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ON THE WATSON-WRIGTH TRANSFORMATION ON CERTAIN WEIGHTED L_p-SPACES

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Abstract .

In this paper we study the boundedness and ranges of the Watson-Wrigth integral transformation on certain weigthed L_p-spaces by using Mellin multipliers techniques. Moreover this integral transforms is extended to several spaces of generalized functions.

Key words : Watson-Wright transformation, weighted L_p spaces, Mellin multiplier techniques, Bessel functions.

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1. Introduction .

The function $w_{\nu,\nu'}(x)$ was defined by G.N. Watson [19,p.308] in 1931 by means of the integral relation

$$w_{\nu,\nu'}(x) = x^{1/2}\int_0^\infty t^{-1}J_\nu(xt)J_{\nu'}(\tfrac{1}{t})dt \quad , \quad \nu,\nu'>-\tfrac{1}{2} \tag{1}$$

where as usual J_ν denotes the Bessel function of the first kind and order ν.

The integral of the right of (1) was first evalued by C.V.H. Rao [11] and that it playes the role of a transform was studied in detail by K.P. Bhatnagar [3].

G.S. Olkha and P.N. Rathie ([9] and [10]) introduced the following generalization of the Watson function $w_{\nu,\nu'}$,

$$w^{\mu,\mu'}_{\nu,\nu',\lambda}(x) = x^{1/2}\int_0^\infty t^{\lambda-1} J^{\mu}_{\nu}(xt) J^{\mu'}_{\nu'}(\tfrac{1}{t})dt \qquad (2)$$

where the integral on the right exists provided that

a) ν, $\nu' > -1$,

b) $0<\mu'<1$ or $\mu'=1$ and $\lambda+\nu+\frac{1}{2}>0$, and

c) $0<\mu<1$ or $\mu=1$ and $-\lambda+\nu'+\frac{1}{2}>0$.

Here J^{μ}_{ν} denotes the Wright's generalized Bessel function (see [20]).Note that $w^{1,1}_{\nu,\nu',0}=w_{\nu,\nu'}$.

Taking the functions defined by (2) as kernels, the last authors defined the Watson-Wrigth transform by the integral equation

$$H^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}(f)(y) = \int_0^\infty (xy)^{\sigma} w^{\mu,\mu'}_{\nu,\nu',\lambda}(x^2y^2)f(x)dx \qquad (3)$$

under the above restrictions and $\sigma\in\mathbb{R}$ suitable. In [10] an inversion formula for (3) was established.

P.G. Rooney [13] considered certain weighted L_p-spaces denoted by $\mathcal{L}_{\gamma,p}$ and defined for real γ and $1\leq p<\infty$ to consist of those complex valued functions f, measurable on $(0,\infty)$ and such that $\|f\|_{\gamma,p}<\infty$, where

$$\|f\|_{\gamma,p} = \left\{\int_0^\infty |x^{\gamma}f(x)|^p x^{-1}dx\right\}^{1/p}$$

In this paper we study the boundedness and ranges of the Watson-Wright transformation on the spaces $\mathcal{L}_{\gamma,p}$, by using the techniques developed by Rooney [13]. Moreover we define the generalized $H^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}$-transformation on the spaces $\mathcal{L}'_{\gamma,p}$, the dual space of $\mathcal{L}_{\gamma,p}$.

Our notation will be that of [13]. Specifically, we say a function m belongs to $\mathcal{A}$ if there are extended real numbers $\alpha(m)$ and $\beta(m)$, with

$\alpha(m)<\beta(m)$, so that

i) $m(s)$ is analytic in the strip $\alpha(m)<\mathrm{Re}\ s<\beta(m)$,

ii) in every closed sub-strip, $\sigma_1\leq \mathrm{Re}\ s\leq\sigma_2$, where $\alpha(m)<\sigma_1\leq\sigma_2<\beta(m)$, $m(s)$ is bounded, and

iii) for $\alpha(m)<\sigma<\beta(m)$, $|m'(\sigma+it)|=O(|t|^{-1})$, as $|t|\to\infty$.

