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LOCAL GRADIENT ESTIMATES AND EXISTENCE OF MINIMAL SOLUTIONS OF SOME NONLINEAR ELLIPTIC EQUATIONS BLOWING UP ON THE BOUNDARY

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Resumen

En este trabajo mostramos una estimación local del gradiente de cualquier solución de la EDP cuasilineal

$$-\Delta u + H(x,u,Du) = f, \ \text{ in } \Omega, \tag{E}$$

bajo adecuadas hipótesis de estructura sobre los datos. Con la estimación anterior demostramos que existe una solución minimal de (E), explosiva sobre $\partial\Omega$.

Abstract

A local estimate of the gradient of every solution to the quasilinear PDE (E) is obtained in this work, under suitable structural conditions of the data. We show the existence of a minimal solution blowing up on the boundary with the help of this estimate.

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I. INTRODUCTION.

In this work we consider the quasilinear equation

$$-\Delta u + H(x, u, Du) = f \quad \text{in } \Omega, \tag{E}$$

where Ω is a bounded open set of $I\!R^N$, $N \geq 1$, $f \in L^{\infty}_{loc}(\Omega)$ and $H : \Omega \times I\!R \times I\!R^N \longrightarrow I\!R^+$ is a function which satisfies the following **structure conditions:**

E_1.- The function $x \longrightarrow H(x, r, q)$ belongs to $L^{\infty}_{loc}(\Omega)$, for all (r, q), and the function $(r, q) \longrightarrow H(x, r, q)$ is continuous for all x in Ω.

E_2.- $|H(x, r, q) - H(x, r, q')| \leq \rho_H(|q - q'|)$, almost everywhere in Ω, for all r and for $|q - q'|$ small, where $\rho_H : I\!R_+ \longrightarrow I\!R_+$ is an increasing continuous function with $\rho_H(0^+) = 0$.

E_3.- $H(x, r, q) - H(x, s, q) \geq \lambda[\beta(r) - \beta(s)]$, $r \geq s$ where $\beta : I\!R_+ \longrightarrow I\!R_+$ is an increasing continuous function with $\beta(0^+) = 0$.

REMARK 1.

Some particular choices of H of relevance in the applications are:

$$-\Delta u + \sum_{i=1}^{N} a_i(x) \frac{\partial u}{\partial x_i} + \beta(u) = f \tag{E_s}$$

$$-\Delta u + a(x) \nabla u^k + \beta(u) = f \tag{E_q}$$

where $\beta : I\!R_+ \longrightarrow I\!R_+$ is an increasing continuous function, with $\beta(0) = 0$; and $f \in L^{\infty}_{loc}(\Omega)$.

DEFINITION.

Let $1 < s \leq \infty$. We say that $u \in W^{2,s}_{loc}(\Omega)$ is a strong solution of the quasilinear equation

$$-\Delta u + H(x, u, Du) = f \quad \text{in } \Omega, \tag{E}$$

if

$$-\Delta u(x) + H(x, u(x), Du(x)) = f(x) \quad \text{almost everywhere in } \Omega.$$

We prove some results on the existence of strong solutions $u \in W^{2,\infty}_{loc}(\Omega)$ of the equation (E), blowing up on the boundary $\partial\Omega$. Note also that only the general condition $u = +\infty$ on $\partial\Omega$ is required.

In order to prove this result, we proceed in several steps.

STEP 1.

In this first step we obtain a local estimate of the gradient in L^∞ norm, useful in the proof of the theorems of existence, where the topological fixed points results of Leray-Schauder will be used (see [Gi-Tr], [La-Ur], [Se] and [Di-Le1]).

STEP 2.

In this step we construct a function $u : \Omega \longrightarrow \overline{I\!R}^+$ as the pointwise limit of solutions of a suitable sequence of approximated problems. This function u is a solution of our problem

$$\begin{aligned} -\Delta u + H(x,u,Du) &= f \quad \text{in } \Omega \\ u &= +\infty \quad \text{on } \partial\Omega \end{aligned} \tag{P}$$

STEP 3.

