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APPLICATION OF MODIFIED JACOBI PARTIAL DIFFERENTIAL OPERATORS TO PARTIAL DIFFERENTIAL EQUATIONS

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1. Introduction.

Recently, Isaac I.H. Chen and T.W. Barrett [1] have solved some second order linear ordinary differential equations with the help of Bessel's ordinary differential operators which raise and lower the index of Bessel's function of the first kind. The object of this paper is to use the partial differential operators in connection with the modified Jacobi polynomials, which are regarded as generators of Lie-algebra, in the determination of some operational results and finally in the solution of these partial differential equations which can be factorised by means of the generators of the Lie-algebra for modified Jacobi polynomials.

Now, we consider

$$(1.1)\quad \begin{cases} \dfrac{\lambda y}{b-a}\left[(x-a)(b-x)\dfrac{\partial}{\partial x} - y(a+b-2x)\dfrac{\partial}{\partial y} + \left\{\alpha(b-x) - \beta(x-a)\right\}\right] X_n = X_{n+1} \\ \left(\dfrac{b-a}{\lambda}\, y^{-1}\dfrac{\partial}{\partial x}\right) X_n = X_{n-1} \end{cases}$$

where

$$X_n = F_n(\alpha-n, \beta-n; x)\, y^n \quad \text{and} \quad F_n(\alpha-n, \beta-n; x)$$

is the extended Jacobi polynomials.

If we substitute

$$(1.2) \quad \begin{cases} R = \dfrac{\lambda y}{b-a}\Big[(x-a)(b-x)\dfrac{\partial}{\partial x} - y(a+b-2x)\dfrac{\partial}{\partial y} \\ \qquad\qquad + \{\alpha(b-x) - \beta(x-a)\}\Big] \\ L = \dfrac{b-a}{\lambda}\, y^{-1}\dfrac{\partial}{\partial x} \end{cases}$$

then

$$(1.3) \quad \begin{pmatrix} 0 & R \\ L & 0 \end{pmatrix} \begin{pmatrix} f_{n+1}(x,y) \\ f_n(x,y) \end{pmatrix} = \begin{pmatrix} -(n+1)\, f_{n+1}(x,y) \\ (1+\alpha+\beta-n)\, f_n(x,y) \end{pmatrix}$$

where

$$f_n(x,y) = F_n(\alpha-n, \beta-n; x)\, y^n .$$

We also note that

$$(1.4) \quad LR\,(f_n(x,y)) = -(1+n)(1+\alpha+\beta-n)\, f_n(x,y)$$

which yields the following relation

$$(1.5) \quad \Big[(x-a)(b-x)\frac{\partial^2}{\partial x^2} + (2x-a-b)\, y\frac{\partial^2}{\partial x \partial y} + \{(\alpha+1)(b-x) - (\beta+1)(x-a)\}\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} - (\alpha+\beta)\Big] u = 0$$

Again

$$\left[R,L \right] = (\alpha+\beta) - 2y \frac{\partial}{\partial y}$$

where

$$\left[R,L \right] = RL - LR$$

Also we have

$$(1.6) \qquad \frac{\lambda}{b-a} y(x-a)(b-x) L - \frac{b-a}{\lambda} y^{-1} R$$

$$= y(a+b-2x) \frac{\partial}{\partial y} - \left\{ \alpha(b-x) - \beta(x-a) \right\}$$

$$\frac{\lambda}{b-a} y L = \frac{\partial}{\partial x} .$$

2. Derivation of Operational Formulas from Modified Jacobi Raising and Lowering Operators:

We consider the partial differential equation

$$(2.1) \qquad \frac{\lambda y}{b-a} \left[(x-a(b-x) \frac{\partial u}{\partial x} - y(a+b-2x) \frac{\partial u}{\partial y} \right.$$

$$\left. + \left\{ \alpha(b-x) - \beta(x-a) \right\} u \right] = f(x,y)$$

which is equivalent to

$$(x-a)(b-x) \frac{\partial u}{\partial x} - y(a+b-2x) \frac{\partial u}{\partial y} + \left\{ \alpha(b-x) - \beta(x-a) \right\} u$$

$$= \frac{(b-a)y^{-1}}{\lambda} f(x,y) .$$

Hence the corresponding system of ordinary differential equations is

$$(2.2)\qquad \frac{dx}{(x-a)(b-x)} = \frac{dy}{-y(a+b-2x)} = \frac{du}{\frac{b-a}{\lambda}\, y^{-1}\, f(x,y) - \{\alpha(b-x)-\beta(x-a)\}\, u}$$

