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SEMI GROUP OF QUASI-PROXIMAL MAPPINGS AND THEIR FIXED POINTS

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Abstract

The purpose of this paper is to obtain some results on fixed point of certain proximally contractive semi-groups of mapping in a suppurated q-proximity space.

Introduction

On a quasi proximity space, concept of quasi-metrices and its associated topologies was first initialed by J. C. Kelly [4], R. D. Holmes [6] had studied semi-group of mappings of proximally mappings on a metric space. Some systematic study of quasi-proximity spaces and its associated topologies was made by Singal and Lal [2] and Jas and Banerjee [1]. Some allied results also appear in Jas and Baisnab [3]. Chikkala and Baisnab [8] had also proved some fixed point theorems in this connection. The aim of this paper is to obtain some results of fixed points of certain proximally contractive semi-groups of mappings in a separated quasi- proximity spaces. Also we demonstrate how the motion of gauges can be fruitfully employed in quasi-proximity spaces with positive character to derive some fixed point Theorems.

Key Words: Semi group, quasi proximity space, proximally contractive mapping, fixed point

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Before proving the main theorems we need some preliminaries.

Definition 1: A binary relation π on the power set $P(X)$ of X is said to be a quasi-proximity (q-proximity) on X if the following axioms are satisfied.

P.1. $A\pi(B \cup C)$ if and only if $A\pi B$ or $A\pi C$ and $(A \cup B)\pi C$ if and only if $A\pi C$ or $B\pi C$

P.2. $A\pi B \Rightarrow A \neq \phi$ and $B \neq \phi$

P.3. $A \cap B \neq \phi \Rightarrow A\pi B$

P.4. $A \not\pi B \Rightarrow$ there is a subset E of X such that $A \not\pi B$ and $(X \mid E) \not\pi B$.

For $A \subset X$, where (X,π) is a q-proximity space

$\Im_{\pi} - cl(A) = \{x \in X : \{x\}\pi A\}$ describes a Kuratowski closure operator on X including a topology $\Im_{\pi}$ on X.

A q-proximity π on X defines a q-proximity π^* on X by $A\ \pi^* B$ iff $B\ \pi\ A$: π^* is called the conjugate of π.

Definition 2. A set $X(\neq \phi)$ on which there are defined two topologies τ and υ, is called a bi-topological space denoted by (X,τ,υ).

Example1. Let π be a quasi-proximity on X, and π^* be its conjugate on X, then $(X,\tau_{\pi},\tau_{\pi^*})$ is a bitopological space, with topologies being $\Im_{\pi}$ and $\Im_{\pi^*}$ being T_1 while (X, π, π^*) is termed as a bi-quasi-proximity space.

Definition 3 : A bitopological space (X,τ,υ) is called pairwise Hausdorff (T_2) if for distinct points $x, y \in X$ there exists a τ-open set U and a υ-open set V such that $x \in U$ and $y \in V$ and $U \cap V = \phi$.

Definition 4. A subfamily $\wp \subset \tau U \upsilon$ of a bitopological space (X, τ, υ) is called a pairwise open cover of X if it covers X.

Definition 5. A q-proximity space (X,τ) is said to be separated if and only if for $x, y \in X$ $\{x\}\pi\{y\}$ implies $x = y$.

Definition 6: A quasi pseudometric d on X is called a gauge in (X,π) if for given $A\ \pi\ B$ and $\varepsilon > 0$ there exists $a \in A$ and $b \in B$ such that $d(a,b) < \varepsilon$.

Definition 7. A bitopological space (X, τ, υ) is said to be compact if each pairwise open cover of it has a finite subcover.

Definition 8. Let (X,π) and (Y,π_1) be two quasi-proximity spaces. A mapping $f:(X,\pi)\to(Y,\pi_1)$ is called quasi-proximal iff $A\pi B$ implies $f(A)\pi_1 f(B)$. Note that f: (X, π) $\to$ (Y, π_1) is quasi-proximal iff $f:(X,\pi^*)\to(Y,\pi_1^*)$ is quasi-proximal.

Definition 9. Γ is called proximally contractive if for any gauge d and any $\varepsilon > 0$, there is a member $g\in\Gamma$ such that

$d(x, y) \leq \varepsilon$ implies $d\left(g(x),\ g(y)\right) < \varepsilon; x, y \in X$.

Definition 10. Two points and x and y of (X,π) is said to be proximal if for any gauge d and for any $\varepsilon > 0$, there is a member $g\in\Gamma$ satisfying $d(g(x), g(y)) < \varepsilon$.

Lemma 1. For any gauge d on a quasi-proximity space (X, π) and $\varepsilon > 0$. Let

$B_{d,\varepsilon}=\{(x,y)\in X\times X : d(x,y)<\varepsilon\}$. Then $B_{d,\in}$ is a $\tau_{\pi\times\pi^*}$ neighborhood base for the diagonal set $\Delta(x)$ in $(X\times X,\tau_{\pi\times\pi^*})$.

Lemma 2. If (X,π) is a separated quasi-proximity space, then

$$\tau_{\pi\times\pi^*}-cl(\Delta(X))=\Delta(X).$$

Theorem 1. Let (X, π) be separated and $\left(X,\tau_\pi,\tau_{\pi^*}\right)$ is compact and pairwise T_2. If for any $x\in X$ and $f\in\Gamma$, x and $f(x)$ are proximal, then f has a fixed point in X.