Also, for X and Y Banach spaces, [X,Y] denotes the bounded linear transformations from X to Y; [X,X] will be shortened to [X].

Let $\mathcal{S}_\gamma$ denote the collection of operators $S\in[\mathcal{L}_{\gamma/2,2}]$, which are defined in terms of a kernel k, associated with S, by an equation of the form

$$(Sf)(x) = x^{-(\mu-1)/2}\frac{d}{dx}x^{-(\mu-1)/2}\int_0^\infty k(xt)f(t)t^{-1}dt$$

The following formula (see [10,p.52]) will be useful in the sequel:

$$\mathcal{M}(w^{\mu,\mu'}_{\nu,\nu',\lambda}(z))(s) = \frac{2^{2s-\lambda-1}\Gamma(\frac{s}{2}+\frac{\nu}{2}+\frac{1}{4})\Gamma(\frac{s}{2}+\frac{\nu'}{2}-\frac{\lambda}{2}+\frac{1}{4})}{\Gamma(1+\nu-\mu(\frac{s}{2}+\frac{\nu}{2}+\frac{1}{4}))\Gamma(1+\nu'-\mu'(\frac{s}{2}+\frac{\nu'}{2}-\frac{\lambda}{2}+\frac{1}{4}))} \quad (4)$$

for $\mathrm{Re}\ s>\max(-\nu-\frac{1}{2},-\nu'+\lambda-\frac{1}{2})$, $\mu>0$ and $\mu'>0$, where $\mathcal{M}$ represents the Mellin transformation.

2. The boundedness and the range of the Watson-Wright transformation.

We now study the boundedness and the range of the $H^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}$-transformation on the spaces $\mathcal{L}_{\gamma,p}$ by using the technique developed in [13] by P.G. Rooney.

Define firstly the function

$$\eta^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}(x) = \int_0^x t^\sigma w^{\mu,\mu'}_{\nu,\nu',\lambda}(t^2)dt$$

According to well known properties of the Mellin transformation and by (4) one follows,

$$\mathcal{M}(\eta^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}(x))(-\tfrac{1}{2}+it) = \Omega^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}(t)/(\tfrac{1}{2}-it) \quad , \text{ for } t \in (-\infty,+\infty)$$

where

$$\Omega^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}(t) = \frac{2^{it+\sigma-\lambda-(s/2)}\Gamma(\frac{\sigma}{4}+\frac{\nu}{2}+\frac{s}{8}+i\frac{t}{4})\Gamma(\frac{\sigma}{4}+\frac{\nu'\lambda}{2}-\frac{s}{2}+\frac{s}{8}+i\frac{t}{4})}{\Gamma(1+\nu-\mu(\frac{\sigma}{4}+\frac{\nu}{2}+\frac{s}{8}+i\frac{t}{4}))\Gamma(1+\nu'-\mu'(\frac{\sigma}{4}+\frac{\nu'\lambda}{2}-\frac{s}{2}+\frac{s}{8}+i\frac{t}{4}))}$$

with $\max(-2\nu-1-\sigma,-2\nu'+2\lambda-1-\sigma)<\frac{1}{2}$.

By [4,1.18(6)],

$$|\Gamma(x+iy)| \cong (2\pi)^{1/2}|y|^{x-(1/2)}\exp(-\pi|y|/2) \ , \text{ as } |y|\to\infty \tag{5}$$

uniformly in x for x in any finite interval, and thus

$$|\Omega^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}(t)| \cong M|t|^{\zeta}\exp(-\pi|t|(2-\mu-\mu')/8), \text{ as } |t|\to\infty$$

for a suitable $M>0$ and being $\zeta=\frac{1}{2}(\frac{s}{2}+(\mu+\mu')(\frac{\sigma}{2}+\frac{s}{4})+\nu(1+\mu)+(\nu'-\lambda)(1+\mu')$.
Hence, $\Omega^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}(t)$ is bounded , provided that $\mu+\mu'<2$ or $\mu+\mu'=2$ and $\frac{s}{2}+\sigma+(\mu+1)\frac{\nu}{2}-\frac{1}{2}(\mu-3)(\nu'-\lambda)\leq 0$.