By using the local estimates obtained in the step 1 and some compactness arguments we pass to the limit in the approximated problems showing the existence of a strong solution of the problem (P).

Finally, we point out also that the **strong solution of** (P) **constructed via approximation is in addition the minimal classical solution of** (P).

II. THE RESULTS.

STEP I.- A LOCAL ESTIMATE OF THE GRADIENT.

THEOREM 1 [Di-Le1].

Let $u_1, u_2 \in W^{2,\infty}_{loc}(\Omega)$, be such that

$$-\Delta u_1(x) + H(x,u_1(x),Du_1(x)) \le -\Delta u_2(x) + H(x,u_2(x),Du_2(x)) \quad a.e. x \text{ in } \Omega, \tag{1}$$

and

$$\limsup \frac{u_1(x)}{u_2(x)} \le 1 \quad \text{when } dist(x,\partial\Omega) \longrightarrow 0. \tag{2}$$

Then

$$u_1(x) \le u_2(x) \text{ a.e. } x \text{ in } \Omega. \tag{3}$$

THEOREM 2 [Di-Le1].

Let us suppose that we have the structural hypotheses E_i, $i = 1, 2, 3$. Let $\gamma : I\!R^+ \longrightarrow I\!R^+$ be a continuous function with $\gamma(0^+) = 0$ and satisfying the following conditions:

$$\int^{\infty} \frac{ds}{(\Gamma(s))^{1/2}} < \infty, \qquad (\mathcal{H})_\gamma$$

where $\Gamma(r) = \int^r \gamma(s)ds$, satisfies the inequalities:

$$\mu\rho_H(r) \leq c\gamma(\Gamma^{-1}((2\nu)^{-1}((\delta/2)r)^2)), \quad r \geq 0, \tag{4}$$

$$\lambda(\beta(r) - \beta(s)) \geq (\nu + c)\gamma(r - s), \text{ if } r \geq s \geq 0, \tag{5}$$

for some positive constants c and ν.

Let $u \in W^{2,\infty}_{loc}(\Omega)$ be a strong solution of (E), with $f \in L^{\infty}_{loc}(\Omega)$ and $f \geq 0$, then for every $\Omega' \subset\subset \Omega$ we have

$$u(x) \leq C \quad \forall\, x \in \Omega' \tag{6}$$

where the constant C depends only on the structural data of (E), Ω' and on the $L^{\infty}(\Omega')$ norm of f.

Furthermore, if C_f is the constant of (6) related to the date f then,

$$f_1 \leq f_2 \text{ implies } C_{f_1} \leq C_{f_2}.$$

REMARK 2.

For the homogeneous choice $\gamma_m(r) = r^m$, $m > 0$, the condition $(\mathcal{H})_\gamma$ is satisfied if $m > 1$, taking (4) the form

$$\mu\rho_H(r) \leq \left[c(2\nu)^{-1}\left[1 + (\nu)^{-1}N(2)^{(m-1)/2}\right]^{m/2(m+1)}\right] r^{2m/(m+1)}, \tag{4'}$$

furthermore, for this particular choice of γ we have that the uniform estimate (6) can be written as:

$$u(x) \leq \left[(c_m(\nu))^{-1}\left(\frac{R^2 - |x - x_0|^2}{\delta_{m,\nu}R}\right)\right]^{2/(1-m)} + \beta^{-1}\left(\frac{||H(\cdot,0,0)||_{\infty,B_{R/2}(x_0)}}{\lambda}\right), x \in B_{R/2}(x_0), \tag{6'}$$

where $c_m(\nu)$ and $\delta_{m,\nu}$ are known constants that can be explicitly evaluated.

