w\}$ (set of subsets non empty of A) into A such that for each element n of H, $\Psi(A_n)$ is an element of A_n. And let's b an element of A.
So, if f is the map from W^+ into A such that for each element n of H, $f(n)=\Psi(A_n)$ and for each element of $W^+\backslash H$, $f(n)=b$ then f is a set of T_N theory such that for each element n of H, f(n) (component n of f) is an element of A_n.

Theorem 1.4. If, in theory T_N, E is a set and $\alpha(x)$ is a formula with free variable x and with no other free variables then,

- $\exists x((x\in E)\wedge\alpha(x))$ is true if and only if there exists some set f of T_N theory such that $(f\in E)\wedge\alpha(f)$ is true.
- $\forall x((x\in E) \Rightarrow \alpha(x))$ is true if and only if, if f is a element (whichever) of E then $\alpha(f)$ is true.

Proof. If for each element n of W^+, E_n is the component n of E and $\alpha_n(x)$ is the component n of the formula $\alpha(x)$ then,

- $\exists x((x\in E_n)\wedge\alpha_n(x))$ is the component n of $\exists x((x\in E)\wedge\alpha(x))$ and $\exists x((x\in E)\wedge\alpha(x))$ is true if and only if the set of elements n of W^+ such that $\exists x((x\in E_n)\wedge\alpha_n(x))$ is a true statement of T_w

model, is an element of F_w. So,

- if there exists a set f of T_N theory such that $(f \in E) \wedge \alpha(f)$ is true then the set of elements n of W^+ such that $(f(n) \in E_n) \wedge \alpha_n(f(n))$ is a true statement of T_w model, is an element of F_w. Then, the set of elements n of W^+ such that $\exists \mathbf{x}((x \in E_n) \wedge \alpha_n(x))$ is a true statement of T_w model, is an element of F_w and, consequently, $\exists \mathbf{x}((x \in E) \wedge \alpha(x))$ is true.
- if, reciprocally, $\exists \mathbf{x}((x \in E) \wedge \alpha(x))$ is true and H is the set of elements of W^+ such that $\exists \mathbf{x}((x \in E_n) \wedge \alpha_n(x))$ is true, then H is an element of F_w. So, if for each element n of H, $A_n = \{x \mid (x \in E_n) \wedge \alpha_n(x)\}$ then, by virtue of Lemma 1.1, there exists (in T_w model) a map f from W^+ into $\bigcup_{n \in W+} A_n$ (union set of E_n), such that for each element n of H, $f(n) \in A_n$ is true and, consequently, for each element n of H, $(f(n) \in E_n) \wedge \alpha_n(f(n))$ is true. Therefore, the set of element n of W^+ such that $(f(n) \in E_n) \wedge \alpha(f(n))$ is true, is an element of F_w and, consequently, there exists a set f (of T_N theory) such that $(f \in E) \wedge \alpha(f)$ is true.

- For each element n of W^+, $\neg \exists \mathbf{x}((x \in E_n) \wedge \neg \alpha_n(x))$ is the component n of $\neg \exists \mathbf{x}((x \in E) \wedge \neg \alpha(x))$, $\neg \exists \mathbf{x}((x \in E_n) \wedge \neg \alpha_n(x))$ is equivalent to $\forall \mathbf{x}((x \in E_n) \Rightarrow \alpha_n(x))$ and this statement is the component n of the statement $\forall \mathbf{x}((x \in E) \Rightarrow \alpha(x))$ (of T_N theory). So, $\forall \mathbf{x}((x \in E) \Rightarrow \alpha(x))$ is equivalent to $\neg \exists \mathbf{x}((x \in E) \wedge \neg \alpha(x))$. Then, the statement $\forall \mathbf{x}((x \in E) \Rightarrow \alpha(x))$ is true if and only if for all set f (of T_N theory), $\alpha(f)$ is a true statement.

Theorem 1.5. If f, g are sets of T_N theory then f=g if and only if, if $\alpha(x)$ is a formula (whichever) of T_N theory with a free variable x then the formula $\alpha(f)$ is equivalent to the formula $\alpha(g)$. (So, if f is a set of T_N theory then f=f and if f, g are sets of T_N theory then,

- if f=g then f=g.
- if h is a set of T_N theory, f=h and h=g then f=g.
- if f=g then f has the same elements and is an element of the same sets as g).

Proof. If f, g are sets of T_N theory and f=g then the set of elements n of $\mathbf{W}^+$ such that f(n)=g(n), is an element of F_W. So, if $\alpha(\mathbf{x})$ is a formula of T_N theory with a free variable x and for each element n of $\mathbf{W}^+$, $\alpha_n(\mathbf{x})$ is the component n of α, then for all element n of $\mathbf{W}^+$, x is free in $\alpha_n(\mathbf{x})$ and the set of elements n of $\mathbf{W}^+$ such that $\alpha_n(f(n))$ is equivalent to $\alpha_n(g(n))$, is an element of F_W [8]. Therefore, $\alpha(f)$ is equivalent to $\alpha(g)$.
Reciprocally, if f, g are sets of T_N theory such that «if $\alpha(\mathbf{x})$ is a formula (whichever) of T_N theory with a free variable x then $\alpha(f)$ is equivalent to $\alpha(g)$», then, since x=g is a formula of T_N theory with a free variable x, the statement f=g is equivalent to the true statement g=g and, consequently, f=g is true.[9]

Theorem 1.6. If f, g are sets of T_N theory then $f \subset g$ if and only if all element of f is an element of g.