Therefore, by virtue of Lemma 4.1 [13], the $H^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}$-transformation belongs to $\mathcal{S}_1$ under the above restrictions.

Also, the Fourier-cosine transformation is a member of $\mathcal{S}_1$, being

$$\mathcal{M}(F_c f)(\tfrac{1}{2}+it) = w(t)\mathcal{M}(f)(\tfrac{1}{2}-it) \quad , \text{ for } t \in (-\infty,+\infty)$$

where $w(t)=2^{it}\Gamma(\frac{1}{2}(\frac{1}{2}+it))/\Gamma(\frac{1}{2}(\frac{1}{2}-it))$ and F_c denotes the Fourier-cosine transformation.

We now consider the function

$$m^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}(s)=\frac{2^{\sigma-\lambda-(s/2)}\Gamma(\frac{s}{4}+\frac{\nu}{2}+\frac{1}{4}+\frac{\sigma}{4})\Gamma(\frac{s}{4}+\frac{\nu'\lambda}{2}-\frac{1}{2}+\frac{1}{4}+\frac{\sigma}{4})\Gamma(\frac{1}{2}(1-s))}{\Gamma(1+\nu-\mu(\frac{s}{4}+\frac{\nu}{2}+\frac{1}{4}+\frac{\sigma}{4}))\Gamma(1+\nu'-\mu'(\frac{s}{4}+\frac{\nu'\lambda}{2}-\frac{1}{2}+\frac{1}{4}+\frac{\sigma}{4}))\Gamma(\frac{1}{2}s)} \tag{6}$$

If $\max(-2\nu-1-\sigma,-2\nu'+2\lambda-1-\sigma)<1$ then $m^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}(s)$ is analytic in the strip

$$\max(-2\nu-1-\sigma,-2\nu'+2\lambda-1-\sigma)<\text{Re } s<1$$

Moreover, by using again (5), if

$$\max(-2\nu-1-\sigma,-2\nu'+2\lambda-1-\sigma)<x_1\leq x\leq x_2<1,$$

then $m^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}(x+it)$ is bounded provided that $\mu+\mu'<2$ or $\mu=\mu'=1$ and

$\frac{a}{2}+\sigma+\nu+\nu'-\lambda<-2$

From (4),

$$\frac{d}{ds}\, m^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}(s) =$$

$$= 2^{\sigma-\lambda-(7/2)} m^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}(s)(\psi(\frac{a}{4}+\frac{\nu}{2}+\frac{1}{4}+\frac{\sigma}{4})+\psi(\frac{a}{4}+\frac{\nu'\lambda}{2}-\frac{}{2}+\frac{1}{4}+\frac{\sigma}{4})-2\psi(\frac{1}{2}(1-s))+$$

$$+\mu\psi(1+\nu-\mu(\frac{a}{4}+\frac{\sigma}{4}+\frac{\nu}{2}+\frac{1}{4}))+\mu'\psi(1+\nu'-\mu'(\frac{a}{4}+\frac{\sigma}{4}+\frac{1}{4}-\frac{\lambda}{2}+\frac{\nu'}{2}))-2\psi(\frac{1}{2}s))$$

where $\psi(s)=\Gamma'(s)/\Gamma(s)$. But from [4,1.18(7)], as $s\to\infty$ in $|\arg\ s|\leq\pi-\delta$, where $0<\delta<\pi$,

$$\psi(s) = \log s - (2s)^{-1} + O(|s|^{-2}) \qquad (7)$$

Hence, if $\mu+\mu'<2$ or $\mu=\mu'=1$ and $\frac{a}{2}+\sigma+\nu+\nu'-\lambda<-2$, then

$$\frac{d}{dt} m^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}(x+it) = O(|t|^{-1}) \quad \text{as} \quad |t|\to\infty,$$

for every x such that $\max(-2\nu-1-\sigma,,-2\nu'-2\lambda-1-\sigma)<x<1$.