To conclude the first step, we show a local estimate of the gradient of any strong solution of (E). As we shall see later on, this will play an essential role on the proof of the existence of solutions of (P). We use the known method due to S.Bernstein in 1910 (see [La-Li], [Se]). It consists in the study of the function

$$\Phi(x) = |Du(x)|^2, \quad x \in \Omega.$$

For the sake of simplicity, we will suppose in the proof, that the data are smooth. We will omit the laborious processes of approximation that are needed by the statement. Since we are interested in local estimates, we may restrict ourselves to the class of functions of the form,

$$w(x) = \varphi(x)\Phi(x), \quad x \in \Omega,$$

with $\varphi \in C_c^\infty(\Omega)$ verifying $0 \leq \varphi \leq 1$ in Ω and $\varphi \equiv 1$ in some subset of Ω.

Our result is the following:

THEOREM 3 (LOCAL ESTIMATE OF THE GRADIENT).

Let $u \in W^{2,s}_{loc}(\Omega) \cap W^{1,\infty}_{loc}(\Omega)$, $1 \leq s \leq \infty$ be a strong solution of (E), with $f \in W^{1,\infty}_{loc}(\Omega)$, $f \geq 0$ and $H : \Omega \times \overline{I\!R}_+ \times I\!R^N \longrightarrow \overline{I\!R}_+$ a differentiable function satisfying the conditions,

$$H_r(x,r,q) \geq 0 \quad q \in I\!R^N \tag{7}$$

and

$$\lim_{|q|\to\infty} \frac{(H(x,r,q))^2 + 4Nq \cdot \frac{\partial H}{\partial x}(x,r,q)}{|q|^2 \left[1 + \left|\frac{\partial H}{\partial q}(x,r,q)\right|\right]} = \infty \tag{8}$$

for every $x \in \Omega, r \in \overline{I\!R}_+$.

Then for every $\delta > 0$ we have that:

$$|Du(x)| \leq c_\delta \ \ if \ \ x \in \Omega_{-\delta} \tag{9}$$

where $\Omega_{-\delta} = \{x \in \Omega : dist(x, \partial\Omega) > \delta\}$ and c_δ is a positive constant which depends only on the upper estimates of u, f and Df in $\Omega_{-\delta}$.

PROOF

As in [La-Li], we consider $\varphi \in C_c^\infty(\Omega)$ verifying,

$$0 \leq \varphi \leq 1, \quad \varphi \equiv 1 \text{ in } \Omega_{-\delta} \quad \text{and} \quad |\Delta\varphi| \leq C\varphi, \quad |D\varphi|^2 \leq C\varphi^2 \text{ in } \Omega,$$

where the constant C depends only on δ. Supposing that all the data are smooth, we can argue about the function

$$w(x) = \varphi(x)\Phi(x), \quad x \in \Omega.$$

Our argument will use the differential expression

$$\begin{aligned}\mathcal{L}w \equiv & - \Delta w + 2\varphi|D^2u|^2 + 2\varphi^{-1}(D\varphi \cdot Dw) + 2\varphi\left(Du \cdot \frac{\partial H}{\partial x}(x,u,Du)\right) \\ & + 2\varphi|Du|^2 H_r(x,u,Du) + Dw \cdot \frac{\partial H}{\partial q}(x,u,Du).\end{aligned}$$

Using the repeated index convention, we will have the relations

$$\Delta w = \Delta\varphi(D_iu)^2 + 4D_j\varphi D_iuD_{ij}u + 2\varphi(D_{ij}u)^2 + 2\varphi D_iuD_{ijj}u,$$

$$2\varphi|D^2u|^2 = 2\varphi(D_{ij}u)^2,$$

$$2\varphi^{-1}(D\varphi \cdot Dw) = \frac{2}{\varphi}(D_j\varphi)^2(D_ju)^2 + 4D_j\varphi D_iuD_{ij}u;$$

$$Dw \cdot \frac{\partial H}{\partial q}(x,u,Du) = \varphi\left(D\Phi \cdot \frac{\partial H}{\partial q}(x,u,Du)\right) + \Phi\left(D\varphi \cdot \frac{\partial H}{\partial q}(x,u,Du)\right).$$