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[8]Bearing in mind that if A, B are sets of T_W model, A=B and $\gamma(\mathbf{x})$ is a formula of T_W model with a free variable x then $\gamma(A)$ is equivalent to $\gamma(B)$.

[9]Of the similar way it is proved that if, in general, $A_1,..,A_p$, $B_1,..,B_p$ are 2p set of T_N theory, then ««$A_1=B_1,..,A_p=B_p$» if and only if, «if $\alpha(\mathbf{x}_1,..,\mathbf{x}_p)$ is a formula of T_N theory with p free variables $\mathbf{x}_1,..,\mathbf{x}_p$ then the formula $\alpha(A_1,..,A_p)$ is equivalent to the formula $\alpha(B_1,..,B_p)$»».

Proof. Then, If $f \subset g$ then the set of elements n of W^* such that $f(n) \subset g(n)$, is an element of F_w. And if $h \in f$ then the set of elements n of W^* such that $h(n) \in f(n)$, is an element of F_w. So, if $f \subset g$ and $h \in f$ then the set of elements n of W^* such that $h(n) \in g(n)$, is an element of F_w and, consequently, $h \in g$. So, if $f \subset g$ then all element h of f is an element of g.

If $f \not\subset g$ then the set of elements n of W^* such that $f(n) \not\subset g(n)$ is an element of F_w. Then, if H is the set of elements of W^* such that there exists a set h_n (of T_w model) such that $h_n \in f(n)$ and $h_n \notin g(n)$, then H is an element of F_w. Then, by virtue of Lemma 1.1, there exists a set h of T_N theory such that for all element n of H, $h(n) \in f(n)$ and $h(n) \notin g(n)$. Consequently, $h \in f$ and $f \notin g$. So, if all element of f is an element of g then $f \subset g$.

Theorem 1.7. If f, g are sets of T_N theory and f has the same elements as g then f=g. [10]

Proof. If $f \neq g$ and H is the set of elements n of W^* such that $f(n) \neq g(n)$ then H is the set of elements n of W^* such that there exists a set h_n of T_w model such that either «$h_n \in f(n)$ and $h_n \notin g(n)$» or «$h_n \notin f(n)$ and $h_n \in g(n)$», and H is an element of F_w. Then, by virtue of Lemma 1.1, there exists a set h of T_N theory such that for each element n of H, either «$h(n) \in f(n)$ and $h(n) \notin g(n)$» or «$h(n) \notin f(n)$ and $h(n) \in g(n)$», and, consequently, either «$h \in f$ and $h \notin g$» or «$h \notin f$ or $f \in g$».

Therefore, «if $f \neq g$ then f has not the same elements as g» and,

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[10]This theorem of T_w model is the extensionality axiom of T_N theory.

consequently, if f has the same elements as g then f=g.

Theorem 1.8. The constant sequence Φ_W, Φ_W, Φ_W,... of T_W model (Φ_W being the empty set of T_W model) is the empty set of T_N theory, which we denote Φ. [11]

Easy proof.

Theorem 1.9. If f is a set of T_N theory then there exists a set P(f) (of T_N theory), that we call "power set of f", such that if g is a set (of T_N theory) then $g \in P(f)$ if and only if $g \subset f$. [12]

Easy proof, taking into account that if f is a set of T_N theory then P(f) is a set of T_N theory such that for each element n of W^+, the component n of P(f) is P(f(n)), power set of f(n) (f(n) being the set of T_W model which is the component n of f).

Theorem 1.10. If f is a set of T_N theory then there exists a set $\bigcup_{x \in f} x$ (of T_N theory) that we call "union set of elements of f", such that if g is a set of T_N theory then $g \in \bigcup_{x \in f} x$ if and only if there exists some set h (of T_N theory) such that $h \in f$ and $g \in h$. [13]

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[11]By virtue of this theorem of T_W model, the statement «there exists the empty set» is an axiom of T_N theory and is easy prove, in a classical way, that this axiom is equivalent to statement of T_N theory «there exists some set».

[12]This theorem of T_W model is the power set axiom of T_N theory.

[13]This theorem of T_W model is the union set axiom of T_N theory.

Easy proof, taking into account that if f is a set of T_N theory then $\bigcup_{x\in f}x$ is a set of T_N theory such that for each element n of W^+, the component n of $\bigcup_{x\in f}x$ is $\bigcup_{x\in f(n)}x$, union set of elements of f(n) (f(n) being the set of T_W model which is the component n of f) and taking into account the Lemma 1.1 as in above proofs.