Therefore, $m^{\mu,\mu',\sigma}_{\nu,\nu',\lambda} \in \mathscr{A}$ with

$$\alpha(m)=\max(-2\nu-1-\sigma,,-2\nu'+2\lambda-1-\sigma) \text{ and } \beta(m)=1,$$

provided that $\mu+\mu'<2$ or $\mu=\mu'=1$ and $\frac{a}{2}+\sigma+\nu+\nu'-\lambda<-2$.

Recall that if $\frac{1}{2}>\alpha(m)$, then

$$\mathcal{M}(H^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}f)(\frac{1}{2}+it) = \Omega^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}(t)(\mathcal{M}f)(\frac{1}{2}-it)$$

for $t \in (-\infty,+\infty)$ and $f \in \mathscr{L}_{1/2,2}$. Also

$$m^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}(\frac{1}{2}+it) = \Omega^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}(t)/w(t) \quad , \quad t \in (-\infty,+\infty)$$

Now, according to [12, Theorem 1], if $1<p<\infty$, $\max(\frac{1}{p},1-\frac{1}{p})\leq\gamma<1$, $p\leq q\leq\frac{1}{1-\gamma}$, then $F_c\in[\mathscr{L}_{\gamma,p},\mathscr{L}_{1-\gamma,q}]$. Hence, from [13, Theorem 3], if $1<p<\infty$, $\max(\frac{1}{p},1-\frac{1}{p})\leq\gamma<1$, $p\leq q\leq\frac{1}{1-\gamma}$, $\alpha(m)<\frac{1}{2}$, $\mu+\mu'<2$ or $\mu=\mu'=1$ and $\frac{a}{2}+\sigma+\nu+\nu'-\lambda<-2$, and $1-\beta(m)<\gamma< <1-\alpha(m)$, then $H^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}$ can be extended as a member of $[\mathscr{L}_{\gamma,p},\mathscr{L}_{1-\gamma,q}]$ and $H^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}(\mathscr{L}_{\gamma,p})\subseteq F_c(\mathscr{L}_{\gamma,p})$.

We now define

$$\ell(s) = 2^{\lambda_1-\lambda+\sigma-\sigma_1} \frac{\Gamma(\frac{s}{4}+\frac{\nu}{2}+\frac{1}{4}+\frac{\sigma}{4})\Gamma(\frac{s}{4}+\frac{\nu'}{2}-\frac{\lambda}{2}+\frac{1}{4}+\frac{\sigma}{4})}{\Gamma(1+\nu-\mu(\frac{s}{4}+\frac{\nu}{2}+\frac{1}{4}+\frac{\sigma}{4}))\Gamma(1+\nu'-\mu'(\frac{s}{4}+\frac{\nu'}{2}-\frac{\lambda}{2}+\frac{1}{4}+\frac{\sigma}{4}))} \cdot$$

$$\cdot \frac{\Gamma(1+\nu_1-\mu_1(\frac{s}{4}+\frac{\nu_1}{2}+\frac{1}{4}+\frac{\sigma_1}{4}))\Gamma(1+\nu_1'-\mu_1'(\frac{s}{4}+\frac{\nu_1'}{2}-\frac{\lambda_1}{2}+\frac{1}{4}+\frac{\sigma_1}{4}))}{\Gamma(\frac{s}{4}+\frac{\nu_1}{2}+\frac{1}{4}+\frac{\sigma_1}{4})\Gamma(\frac{s}{4}+\frac{\nu_1'}{2}-\frac{\lambda_1}{2}+\frac{1}{4}+\frac{\sigma_1}{4})}$$

where $(\nu,\nu',\mu,\mu',\lambda)$ and $(\nu_1,\nu_1',\mu_1,\mu_1',\lambda_1)$ satisfie the conditions a), b) and c).