Then, we can replace those terms in the definition of $\mathcal{L}$ and obtain

$$\begin{aligned}\mathcal{L}w = & - \Delta\varphi(D_iu)^2 - 2\varphi D_iuD_{ijj}u + 2\varphi^{-1}(D_j\varphi)^2(D_ju)^2 \\ & + 2\varphi\left(Du \cdot \frac{\partial H}{\partial x}(x,u,Du)\right) + 2\varphi\Phi H_r(x,u,Du) \\ & + \varphi\left(D\Phi \cdot \frac{\partial H}{\partial q}(x,u,Du)\right) + \Phi\left(D\varphi \cdot \frac{\partial H}{\partial q}(x,u,Du)\right).\end{aligned} \tag{10}$$

On the other hand, differentiating the equation (E) with respect to x_i and after multiplying by $2\varphi D_i u$, we get

$$-2\varphi D_i u D_{ijj} u = 2\varphi Du \cdot Df - 2\varphi \left(Du \cdot \frac{\partial H}{\partial x}(x,u,Du) \right) - 2\varphi \Phi H_r(x,u,Du) - \varphi \left(D\Phi \cdot \frac{\partial H}{\partial q}(x,u,Du) \right)$$

after some simple simplifications, this last term leads to the expression

$$\mathcal{L}w = -(\Delta\varphi)\Phi + 2\varphi(Du \cdot Df) + |Du|^2 \left(D\varphi \cdot \frac{\partial H}{\partial q}(x,u,Du) \right) + 2\varphi^{-1}|D\varphi|^2\Phi.$$

As w is continuous in $\overline{\Omega}$ and vanishes on $\partial\Omega$, there exists $x_0 \in \Omega$ such that $w(x) \leq w(x_0)$, $\forall x \in \Omega$. Without loss of generality, we can suppose that x_0 belongs to the support of φ, since, on the contrary the estimate (9) results clear. We will show that $w(x_0)$ is bounded by a constant c_δ, in terms of the statement, since under these conditions we will have

$$|Du(x)|^2 \leq w(x) \leq w(x_0) \leq c_\delta, \quad \forall x \in \Omega_{-\delta}, \tag{11}$$

which is what we want to prove in (9).

To simplify the notation, we can assume particularized all the computations which follow in x_0 and we omit the x_0 term. As x_0 is an stationary point of the function w, the definition of $\mathcal{L}w$ leads to

$$\begin{aligned}\mathcal{L}w \equiv & -\Delta w + 2\varphi|D^2u|^2 + 2\varphi\left(Du \cdot \frac{\partial H}{\partial x}(x,u,Du) \right) + 2\varphi\Phi H_r(x,u,Du) \\ & = -(\Delta\varphi)\Phi + 2\varphi(Du \cdot Df) + \Phi\left(D\varphi \cdot \frac{\partial H}{\partial q}(x,u,Du) \right) + 2\varphi^{-1}|D\varphi|^2\Phi.\end{aligned} \tag{12}$$

Now, we try to get an upper bound of the right hand size of (12). By using $|\Delta\varphi| \leq C\varphi$, $|D\varphi|^2 \leq C\varphi^2$ and the Cauchy-Schwarz inequality, we get the relations

$$2\varphi(Du \cdot Df) \leq 2\varphi|Du|\,|Df| \leq 2\|Df\|\varphi\Phi^{1/2},$$

$$-(\Delta\varphi)\Phi \leq |\Delta\varphi|\Phi \leq C\varphi\Phi,$$

and

$$\begin{aligned}\Phi\left(D\varphi \cdot \frac{\partial H}{\partial q}(x,u,Du) \right) &\leq |D\varphi| \left| \frac{\partial H}{\partial q}(x,u,Du) \right| \Phi \\ &\leq C\varphi^2 \left| \frac{\partial H}{\partial q}(u,x,Du) \right| \Phi \leq C\varphi \left| \frac{\partial H}{\partial q}(x,u,Du) \right| \Phi\end{aligned}$$

(in this last estimate we use that $0 \leq \varphi \leq 1$).