Theorem 1.11. If $\alpha(x,u)$ is a formula of T_N theory with free variables x, u and no other free variables, $\alpha(x,u)$ is a functional formula of function u [14] and f is a set of T_N theory then there exists a set f' of T_N theory such that if g' is a set of T_N theory then g' is an element of f' if and only if there exists an element g of f such that the statement $\alpha(g,g')$ (of T_N theory) is true.[15]

Proof. It $\alpha(x,u)$ is a formula of T_N theory with free variables x, u, no other free variables and for each element n of W^+, the component n of $\alpha(x,u)$ is a formula $\alpha_n(x,u)$ then, for each element n of W^+, $\alpha_n(x,u)$ is a formula of T_W model with free variables x, u and no other free variables. If is not an element of F_W the set of element n of W^+ such that $\alpha_n(x,u)$ is a functional formula of function u then is an element of F_W the set of element of W^+ such that (t being a variable no en $\alpha(x,u)$) the statement (of T_W model) $\exists x\exists u\exists t(\alpha_n(x,u)\wedge\alpha(x,t)\wedge(u\neq t))$ is true. Then the statement

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[14]If $\alpha(x,u)$ is a formula of T_N theory with free variables x, u and no other free variables, then (in a similar way to T_W model) we say that "$\alpha(x,u)$ is a formula functional of function u" if and only if for all set g (of T_N theory) it does not exist more than one set g' (of T_N theory) such that the statement $\alpha(g,g')$ (of T_N theory) is true.

[15]This theorem of T_W model is the class of axioms of sustitution of T_N theory.

(of T_N theory) $\exists x \exists u \exists t(\alpha(\mathbf{x},u) \wedge \alpha(\mathbf{x},t) \wedge (u \neq t))$is true and, consequently, $\alpha(\mathbf{x},u)$ is not a functional formula of fuction u. Therefore, if $\alpha(\mathbf{x},u)$ is a fuctional formula of fuction u (of T_N theory) then the set of element n of $\mathbf{W}^+$ such that $\alpha_n(\mathbf{x},u)$ is a fuctional formula of fuction u (of T_W model), is an element of F_W. So, if the formula $\alpha(\mathbf{x},u)$ is a fuctional formula of fuction u and f is a set of T_N theory, then is an element of F_W the set H of elements n of $\mathbf{W}^+$ such that there exists a set f'_n of T_W model such that if g'_n is a set of T_W model then g'_n is an element of f'_n if and only if there exists an element g_n of f(n) such that $\alpha(g_n,g'_n)$. Then, by virtue of Lemma 1.1, there exists a set f' of T_N theory such that if g' is a set of T_N theory then g' is an element of f' if and only if there exists an element g of f such that $\alpha(g,g')$ is a true statement (of T_N theory).

Lemma 1.2. If f is a set of T_N theory then there exists a set of T_N theory, which we denote "{f}", whose only element is f. [16]

Easy proof taking into account that {f} is the set of T_N theory such that for each element n of $\mathbf{W}^+$, the component n of {f} is {f(n)}.

Lemma 1.3. If f, g are sets of T_N theory then there exists a set (of T_N theory) which we denote "f$\cup$g" and we call "set f union

[16]This theorem of T_W model is a theorem of T_N theory (it is proved in the mentioned T_N theory, in a classical way, considering the power set axiom, the class of axioms of substitution and that the empty set Φ is a set of T_N theory).

g", whose elements are the elements of f and the elements of g. [17]

Easy proof taking into account that f∪g is the set of T_N theory such that for each element n of $\mathbf{W}^+$. the component n of f∪g is f(n)∪g(n).

Definition 1.1. If A is a set of T_W model or of T_N theory then we denoted s(A) and we say "successor of A" to A∪{A}.

Theorem 1.12. There exists a set **N** of T_N theory, whose elements we call "natural number" (of T_N theory), such that,

- Φ∈**N** and we call "zero" (and "empty set" of T_N theory) to natural number Φ.
- if p∈**N** then s(p)∈**N**.
- (Recurrence principle of T_N theory) if f⊂**N**, Φ∈f and «if p∈f then s(p)∈f», then f=**N**. [18]

Proof. We suppose that **N** is a set of T_N theory such that for each element n of $\mathbf{W}^+$, the component n of **N** is the set **W** of natural number of T_W model (that is, we suppose that **N** is the sequence **W**, **W**, **W**,... of T_W model). Therefore,

- Easy proof that Φ∈**N**, taking into account that Φ_W is an element of **W** (since Φ_W is the natural number of T_W model that we call

[17]This theorem of T_W model is a theorem of T_N theory (it is proved in the mentioned T_N theory, in a classical way, considering the power set axiom, the union set axiom, the class of axioms of sustitution and that the empty set Φ is a set of T_N theory).