By using again (5) and (7) we can prove that if $\max(-2\nu-1-\sigma,-2\nu'+2\lambda-1-\sigma)$ is less than $\min(\frac{4}{\mu_1}(1+\nu_1)-\sigma_1-2\nu_1-1, \frac{4}{\mu_1'}(1+\nu_1')-\sigma_1-1+2\lambda_1-2\nu_1')$, $\mu_1+\mu_1'>\mu+\mu'$ or $\mu_1+\mu_1'=\mu+\mu'$ and $\frac{\lambda_1}{2}(1+\mu_1')+\frac{\nu_1}{2}(1-\mu_1)+\frac{\nu_1'}{2}(1-\mu_1')-\frac{\sigma_1}{2}+\frac{\sigma}{2}-\frac{\lambda}{2}(1+\mu')-\frac{\nu}{2}(1-\mu)-\frac{\nu'}{2}(1-\mu')++(\mu+\mu')\frac{\sigma-\sigma_1}{4}<-1$, then $\ell \in \mathcal{A}$ with $\alpha(\ell)=\max(-2\nu-1-\sigma,-2\nu'+2\lambda-1-\sigma)$ and $\beta(\ell)$ = $=\min(\frac{4}{\mu_1}(1+\nu_1)-\sigma_1-2\nu_1-1,\frac{4}{\mu_1'}(1+\nu_1')-\sigma_1-1+2\lambda_1-2\nu_1')$.

Also

$$\ell(\tfrac{1}{2}+it) = \Omega^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}(t)/\Omega^{\mu_1,\mu_1',\sigma}_{\nu_1,\nu_1',\lambda_1}(t) \quad \text{, for every } t \in (-\infty,+\infty)$$

By invoking again [13, Theorem 3], if $1<p<\infty$, $\max(\frac{1}{p},1-\frac{1}{p})\leq\gamma<1$, $p\leq q\leq\frac{1}{1-\gamma}$,, $\alpha(\ell)<\frac{1}{2}<\beta(\ell)$, $1-\beta(\ell)<\gamma<1-\alpha(\ell)$, $\mu_1+\mu_1'>\mu+\mu'$ or $\mu_1+\mu_1'=\mu+\mu'$ and $\frac{\lambda_1}{2}(1+\mu_1')+\frac{\nu_1}{2}(1-\mu_1)+\frac{\nu_1'}{2}(1-\mu_1')-\frac{\sigma_1}{2}+\frac{\sigma}{2}-\frac{\lambda}{2}(1+\mu')-\frac{\nu}{2}(1-\mu)-\frac{\nu'}{2}(1-\mu')+(\mu+\mu')\frac{\sigma-\sigma_1}{4}<-1$, then

$$H^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}(\mathcal{L}_{\gamma,p}) \subseteq H^{\mu_1,\mu_1',\sigma_1}_{\nu_1,\nu_1',\lambda_1}(\mathcal{L}_{\gamma,p})$$

Moreover, if $\mu=\mu_1$, $\mu'=\mu_1'$,

$$(1-\mu)(\nu_1-\nu)+(1-\mu')(\nu_1'-\nu')+\lambda_1(1+\mu')-\lambda(1+\mu)+(\frac{\mu+\mu'}{2}+1)(\sigma-\sigma_1)=0$$

and

$$\max(-2\nu_1-1-\sigma_1,-2\nu_1'+2\lambda_1-1-\sigma_1)<\min(\frac{4}{\mu}(1+\nu)-\sigma-2\nu-1,\frac{4}{\mu'}(1+\nu')-\sigma-1+2\lambda-2\nu'),$$

then $\ell^{-1} \in \mathcal{A}$, with $\alpha(\ell^{-1})=\max(-2\nu_1-1-\sigma_1,-2\nu_1'+2\lambda_1-1-\sigma_1)$ and $\beta(\ell^{-1})=$ $=\min(\frac{4}{\mu}(1+\nu)-\sigma-2\nu-1,\frac{4}{\mu'}(1+\nu')-\sigma-1+2\lambda-2\nu')$.