Solving (2.2) we get

$$(x-a)(x-b)y = c_1$$

and

$$(2.3)\qquad u(x-a)^{\alpha}(x-b)^{\beta} = -\frac{1}{c_1}\,\frac{b-a}{\lambda}\int (x-a)^{\alpha}(x-b)^{\beta} \times f\left(x, \frac{c_1}{(x-a)(x-b)}\right) dx + c_2$$

$$= \psi_1(x,c_1) + c_2 \quad \text{(say)} .$$

Hence

$$(2.4)\qquad \frac{1}{R}\left[f(x,y)\right] = (x-a)^{-\alpha}(x-b)^{-\beta}\,\psi_1(x,c_1)\Big|_{c_1=(x-a)(x-b)} + (x-a)^{-\alpha}(x-b)^{-\beta}\,\Phi_1((x-a)(x-b)y)$$

where ψ_1 is given in (2.3) and Φ_1 is arbitrary.

Corollaries :

(i) If we take

$$y^{-1} f(x,y) = \text{a function of } x \text{ only}$$
$$= P(x)$$

then

$$\frac{1}{R} f(x,y) = (x-a)^{-\alpha} (x-b)^{-\beta} \left(\frac{a-b}{\lambda}\right) \int (x-a)^{\alpha-1} (x-b)^{\beta-1} \tag{2.5}$$
$$\times P(x)\, dx + (x-a)^{-\alpha} (x-b)^{-\beta} \Phi_1((x-a)(x-b)y)$$

where ψ_1 is given in (2.3) and Φ_1 is arbitrary.

(ii) If we take

$$\frac{(x-a)^{\alpha-2} (x-b)^{\beta-2}}{(2x-a-b)} f(x,y) = \text{a function of } y \text{ only}$$
$$= Q(y) \quad (\text{say})$$

then

$$\frac{1}{R} f(x,y) = (x-a)^{-\alpha} (x-b)^{-\beta} \left(\frac{a-b}{\lambda}\right) \int y^{-2} Q(y)\, dy \tag{2.6}$$
$$+ (x-a)^{-\alpha} (x-b)^{-\beta} \Phi_1((x-a)(x-b)y)$$

where Φ_1 is arbitrary.

Next we consider the partial differential equation

$$\frac{b-a}{\lambda} y^{-1} \frac{\partial u}{\partial X} = F(x,y) \tag{2.7}$$

which is equivalent to

$$\frac{du}{dx} + 0 \cdot \frac{\partial u}{\partial y} = \frac{\lambda}{b-a} \, y \, F(x,y) \ .$$

Hence the corresponding system of ordinary differential equation is

$$(2.8) \qquad \frac{dx}{1} = \frac{dy}{0} = \frac{du}{\left(\frac{\lambda}{b-a}\right) y \, F(x,y)}$$

Solving (2.8) we get

$$y = c_1$$

and

$$(2.9) \qquad u = \frac{\lambda c_1}{b-a} \int F(x,c_1) \, dx + c_2$$

$$= \psi_2(x,c_1) + c_2 \quad \text{(say)}$$

Hence

$$(2.10) \qquad \frac{1}{L}\,(F(x,y)) = \psi_2(x,c_1)\Big|_{c_1=y} + \Phi_2(y)$$

where ψ_2 is given in (2.9) and Φ_2 is arbitrary.