[18]This theoren of T_W model is the axiom of infinity of T_N theory.

"zero" and also we denote 0_W).

- If p is an element of **N** then the set of elements n of $\mathbf{W}^+$ such that $p(n) \in W$, is an element of F_W. So, since if $p(n) \in \mathbf{W}$ then $s(p(n)) \in \mathbf{W}$, the set of elements n of $\mathbf{W}^+$ such that $s(p(n)) \in \mathbf{W}$, is an element of F_W. So, s(p) is an element of **N**.
- If $f \subset \mathbf{N}$, $\Phi \in f$, $f \neq \mathbf{N}$ and H is the set of elements n of $\mathbf{W}^+$ such that for each element n of H there exists the first element p_n of f(n) such that $s(p_n)$ is not element of f(n) then H is an element of F_W. So, if p is a set of T_N theory such that for each element n of H, $p(n)=p_n$ then $p \in f$ and $s(p) \notin f$.

  Therefore, if f is a subset of **N**, $\Phi \in f$ and «if $p \in f$ then $s(p) \in f$», then $f=\mathbf{N}$.

Theorem 1.13. If f is a non empty set of T_N theory then there exists a map Ψ from $P(f) \setminus \{\Phi\}$ (set of non empty subsets of f) into f such that if $g \in P(f) \setminus \{\Phi\}$ then Ψ (as to map of T_N theory) assigns to g an element of g. [19]

Proof. Since the T_W model satisfies the axiom of choice, for each element n of $\mathbf{W}^+$ such that $f(n) \neq \Phi_W$ (Φ_W being the empty set of T_W model) there exists a map Ψ_n from $P(f(n)) \setminus \{\Phi_W\}$ (set of non empty subsets of f(n)) into f(n) such that, if g_n is an element of $P(f(n)) \setminus \{\Phi_W\}$ then Ψ_n assigns to g_n an element of g_n. And, since $f \neq \Phi$ (Φ being the empty of T_N theory), if H is the set of elements n of $\mathbf{W}^+$ such that $f(n) \neq \Phi_W$ then H is a element of F_W. So, by virtue of Lemma 1.1, there exists a set Ψ of T_N theory such that for

[19]This theorem of T_W model is the axiom of choice of T_N theory.

each element n of H, $\Psi(n)$ (component n of Ψ) is a map from $P(f(n))\backslash\{\Phi_w\}$ into f(n) such that if $g_n \in P(f(n))\backslash\{\Phi_w\}$ then $\Psi(n)$ assigns to g_n a element of g_n. So, since H is a element of F_w, Ψ is a map from $P(f)\backslash\{\Phi\}$ into f such that, if g is an element of $P(f)\backslash\{\Phi\}$ then Ψ, as to map of T_N theory, assigns to g, an element of g.

Therefore, if T_w is a model of ZFC then the empty set is a set of T_N theory and, the mentioned T_N theory, satisfies the extensionality axiom, the power set axiom, the union set axiom, the class of axioms of sustitution, the axiom of infinity and the axiom of choice and, consequently, if T_w is a model of ZFC then T_N theory is also a model of ZFC, that we call "T_N model".

Definition 1.2. If p is an element of **N** (set of natural number of T_N model) then we say that,

- "p is limited" if and only if there exists an element q of **W** (set of natural number of T_w model) such that the set of elements n of $\mathbf{W}^+$ such that p(n)=q, is an element of F_w.
- "p is unlimited" if and only if p is not limited.

Theorem 1.14. Φ (zero of **N**) is limited. If f is a limited natural number then s(f) is limited. And there exists unlimited elements of N.

Proof. Φ is limited since Φ is the sequence 0_w, 0_w, 0_w,... of T_w model (0_w being the zero of **W**, $0_w=\Phi_w$ and Φ_w the empty set of T_w model). If f is a limited natural number then there exists an element p of W such that f is equal to the sequence p, p, p,... of T_w model. So, s(f), sucessor of f, is equal of sequence s(p), s(p), s(p),... of T_w model (s(p) being the sucessor of p) and, consequently, s(f) is limited.

And an unlimited natural number is, for example, g such that for all element n of W^*, g(n)=n (that is, g is equal to the sequence $s(\Phi_w)$, $s(s(\Phi_w))$, $s(s(s(\Phi_w)))$,... of T_w model).

Appendix of first part.

A "filter" on a non empty set E is, by definition, a subset F of P(E), such that F is not the empty set, the empty set is not an element of F, «if A, B are elements of F then $A \cap B$ is an element of F» and if A is an element of F, B is a subset of E and A is a subset of B then B is a element of F.

So, if F is a filter on a set E and A is an element of F then ${}^cA$ (complementary of A with respect to E) is not an element of F.

An "ultrafilter" on a non empty set E is a filter U such that if F is a filter on E and $U \subset F$ then U=F.