Therefore, if $1<p<\infty$, $\max(\frac{1}{p},1-\frac{1}{p})\leq\gamma<1$, $p\leq q\leq\frac{1}{1-\gamma}$, $\mu=\mu_1$, $\mu'=\mu_1'$,

$$\max(1-\beta(\ell),1-\beta(\ell^{-1}))<\gamma<\min(1-\alpha(\ell),1-\alpha(\ell^{-1})),$$

$$(1-\mu)(\nu_1-\nu)+(1-\mu')(\nu_1'-\nu')+\lambda_1(1+\mu')-\lambda(1+\mu)+(\frac{\mu+\mu'}{2}+1)(\sigma-\sigma_1)=0$$

and $\alpha(\ell)<\frac{1}{2}<\beta(\ell)$, then

$$H^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}(\mathcal{L}_{\gamma,p}) = H^{\mu_1,\mu_1',\sigma_1}_{\nu_1,\nu_1',\lambda_1}(\mathcal{L}_{\gamma,p})$$

The above results can be summarized in the following

Theorem 1 : If $1<p<\infty$, $\max(\frac{1}{p},1-\frac{1}{p})\leq\gamma<1$, $p\leq q\leq\frac{1}{1-\gamma}$, and $(\nu,\nu',\mu,\mu',\lambda)$ and $(\nu_1,\nu_1',\mu_1,\mu_1',\lambda_1)$ satisfie the conditions a), b) and c).

i) $\alpha<\frac{1}{2}$, $\mu+\mu'<2$ or $\mu=\mu'=1$ and $\frac{3}{2}+\sigma+\nu+\nu'-\lambda<-2$, and $0<\gamma<1-\alpha$, where $\alpha=\max(-2\nu-1-\sigma,,-2\nu'+2\lambda-1-\sigma)$, then

$$H^{\mu,\mu',\sigma}_{\nu,\nu',\lambda} \in [\mathcal{L}_{\gamma,p},\mathcal{L}_{1-\gamma,q}] \text{ and } H^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}(\mathcal{L}_{\gamma,p}) \subseteq F_c(\mathcal{L}_{\gamma,p})$$

ii) $\alpha_1<\frac{1}{2}<\beta_1$, $1-\beta_1<\gamma<1-\alpha_1$, $\mu_1+\mu_1'>\mu+\mu'$ or $\mu_1+\mu_1'=\mu+\mu'$ and $\frac{\lambda_1}{2}(1+\mu_1')+\frac{\nu_1}{2}(1-\mu_1)+\frac{\nu_1'}{2}(1-\mu_1')-\frac{\sigma_1}{2}+\frac{\sigma}{2}-\frac{\lambda}{2}(1+\mu')-\frac{\nu}{2}(1-\mu)-\frac{\nu'}{2}(1-\mu')++(\mu+\mu')\frac{\sigma-\sigma_1}{4}<-1$ where $\alpha_1=\max(-2\nu-1-\sigma,-2\nu'+2\lambda-1-\sigma)$ and

$$\beta_1=\min(\frac{1}{\mu_1}(1+\nu_1)-\sigma_1-2\nu_1-1,\frac{1}{\mu_1'}(1+\nu_1')-\sigma_1-1+2\lambda_1-2\nu_1'),$$

then

$$H^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}(\mathcal{L}_{\gamma,p}) \subseteq H^{\mu_1,\mu_1',\sigma_1}_{\nu_1,\nu_1',\lambda_1}(\mathcal{L}_{\gamma,p})$$

iii) $\alpha_2<\frac{1}{2}<\beta_2$, $\max(1-\beta_2,1-\beta_2')<\gamma<\min(1-\alpha_2,1-\alpha_2')$, where

$$\alpha_2=\max(-2\nu-1-\sigma,-2\nu'+2\lambda-1-\sigma),$$

$$\beta_2=\min(\frac{1}{\mu}(1+\nu_1)-\sigma_1-2\nu_1-1,\ \frac{1}{\mu'}(1+\nu_1')-\sigma_1-1+2\lambda_1-2\nu_1'),$$

$$\alpha_2'=\max(-2\nu_1-1-\sigma_1,-2\nu_1'+2\lambda_1-1-\sigma_1) \text{ and}$$