<u>Corollaries</u> :

(i) If $y\,F(x,y)$ = a function of x only

$= P'(x)$ (say)

then

$$(2.11) \qquad \frac{1}{L}\,(F(x,y)) \;=\; \frac{\lambda}{b-a}\int P'(x)\,dx + \Phi_2(y)$$

where Φ_2 is arbitrary.

(ii) If $F(x,y)$ = a function of y only

$= Q'(y)$ (say)

then

$$(2.12) \qquad \frac{1}{L}\,(F(x,y)) \;=\; \frac{\lambda}{b-a}\int y\,Q'(y)\,dy + \Phi_2(y)$$

where Φ_2 is arbitrary.

3. Application of Operational Formulas to Second Order Partial Differential Equations.

We consider the partial differential equation

$$(3.1) \qquad (x-a)(b-x)\frac{\partial^2 u}{\partial x^2} + (2x-a-b)\,y\,\frac{\partial^2 u}{\partial x \partial y}$$

$$+\left\{(\alpha+1)(b-x) - (\beta+1)(x-a)\right\}\frac{\partial u}{\partial x} + 2y\,\frac{\partial u}{\partial y} - (\alpha+\beta)\,u$$

$$= F(x,y)$$

Since

$$\left[\frac{b-a}{\lambda}\,y^{-1}\frac{\partial}{\partial x}\right]\left[\frac{\lambda}{b-a}\,y\left\{(x-a)(b-x)\frac{\partial}{\partial x} - y(a+b-2x)\frac{\partial}{\partial y}\right.\right.$$

$$\left.\left. + \alpha(b-x) - \beta(x-a)\right\}\right] =$$

$$= (x-a)(b-x)\frac{\partial^2}{\partial x^2} + (2x-a-b)\; y\,\frac{\partial^2}{\partial x \partial y}$$

$$+ \left\{(\alpha+1)(b-x) - (\beta+1)(x-a)\right\}\frac{\partial}{\partial x} + 2y\,\frac{\partial}{\partial y} - (\alpha+\beta) .$$

The above mentioned partial differential equation (3.1) takes the following form

$$LRu = F(x,y) .$$

It follows from (2.10) that

(3.2) $$Ru = \psi_2(x,c_1)\Big|_{c_1 = y} + \Phi_2(y)$$

(3.3) $$= f(x,y) \quad (\text{say})$$

where ψ_2 is given in (2.9) and Φ_2 is arbitrary.

Again it follows from (2.4) that

(3.4) $$u = (x-a)^{-\alpha}(x-b)^{-\beta}\,\psi_1(x,c_1)\Big|_{c_1 = (x-a)(x-b)y}$$

$$+ (x-a)^{-\alpha}(x-b)^{-\beta} + \Phi_1((x-a)(x-b)y)$$

where ψ_1 is given in (2.3) and Φ_1 is arbitrary.

On the otherhand, if we consider the equation

$$R\,L\,u = f(x,y)$$

which is equivalent to

$$(3.5)\qquad (x-a)(b-x)\frac{\partial^2 u}{\partial x^2} + (2x-a-b)\,y\,\frac{\partial^2 u}{\partial x \partial y}$$

$$+ \left\{(\alpha+1)(b-x) - (\beta+1)(x-a)\right\}\frac{\partial u}{\partial x} = f(x,y)$$

then it follows from (2.4) that

$$(3.6)\qquad Lu = (x-a)^{-\alpha}(x-b)^{-\beta}\,\psi_1(x,c_1)\Big|_{c_1} = (x-a)(x-b)y$$

$$+ (x-a)^{-\alpha}(x-b)^{-\beta}\,\Phi_1((x-a)(x-b)y)$$

$$(3.7)\qquad = F(x,y) \quad \text{(say)}$$

where ψ_1 is given in (2.3) and Φ_1 is arbitrary.

Hence by (2.10) it follows that

$$(3.8)\qquad u = \psi_2(x,c_1)\Big|_{c_1 = y} + \Phi_2(y)$$

where ψ_2 is given in (2.9) and Φ_2 is arbitrary.

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