It can be easily proved that if U is an ultrafilter on a set E, A, B are subset of E, $A \cup B$ is an element of U and A is not an element of U then B is an element of U (So, if A is a subset of E and A is not an element of U then ${}^cA$ is an element of U).

Using Zorn's lemma (valid to use in ZFC, because ZFC satisfies the axiom of choice), it is proved that if F is a filter on a set E then there exists a ultrafilter U containing F.

If **W** is the set of natural numbers, we say that "F is a Frechét's filter" if and only if F is a filter on **W** such that A is an element of F if and only if A is a subset of **W** such that ^{c}A (complementary of A with respect to **W**) is a finite subset of W.

We say that "F_w is a Frechét's ultrafilter" on **W** if and only if F_w is a ultrafilter on **W** containing the Frechét's filter on **W**.

Second part: Support of some principles for nonstandard analysis.

In this Second part we denote 0_w, 1_w, 2_w, 3_w,... to natural numbers of T_w model (that is, to elements of **W**) and we denote 0, 1, 2, 3,... to natural numbers the T_N model (that is, to the elemens of **N**).

In the mentioned T_w model and T_N model, we suppose defined in a classical way, the structure of natural numbers and, with its respective structures, the sets of the integer, rational, real and complex numbers.[20]

Definition 2.1. If f is a set of T_N model then we say that f is a "classic" set if and only if there exists a set A of T_w model

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[20]In the T_N model, apart from this classical definitions, there exist the corresponding no classical interpretations, specific of this mentioned model of ZFC.
For example, the non classical interpretations (specific of T_N model) of order and of addition on the set **R** (of real numbers of T_N model), are the following ones:
If f, g are elements of **R** then we say that,

- $f \leq g$ if and only if the set of elements n of W^+ such that $f(n) \leq g(n)$, is an element of F_w.
- if h is an element of **R** then $f+g=h$ if and only if the set of elements n of W^+ such that $f(n)+g(n)=h(n)$, is an element of F_w.

such that the set of elements n of $\mathbf{W}^*$ such that f(n)=A, is an element of F_W. And we say that a set of T_N model is a "nonclassic" set if and only if is not a classic set.

Theorem 2.1. If f is a natural number of T_N model (is that, if f is an element of **N**) then f is classic if and only if f is limited (so, an element of **N** is nonclassic if and only if is unlimited).

Easy proof, considering Definition 1.2 and Definition 2.1.

Theorem 2.2. If f, g are natural numbers, f is classic and g is nonclassic then f≤g.

Proof. If f is a classic natural number, then there exists a natural number p of T_W model such that the set of elements n of $\mathbf{W}^*$ such that f(n)=p is an element of F_W. And if h is a natural number of T_N model such that h≤f then the set of element n of W such that h(n)≤f(n) is an element of F_W. So, there exists an element q of **W** such that q≤p and the set of element n of **W** such that h(n)=q, is an element of F_W. So, h is limited and, consequently, if g is a nonclassic natural number then f≤g.

Theorem 2.3. If f is a classic non empty set (of T_N model) then there exists a classic set g (of T_N model) such that g is element of f.

Proof. If f is a classic non empty set then there exists a non empty set A of T_W model such that the set of elements n of $\mathbf{W}^*$

such that f(n)=A, is an element of F_w. Then, there exists a set B (of T_w model) such that B is an element of A and, consequently, if g is a set of T_N model such that for all element n of W^+, g(n)=B then g is a classic set and g is and element of f.

Theorem 2.4. If f is a real number of T_N model such that there exists a classic natural number p such that $|f|\leq p$ then there exists an only classic real number g such that for all classic natural number $q\neq 0_w$, $|f-g|\leq(1/q)$.

Proof. If p is a classic natural number then there exists a natural number h of T_w model (that is, an element of W) such that the set of elements n of W^+ such that p(n)=h, is an element of F_w. And if f is a real number of T_N model such that $|f|\leq p$ then the set of elements n of W^+ such that $-h\leq f(n)\leq h$, is an element of F_w. So, if $a_1=-h$ and $b_1=h$ then the set of elements n of W^+ such that $a_1\leq f(n)\leq b_1$, is an element of F_w. So (since F_w is an ultrafilter), the set of elements n of W^+ such that either $a_1\leq f(n)\leq(a_1+b_1)/2_w$ is an element of F_w or the set of elements n of W^+ such that $(a_1+b_1)/2_w\leq f(n)\leq b_1$, is an element of F_w. Continuing this way, we define two sequences a_1, a_2, a_3,... and b_1, b_2, b_3,..., of real numbers of T_w model such that $a_1\leq a_2\leq a_3\leq......\leq b_3\leq b_2\leq b_1$ and there exists one and only one real number r (of T_w model) such that for all element n of W^+, $a_n\leq r\leq b_n$. Therefore, if g is a set of T_N model such that for all element n of W^+, g(n)=r then (since for all natural number $p\neq 0$ of T_w model, the set of elements n of W^+ such that $-(1_w/2_w^P)\leq f(n)-r\leq(1_w/2_w^P)$, is an element of F_w) g is a classic real number such that for all classic natural number $q \neq 0_W$, $|f-g| \leq (1/q)$.