$$\beta_2'=\min(\frac{1}{\mu}(1+\nu)-\sigma-2\nu-1,\frac{1}{\mu'}(1+\nu')-\sigma-1+2\lambda-2\nu'), \text{ and}$$

$$(1-\mu)(\nu_1-\nu)+(1-\mu')(\nu_1'-\nu')+\lambda_1(1+\mu')-\lambda(1+\mu)+(\frac{\mu+\mu'}{2}+1)(\sigma-\sigma_1)=0$$

then

$$H^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}(\mathcal{L}_{\gamma,p}) = H^{\mu\ ,\mu',\sigma_1}_{\nu_1,\nu_1',\lambda_1}(\mathcal{L}_{\gamma,p})$$

3. The generalized Watson-Wright transformation.

In this section we define the $H^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}$-transformation on certain spaces of generalized functions. To this end, we first establish a Parseval equation for the integral transformation under consideration.

Theorem 2 : If $f \in \mathcal{L}_{\gamma,p}$ and $g \in \mathcal{L}_{\gamma,q}$, where $1<p<\infty$, $1<q<\infty$, $\frac{1}{\gamma}\le p\le\frac{q}{q-1}\le\frac{1}{1-\gamma}$, $\alpha<\frac{1}{2}$, $\mu+\mu'<2$ or $\mu=\mu'=1$ and $\frac{3}{2}+\sigma+\nu+\nu'-\lambda<-2, 0<\gamma<1-\alpha$, being $\alpha=\max(-2\nu-1-\sigma,,-2\nu'-2\lambda-1-\sigma)$, then

$$\int_0^\infty H^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}(f)(x)g(x)dx = \int_0^\infty f(y)H^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}(g)(y)dy \qquad (8)$$

Proof : Let C_0 denote the collection of functions, continuous on $(0,\infty)$ and vanishing outside some interval (a,b), where $0<a<b<\infty$. If $f \in C_0$ and $g \in C_0$, then from (3),

$$\int_0^\infty H^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}(f)(x)g(x)dx = \int_0^\infty g(x)dx\int_0^\infty (xy)^{\sigma}w^{\mu,\mu'}_{\nu,\nu',\lambda}(x^2y^2)f(y)dy =$$

$$= \int_0^\infty f(y)dy\int_0^\infty (xy)^{\sigma}w^{\mu,\mu'}_{\nu,\nu',\lambda}(x^2y^2)g(x)dx = \int_0^\infty f(y)H^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}(g)(y)dy$$

the interchange of the orders of the integration being easily justified by Fubini's theorem. Thus (8) is true if $f \in \in C_0$ and $g \in C_0$, and hence, since from [13, Lemma 2.2], C_0 is dense in $\mathcal{L}_{\gamma,p}$ and $\mathcal{L}_{\gamma,q}$, the general result will be true if we show that both sides of (8) represent bounded bilinear functionals on $\mathcal{L}_{\gamma,p} \times \mathcal{L}_{\gamma,q}$.

Now, since in the imposed conditions $H^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}$ is in $[\mathcal{L}_{\gamma,q},\mathcal{L}_{1-\gamma,p'}]$, where $p'=\frac{p-1}{p}$, by using Holder's inequality

$$\left|\int_0^\infty f(y)H^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}(g)(y)dy\right| \le \int_0^\infty |x^{\gamma}f(x)|\,|x^{1-\gamma}H^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}(g)(x)|\frac{dx}{x}\le$$

$$\le \|f\|_{\gamma,p}\|H^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}g\|_{1-\gamma,p'} \le K\,\|f\|_{\gamma,p}\|g\|_{\gamma,q}$$

where K is a bound of $H^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}$ as an element of $[\mathcal{L}_{\gamma,q},\mathcal{L}_{1-\gamma,p'}]$, so that the right hand side of (8) is a bounded bilinear functional on $\mathcal{L}_{\gamma,p} \times \mathcal{L}_{\gamma,q}$, as is the left hand side of (8) by a similar calculation, and the resuls follows.