If we call $\mathcal{C}$ the maximum of the constants $2\|Df\|$ and $3C$, the expressions (7) and (12) lead to

$$-\Delta w + 2\varphi|D^2u|^2 + 2\varphi\left(Du\cdot\frac{\partial H}{\partial x}(u,x,Du)\right) \leq \mathcal{C}\varphi\Phi^{1/2} + \mathcal{C}\varphi\Phi$$

$$+\mathcal{C}\varphi\left|\frac{\partial H}{\partial q}(x,u,Du)\right|\Phi.$$

The nature of the point x_0 implies that

$$2\varphi|D^2u|^2 + 2\varphi\left(Du\cdot\frac{\partial H}{\partial x}(x,u,Du)\right) \leq \mathcal{C}\varphi\Phi + \mathcal{C}\varphi\Phi^{1/2} + \mathcal{C}\varphi\left|\frac{\partial H}{\partial q}(x,u,Du)\right|\Phi. \tag{13}$$

Due to Cauchy-Schwarz and Young inequalities and the equation (E) we have

$$2\varphi|D^2u|^2 \geq \frac{2\varphi}{N}(\Delta u)^2 = \frac{2\varphi}{N}\{H(x,u,Du) - f\}^2$$

$$\geq (1/2N)\varphi(H(x,u,Du))^2 - (1/N)\varphi f^2$$

$$= (1/2N)\varphi(H(x,u,Du))^2 - (1/N)\varphi\|f\|^2.$$

Now, choosing c as the maximum of the constants $2\|f\|^2$ and $2N\mathcal{C}$, from the previous inequality and (14) we have

$$\varphi(H(x,u,Du))^2 + 4N\varphi\left(Du\cdot\frac{\partial H}{\partial x}(x,u,Du)\right) \leq c\varphi + c\varphi\Phi^{1/2}$$

$$+ c\varphi\Phi + c\varphi\Phi\left|\frac{\partial H}{\partial q}(x,u,Du)\right|,$$

and therefore,

$$(H(x,u,Du))^2 + 4N\left(Du\cdot\frac{\partial H}{\partial x}(x,u,Du)\right) - c\Phi\left|\frac{\partial H}{\partial q}(x,u,Du)\right| - c\Phi - c\Phi^{1/2} \leq c \tag{15}$$

(recall that $\varphi(x_0) > 0$).

Finally, we can use a coercitivity argument to obtain (11). In fact, we consider the continuous function $\mathcal{F} : I\!R^N \longrightarrow I\!R$, defined by

$$\mathcal{F}(q) \equiv (H(x_0, u(x_0), q))^2 + 4N \left(q \cdot \frac{\partial H}{\partial x}(x_0, u(x_0), q) \right)$$
$$- c|q|^2 \left| \frac{\partial H}{\partial q}(x_0, u(x_0), q) \right| - c|q|^2 - c|q|$$

from which (14) takes the form

$$\mathcal{F}(|Du(x_0)|) \leq c.$$

The regularity hypotheses of H and (8) imply the relations:

$$\lim_{|q| \to 0} |q|^2 \left| \frac{\partial H}{\partial q}(x, r, q) \right| = 0,$$

and

$$\lim_{|q| \to \infty} \frac{(H(x, r, q))^2 + 4Nq \cdot \frac{\partial H}{\partial x}(x, r, q)}{|q|^2 \left| \frac{\partial H}{\partial q}(x, r, q) \right|} = \infty.$$

Thus, we have proved that,

$$\mathcal{F}(0) = (H(x_0, u(x_0), 0))^2 \text{ and } \lim_{|q| \to \infty} \mathcal{F}(q) = \infty,$$

concluding (11) from the bound of the right hand size of (14).