Theorem 2.5. If f is a set of T_N model then f is classic if and only if there exists a formula $\alpha(x)$ with a free variable **x**, with no other free variables, with no nonclassic constants[21] and such that the statements $\exists_1 x\alpha(x)$ [22] and $\alpha(f)$ are true.

Proof. If $\alpha(x)$ is a formula of T_N model, with a free variable x, with no other free variables and with no nonclassic constants and for each element n of W^+, $\alpha_n(x)$ is the component n of $\alpha(x)$, then there exists a element p of W^+ such that the set of elements n of W^+ such that $\alpha_n(x)$ is (identical to) $\alpha_p(x)$ is an element of F_W. So, if the statement (of T_N model) $\exists_1 x\alpha(x)$ is true then the statement (of T_W model) $\exists_1 x\alpha_p(x)$ is true and, consequently, there exists one and only one set A (of T_W model) such that $\alpha_p(A)$ is true. Therefore, if f is a set of T_N model such that for all element n of W^+, $f(n)=A$ then f is classic and $\alpha(f)$ is true.
If, reciprocally, we suppose that f is a classic set (of T_N model) then x=f is a formula (of T_N model) with a free variable x, with no other free variables, with no nonclassic constants (since f is a classic constant) and such that the statement $\exists_1 x(x=f)$ and f=f are true.

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[21]We call "classic constants" to the signs which denote classic sets and "nonclassic constants" to the signs which denote nonclassic sets.

[22]Bearing in mind that the statement $\exists_1\alpha(x)$ is equivalent of statement $\exists x(\alpha(x) \wedge \forall t(\alpha(t) \Rightarrow t=x))$.

Third part: Some principles for nonstandard analysis.

We suppose all what follows in a model T_N of ZFC in which some principles for nonstandard analysis (compatible with the axioms of the classic ZFC) are proposed and, as an example, there appear some non standard definitions and theorems of some basic concepts of mathematical analysis on sequences or standard functions.[23] But, what it not possible at this point (because it would make the paper too long) is to make a study of the properties corresponding to the before mentioned concepts which we define. Also we suppose all what follows, that 0, 1, 2, 3,... are the natural numbers, **N** is the set of natural numbers and **R** is the set of real numbers, of mentioned T_N model (and that $\mathbf{N}\subset\mathbf{R}$).

Definition 3.1. If A is a set then we say that "$\alpha(x)$ is a formula defining A" if and only if $\alpha(x)$ is with a free variable x and with no other free variables and the statements $\exists_1 x\alpha(x)$ and $\alpha(A)$ are true.

Definition 3.2. If A is a set (whichever), we say that,

- "A is standard" if and only if there exists a formula

[23]Such concepts are, for the sequences of real numbers, those of regular sequence and limit. And, for real functions of one real variable, those of limit, continuity at a point, uniform continuity, derivative at a point, derivative function and Riemann integral.

defining A, with no nonstandard constants.

- "A is nonstandard" if and only if A is not standard.[24]

Definition 3.3. If α is a formula (whichever) then we say that "α is standard" if and only if α is with no nonstandard constants.[25]

Principle 1. There exists a nonstandard natural number (that is, there exists a natural number that is a nonstandard set).[26]

Principle 2. If f, g are natural numbers, f is standard and g is nonstandard then $f \leq g$.[27]

Principle 3. If A is a non empty standard set then there exists a standard set which an element of A.[28]

Definition 3.4. If r is a real number then we say that,

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[24]In this Third part we call "standard sets" and "nonstandard sets" to the set that in Second part we call, respectively, "classic set" and "nonclassic sets". And (in this Third part) we call "standard constans" and "nonstandard constants" to the constants that in Second part, we call, respectively, "classic constants" and "nonclassic constants".

[25]So, if α is a formula with no constants then α is a standard formula. And if A is a set such that there exists a formula defining A with no constants then A is a standard set.

[26]The Principle 1 is supported in Theorem 1.14, Theorem 2.1, Theorem 2.5, Definition 3.1 and Definition 3.2.

[27]The Principle 2 is supported in Theorem 2.2, Theorem 2.5, Definition 3.1 and Definition 3.2.

[28]The Principle 3 is supported in Theorem 2.3, Theorem 2.5, Definition 3.1 and Definition 3.2.

- "r is limited" if and only if there exists a standard natural number p such that $|r| \leq p$.
- "r is unlimited" if and only if r is not limited.
- "r is infinitesimal" if and only if either r=0 or 1/r is unlimited.
- "r is appreciable" if and only if r is limited and is not infinitesimal.

Principle 4. If r is a limited real number then there exists an only standard real number °r, that we call "standard part of r", such that r-°r is infinitesimal.[29]

Theorem 3.1. If E is a set then «E is standard if and only if there exists a standard formula $\beta(t)$ with only a free variable t and such that $E=\{t|\beta(t)\}$».