Now we define the generalized $H^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}$-transformation on $\mathcal{L}'_{\gamma,p}$ as the adjoint of the classical transformation on $\mathcal{L}_{1-\gamma,q}$, through

$$<(H^{\mu,\mu',\sigma}_{\nu,\nu',\lambda})'f,\phi> = <f,H^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}\phi> , \qquad (9)$$

for $f \in \mathcal{L}'_{\gamma,p}$ and $\phi \in \mathcal{L}_{1-\gamma,q}$. Last expression can be unders- tood as a generalization of the Parseval equation (8). The following assertion is a simple consequence of Theorem 1,i),

Theorem 3 : The generalized $(H^{\mu,\mu',\sigma}_{\nu,\nu',\lambda})'$-transformation, defined by (9), is a continuous linear mapping of $\mathcal{L}'_{\gamma,p}$ into $\mathcal{L}'_{1-\gamma,q}$, provided that $1<q<\infty$, $\max(\frac{1}{q},1-\frac{1}{q})\leq 1-\gamma<1$, $q\leq p\leq\frac{1}{\gamma}$, $\frac{1}{2}>\max(-2\nu-1-\sigma,,-2\nu'+2\lambda-1-\sigma)$, $\mu+\mu'<2$ or $\mu=\mu'=1$ and $\frac{3}{2}+\sigma+\nu+\nu'-\lambda<-2$ and $\max(-2\nu-1-\sigma,,-2\nu'+2\lambda-1-\sigma)<\gamma<1$.

By using the Holder's inequality we can see that if f is in $\mathcal{L}_{1-\gamma,p'}$, being $p'=\frac{p}{p-1}$, then f defines a regular generalized function on $\mathcal{L}'_{\gamma,p}$. Hence, $\mathcal{L}_{1-\gamma,p'}\subset\mathcal{L}'_{\gamma,p}$ and if f belongs to $\mathcal{L}_{1-\gamma,p'}$, then $(H^{\mu,\mu',\sigma}_{\nu,\nu',\lambda})'f$ exists provided that the conditions in Theorem 3 hold, when q is replaced by p'. We have, according to our definition (9)

$$<(H^{\mu,\mu',\sigma}_{\nu,\nu',\lambda})'f,\phi> = <f,H^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}\phi> = \int_0^\infty f(x)H^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}(\phi)(x)dx$$

where $\phi \in \mathcal{L}_{1-\gamma,p'}$. By applying (8) we arrive at

$$\int_0^\infty f(x)H^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}(\phi)(x)dx = \int_0^\infty H^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}(f)(x)\phi(x)dx = <H^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}f,\phi>$$

Therefore, when $f \in \mathcal{L}_{1-\gamma,p'}$, it is proven that

$$(H^{\mu,\mu',\sigma}_{\nu,\nu',\lambda})'f = H^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}f$$

in other words, the classical $H^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}$-transformation $H^{\mu,\mu',\sigma}_{\nu,\nu',\lambda}f$ is a special case of the generalized transformation $(H^{\mu,\mu',\sigma}_{\nu,\nu',\lambda})'$.

Remark : A generalization of our work to transformations of the form

$$(Tf)(y) = \int_0^\infty (xy)^\sigma \, {}_\rho^{\rho'}w_{\nu,\nu',\lambda}^{\mu,\mu'}(x^2y^2)f(x)dx$$

where

$${}_\rho^{\rho'}w_{\nu,\nu',\lambda}^{\mu,\mu'}(x) = x^{1/2}\int_0^\infty t^{\lambda-1}J_{\nu,\rho}^{\mu}(xt)J_{\nu',\rho'}^{\mu'}(\tfrac{1}{t})dt$$

and $J_{\nu,\rho}^{\mu}$ denotes the Lommel-Maitland function defined by the series ([6])

$$J_{\nu,\rho}^{\mu}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k+\nu+2\rho}}{\Gamma(1+\rho+k)\Gamma(1+\rho+\nu+\mu k)}$$

is possible. By using the techniques of this paper one obtains similar results (see [2]).

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