REMARK 3.

The last result coincides with the result obtained in [La-Li, Appendix] for the homogeneous choice $H(x, r, q) = |q|^k + \lambda r, k > 1$ and $\lambda > 0$, considered there. Results about the sublinear case are treated in [Ba-Di-Di].

REMARK 4.

The hypotheses (8) and (9) are close to the ones used in [Se], [Li1] and [Li2].

REMARK 5.

Using approximation methods, we can extend the last result to the choice $H(x, r, q) = a(x)|q|^k + \lambda\beta(r)$, where $k > 0, \lambda > 0$, β is an increasing continuous function and $a \in L^\infty(\Omega)$ is a continuous function.

STEP II.- CONSTRUCTION OF THE FUNCTION U

PROPOSITION 4.

Let Ω be a $C^{0,1}$ domain in $I\!R^N$, suppose also that H verifies the structural hypothesis E_i, $i = 1, 2, 3$, (7) and (8),

$$H(x, 0, 0) = 0 \quad \forall x \in \Omega$$

and $H(x, r, q) \leq \eta(r)(1+|q|^2)$. Suppose furthermore that for some non decreasing function $\eta : I\!R^+ \longrightarrow I\!R^+$, $(\mathcal{H})_\gamma$, (4) and (5) hold.

If $f \in W^{1,\infty}_{loc}(\Omega), f \geq 0$, then the approximated problem

$$\begin{aligned} -\Delta u_R + H(x, u, Du_R) = f_R, \quad \text{a.e. in} \quad \Omega \\ u_R = R, \quad \text{on} \quad \partial\Omega\,, \end{aligned} \qquad (P_R)$$

with $f_R(x) = \min\{f(x), R\}$, has a unique solution $u_R \in \cap C^2(\overline{\Omega})$.

PROOF.

Clearly $f_R \in W^{1,\infty}(\Omega)$, the sequence $\{f_R\}$ is increasing in R and $f_R \longrightarrow f$, in $W^{1,\infty}_{loc}(\Omega)$.

On the other hand, the properties of H and the bounded nature of the problem (P_R) allow us to say (via sub and super solutions method) that (P_R) admits, at least, one classical solution $u_R \in C^2(\overline{\Omega})$ (see [AmH] and [Gi-Tr]). Also, from Theorem 1, it follows that u_R is the unique solution.

Moreover, the Comparison Theorem shows also that the sequence $\{u_R\}_R$ is increasing on R. That is,

$$R_1 < R_2 \Longrightarrow u_{R_1} < u_{R_2}, \quad \text{in} \ \Omega.$$

Now let $x \in \Omega$, fixed but arbitrary. We define $u(x)$ by

$$u(x) = \lim_{R \to \infty} u_R(x) = \sup_{R>0} u_R(x).$$

Since the sequence $\{u_R\}_R$ is bounded by a constant which depends only on L^∞ norm of f (see Theorem 2), the monotonicity of the sequence $\{u_R\}_R$ allow us to conclude that u is a well defined function in the open set Ω.

STEP III.- EXISTENCE OF THE STRONG SOLUTIONS OF (P)

THEOREM 5.

Let us suppose that H verifies the structural hypotheses E_i, $i = 1, 2, 3$, (7), (8),

$$H(x, 0, 0) = 0, \quad \forall x \in \Omega$$

and $H(x, r, q) \leq \eta(r)(1 + |q|^2)$, for some increasing function $\eta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$. Suppose also that $(\mathcal{H})_\gamma$, (4) and (5) hold.

If $f \in W^{1,\infty}_{loc}(\Omega)$, $f \geq 0$, then the problem (P) has, at least, one strong solution $u \in W^{2,\infty}_{loc}(\Omega)$.

PROOF.

We will prove that the function u constructed in the previous step is a strong solution of (P). To do this, it is enough to prove that:

a) $u \in W^{2,p}_{loc}(\Omega)$ $1 < p < \infty$.

b) u verifies, in strong sense, the equation (E).

c) u satisfies the "boundary condition".