Proof. If E is a standard set then $t \in E$ is a standard formula with only a free variable t and such that $E=\{t|t \in E\}$.
Reciprocally, if E is a set and $\beta(t)$ is a standard formula with only a free variable t and such that $E=\{t|\beta(t)\}$ then $\forall t(t \in x \Leftrightarrow \beta(t))$ is a standard formula with only a free variable x and is a formula defining E. So, E is standard.

Theorem 3.2. 0 is a standard natural number, 1 is a standard natural number and if n, p are standard natural numbers then,

- n+p, n.p are standard.

[29]The Principle 4 is supported in Theorem 2.4, Theorem 2.5, Definition 3.1, Definition 3.2 and Definition 3.4.

• if $r\neq 0$ then n^p is standard.

Poof. $\forall t(t\notin x)$ is a standard formula (since is with no constants) defining 0 (since $0=\Phi$ and the statements $\exists_1 x\forall t(t\notin x)$ and $\forall t(t\notin\Phi)$ are true) and, consequently, 0 is standard.

$\forall t(t\in x \Leftrightarrow t=0)$ is a standard formula defining 1 (since 0 is a standard constant and $1=\{0\}$). So, 1 is standard.

And if $\alpha(x)$ is a standard formula defining n (with free variable x and with no variables t, u) and $\beta(t)$ is a standard formula defining p (with free variable t and with no variables x, u) then,

• $\exists x\exists t((u=x+t)\wedge\alpha(x)\wedge\beta(t))$ is a standard formula defining n+p (with free variable u) and $\exists x\exists t((u=x.t)\wedge\alpha(x)\wedge\beta(t))$ is a standard formula defining n.p (with free variable u). Consequently, n+p and n.p are standard.
• Similar proof to the preceding.

Theorem 3.3. **N** and **R** are standard sets and the class of standard natural number is not a set (of T_N model).

Proof. By virtue of axiom of infinity, there is a standard formula defining **N** [30] and, consequently, **N** is standard.

By virtue of classic construction of real numbers, there is a standard formula defining **R** and, consequently, **R** is standard.

By virtue of Thorem 3.1, 0 is standard and if n is a standard

[30] So, for example, the formula $(\Phi\in x)\wedge\forall t(t\in x \Rightarrow s(t)\in x)\wedge\wedge\forall u(((u\subset x)\wedge((\Phi\in u)\wedge\forall y(y\in u \Rightarrow s(y)\in u))) \Rightarrow u=x)$ (with only free variable x and with only standard constant Φ) is a standard formula defining **N**.

natural number then n+1 is standard. And, by virtue of Principle 1, there exists a nonstandard natural number. Then, considering the classic recurrence principle, the class of standard natural number is not a set (of T_N model).

Theorem 3.4. If p is a natural number then p is (a real number) limited if and only if p is standard.

Easy proof considering Principle 3.2 and Definition 3.4.

Theorem 3.5. If r is a real number then,

- r is unlimited if and only if there exists a nonstandard natural number p such that $p<|r|$.
- r is infinitesimal if and only if there exists a nonstandard natural number such that $|r|<1/p$.
- r is appreciable if and only if there exists a standard natural number $p\neq 0$ such that $1/p<|r|<p$.

Easy proof considering Definition 3.4, Theorem 3.2 and recurrence principle.

Theorem 3.6. If r is a real number then,

- if r is limited then -r is limited.
- if r is limited and s is a limited real number then r+s, r.s are limited.
- if r is infinitesimal then r es limited.
- if r is infinitesimal then -r is infinitesimal.
- if r is infinitesimal and s is an infinitesimal real number

then r+s is infinitesimal.

- if r is infinitesimal and s is limited real number then r.s is infinitesimal.
- if r is appreciable then 1/r is appreciable.
- if r is appreciable and s is an appreciable natural number then r.s is appreciable.
- if r is standard then either r=0 or r is appreciable.
- if r is standard and s is a standard real number then r+s, r.s are standard.
- if r is standard then «-r is standard and if r≠0 then 1/r is standard».

Easy proof considering Definition 3.1, Definition 3.2, Definition 3.4, Principle 2, Principle 4, Theorem 3.2 and Theorem 3.5.

Definition 3.5. If r, s are real numbers we say tath "r is almost s" if and only if r-s is infinitesimal.

Definition 3.6. If r_1, r_2, r_3,... is a standard sequence of real numbers then we say that "r_1, r_2, r_3,... is regular" if and only if, for all unlimited natural numbers n, m, r_n is almost r_m.

Definition 3.7. If r_1, r_2, r_3,... is a standard sequence of real numbers then we say that "r_1, r_2, r_3,... is convergent to r", "$r=\lim_n r_n$" and "r is limit of r_1, r_2, r_3,..." if and only if r is a standard real number and for all ilimited natural number n, r_n is almost r.

Definition 3.8. If f is a standard function from an open real set D into **R** and s is a standard element of adherence of D then we say that $r=\lim x_{x \to s} f(x)$ and "r is the limit of f at point s", if and only if r is a standard real number and for all element u of D such that u is almost s and u≠s, f(u) is almost r.

Definition 3.9. If f is a standard function from an open real set D into **R** and s is a standard element of D then we say that "f is continuous at point s" if and only if there exists $\lim_{x \to s} f(x)=f(s)$.

Definition 3.10. If f is a standard function from an open real set D into **R** then we say that "f is uniformly continuous" if and only if, if r, s are points of D and r is almost s then f(r) is almost f(s).

Definition 3.11. If f is a standard real function from an open real set D into **R** and s is a standard element of D then we say "derivative of f at point s" to $\lim x_{x \to s}((f(x)-f(s))/(x-s))$, if there exists this limit.

Definition 3.12. If f is a standard real function from an open real set D into **R** then we say that "f' is the derivative function of f" if and only if f' is a standard function from D into R such that for all standard point s of D
$f'(x)=\lim_{x \to s}((f(x)-f(s))/(x-s))$.

Theorem 3.7. If f is a standard function from a open real set D into **R** g is a derivative function of f and h is a derivative function of f then g=h.

Proof. Then, if $E=\{x \mid (x \in D) \wedge (g(x) \neq h(x)\}$ then E is standard (since g and h are standard). So, if $E \neq \Phi$ then (Principle 3) there exists a standard element s of E and, consequently, $g(s) \neq h(s)$. So (since if s is standard then $f(s)=g(s)$) $E=\Phi$ and, consequently, g=h.

Definition 3.13. If b, c are standard real numbers, b<c and f is a standard function from [b,c] into R then we say that $r=\int_{[b,c]} f(x)dx$ (x being a variable) and that "r is the Riemann integral of f on [b,c]" if and only if r is a standard real number such that if x_0, $x_1, \ldots, x_n$ are n+1 real numbers (n being a natural number) such that $b=x_0<x_1<\ldots<x_n=c$ and for all natural number j such that $1 \leq j \leq n$, x_{j-1} is almost x_j and ξ_j is a real number such that $x_{j-1} \leq \xi_j \leq x_j$, then r is almost $\sum_{j \in \{1,\ldots,n\}} f(\xi)(x_j - x_{j-1})$ (that is, $\int_{[b,c]} f(x)dx =^0 (f(\xi_1).(x_1-x_0)+\cdots+f(\xi_n).(x_n-x_{n-1})))$.

Remarks.

It is easy to prove that the class of real limited numbers, of real infinitesimal numbers, of real appreciable numbers, real standard numbers and the real unlimited numbers are not sets. And the same happens with many other class whose elements are sets.[31]

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[31]Since if, for example, there existed the set $\mathbf{C}_L$ of the limited complex numbers then $\{x \mid (x \in \mathbf{N}) \wedge (x \in \mathbf{C}_L)\}$ would be the set of the standard natural numbers, a set which does not exist.

If in definitions 3.6, 3.7, 3.8, 3.10 and 3.13 we replace the terms "unlimited" and "is almost" for, respectively, the terms "very advanced" and "very close", we get definitions often used by teachers of mathematics and by other professionals who, as physicists, chemists, engineers and economists, make use of mathematics, even thoughsuch definitions of "their" mathematical analysis (the standard one), what they are forgiven on account of how intuitive and simple they turn out to be. And the proverbial nuisanse of reasoning in standard analysis due to the "paces within limit" because of the resulting consideration of the traditional "very big" natural number n_0 and "very small" real numbers ϵ (epsilon) and δ (delta), can be avoided in nonstandard analysis and doing so get that reasoning be "more fluent" (the way algebra does) if, for example, the definitions expressed are used and, in particular, the Riemann integral is defined as the standard part of a certain finite addition (unlimited but finite).

The non standard definitions 3.6, 3.7, 3.8, 3.10 and 3.13 are equivalent to the respective standard (or classical), notwithstanding, to use such non standard difinitions, it is not necessary to take into account such equivalencies, neither is, to use the referred to classical definitions.

We will not prove here the before mentioned equivalences, but let us say at least, that are a consequence of the above mentioned priciples.

REFERENCES

[1] KRIVINE, Jean-Louis: "Théorie axiomatique des ensembles". Presses Universitaires de France. Paris. 1969.

[2] NELSON, Edward: "Internal set theory: A new approach to nonstandard analysis". Bulletin of the American Mathematical Society. Volume 83. Number 6. Page 1165-1198. November 1977.

[3] ROBINSON, Abraham: "Non standard Analysis". North Holland. 1974.

Manuel Suárez Fernández
Profesor de Matemáticas
de la
Facultad de Ciencias Económicas y Empresariales.
Campus Universitario
Plaza de la Universidad, 1
02071-ALBACETE

e-mail: msuarez @ ecem-ab.uclm.es