Proof of a).

Let $\Omega' \subset\subset \Omega$ be fixed but arbitrary, we shall prove that $u \in W^{2,p}(\Omega')$, $1 < p < \infty$.

From the definition of u and Dominated Convergence Theorem, we obtain that:

$$u_R \longrightarrow u, \quad \text{in } L^p(\Omega'),$$

On the other hand, the local estimates of u_R in $W^{1,\infty}(\Omega')$ and the hypotheses $H(x, r, q) \leq \eta(r)(1 + |q|^2)$, allow us to obtain that:

$$u \in W^{1,\infty}(\Omega')$$

and

$$u_R \longrightarrow u, \quad \text{in } W^{1,\infty}(\Omega') \qquad (i)$$

Furthermore, since

$$\Delta u_R = H(x, u_R, Du_R) - f_R$$

is bounded in $L^\infty(\Omega')$ independent of R. From Theorem 9.11, [Gi-Tr], we have that if $\Omega'' \subset\subset \Omega'$ then:

$$||u_R||_{W^{2,p}(\Omega'')} \leq C\{||u_R||_{L^p(\Omega')} + ||\Delta_R||_{L^p(\Omega')}\} \leq M,\ 1 < p < \infty, \qquad (ii)$$

where M is independent of R.

Therefore, from (i) and (ii) (see [Br]), we have that

$$u \in W^{2,p}(\Omega''),\ \ 1 < p < \infty$$

and

$$u_R \longrightarrow u,\ \text{ in } W^{2,p}(\Omega''),\ \ 1 < p < \infty \qquad (iii)$$

Proof of b).

Let $\Omega' \subset\subset \Omega$.

From (iii), the continuity of the function H and from the convergence

$$f_R \longrightarrow f,\ \text{ in }\ L^p(\Omega'),$$

we have that

$$\Delta u_R - H(x, u_R, Du_R) - f_R$$

converges in $L^p(\Omega')$, $1 < p < \infty$, to

$$\Delta u - H(x, u, Du) - f$$

now, from the equation of the problem (P_R) and the uniqueness of the limit, it follows that u verifies the equation (E) , in $L^p(\Omega'), 1 < p < \infty$, and therefore u is a strong solution of the equation. Moreover, from the equation (E) we obtain that $u \in W^{2,\infty}_{loc}(\Omega)$.

Proof of c).

Finally, the condition

$$\lim_{dist(x,\partial\Omega)\longrightarrow 0} u(x) = \infty$$

follows from the monotonicity of the sequence $\{u_R\}_R$, the fact that $u_R = R$ on $\partial\Omega$ and from the continuity of u in Ω.
This concludes the proof.

PROPOSITION 6.

The function costructed in (15) is the minimal solution of (E). That is, if v is a strong solution of (E) such that,

$$\lim_{dist(x,\partial\Omega)\longrightarrow 0} v(x) = \infty$$

then,

$$u(x) \leq v(x), \quad \text{a.e. } x \text{ in } \Omega.$$

PROOF

If v is a strong solution of the problem (P), then from Theorem 1,

$$u_R(x) \leq v(x), \quad \text{a.e. } x \text{ in } \Omega.$$

therefore,

$$sup\{u_R(x) : R > 0\} = u(x) \leq v(x), \quad \text{a.e. } x \text{ in} \Omega.$$

REMARK 6.

Since $f \in C^\alpha(\Omega')$, for some $\alpha \in (0,1)$, and $x \longrightarrow H(x,\cdot,\cdot)$ is Hölder continuous (see Theorem 3), we have that $\Delta u \in C^\beta(\Omega')$, $\forall \Omega' \subset\subset \Omega$, for some $\beta \in (0,1)$. From Schauder's Theory (see for instance [Gi-Tr]) we also have that the solution u of the problem (P), is a **classical solution.**

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