Proof: Let Δ denote the family of gauges to generate π in X. Since x and $f(x)$ are proximal we find a member, say, $g_k\in\Gamma$ satisfying $d\left(g_k(x),\ g_k\left(f(x)\right)\right)<\varepsilon$.

Put $N=\{B_{d,\varepsilon}: d\in\Delta, \varepsilon>0\}$

where $B_{d,\varepsilon}=\{(x,y)\in X\times X : d^*(x,y)<\varepsilon\}$

is a neighbourhood of (X). Then as in Lemma 1 $B_{d,\in}$ is a neighbourhood base for (X). ***N*** is directed by set inclusion relation $\subset$.. So we consider $\left(g_k(x),\ g_k\left(f(x)\right)\right)<\varepsilon$ as a net in $(X\times X,\tau_{\pi\times\pi^*})$, which is compact by virtue of (X,τ_π,τ_π^*) being assumed to be compact. Let $\{g_{k_i}(x),\ g_{k_i}(f(x)) < \varepsilon\}$ be a convergent subnet of

$\{g_k(x),\ g_k(f(x))\}$ in $X \times X$.

Put $\lim_i \{(g_{k_i}(x),\ g_{k_i}(f(x))) = (z,u) \in X \times X\}$.

Since the net is frequently in *N*, it follows that

$(z,u) \in_N \tau_{\pi\times\pi^*} - Cl(B_{d,\varepsilon})$ i.e., $(z,u) \in \tau_{\pi\times\pi} - Cl(\Delta(x))$

i.e., $(z,u) \in \Delta(x)$, by Lemma 2 …….. (1)

Hence $z = u$. From above we have $\lim_i g_{k_i}(x) = z$ and $\lim_i g_{k_i}(f(x)) = u$.

By continuity of $d : (X \times X, \tau_{\pi\times\pi^*}) \to (\Re, \tau_u)$ and using the fact that convergence of a sequence of reals to a real number with respect to usual topology of reals implies its convergence with respect to τ_π with limit unchanged.

For $d \in \Delta$ we have

$$d(z,u) = d\left(\lim_i g_{k_i}(x), \lim_i g_{k_i}(f(x))\right)$$
$$= d\left(\lim_i g_{k_i}(x), f\left(\lim_i g_{k_i}(x)\right)\right)$$
$$= d(z, f(z)).$$

As $z = u$ by (1), we have $d(u, f(u)) = 0$.

That means $\{u\}\pi\{f(u)\}$. As π is separated we have $f(u) = u$.

Theorem 2: If π is a quasi-proximity on X and (X, τ_π, τ_{π^*}) is pairwise T_2 and compact and Γ is commutative semigroup of quasi-proximal mapping, then each pair of points in (X, π) is proximal.

Proof: Let Δ denote the family of all gauges to generate π on X.

Let $x, y \in X$ and assume that they are not proximal. Then for some $d \in D$ and $r > 0$, we have $\mu = \inf\{d(g(x), g(y)) : g \in G\}$. Then clearly, we have $\mu \geq r$. Take a member $g_1 \in G$ such that $\mu \leq d(g_1(x), g_1(y)) < 2\mu$.

Put $\mu_1 = \inf\left\{d\left(g\ \ g_1(x), g\ \ g_1(y)\right) : g \in G\right\}$. Then we have $r \le \mu \le \mu_1 < 2\mu_1$. Next take a g_2 from Γ such that $\mu_1 \le d\left(g_2 g_1(x),\ g_2 g_1(y)\right) < \min\ \ 1+\frac{1}{2}\ \ \mu_1, 2\mu$.

Put $\mu_2 = \inf\left\{d\left(g\ \ g_2 g_1(x), g\ \ g_2\ g_1(y)\right) : g \in G\right\}$.

Then we have $\mu_1 \le \mu_2 < \min\ \ 1+\frac{1}{2}\ \ \mu_1, \mu_2$, continuing this process we produce a sequence $\{f_n\} \in G$ and a sequence satisfying

(i) $\mu \le \mu_n \le \mu_{n+1} < 2\mu$

(ii) $\mu_n = \inf\{d(gf_n(x), gf_n(y) : g \in \quad\}$, where $f_n = g_n g_{n-1} \cdots g_2 g_1$;

(iii) $\mu_{n+1} <\ 1+\frac{1}{n+1}\ \ \mu_n$.

Let us now consider sequences $\{f_n(x)\}$ and $\{f_n(y)\}$. Since $(X, \tau_\pi, \tau_{\pi^*})$ is compact, (X, τ_d) is also compact where (X, τ_d) is the induced topology on X by d.

Hence there is a subsequence $\{n_i\}$ of positive integers such that $\left\{f_{n_i}(x)\right\}$, $\left\{f_{n_i}(y)\right\}$ and $\left\{\mu_{n_i}\right\}$ are convergent.

Put $\lim_i f_{n_i}(x) = u$, $\lim_i f_{n_i}(y) = v$, and $\lim_i \mu_{n_i} = \alpha$. .

Clearly, $\alpha \ge \mu \ge r$.

Also we have

$$\frac{\mu_{n_i}}{1+\frac{1}{n_i}} \le \mu_{n_i-1} \le d\left(f_{n_i}(x), f_{n_i}(y)\right)$$

$$\le\ 1+\frac{1}{n_i}\ \ \mu_{n_i} \qquad ....(2)$$

Taking limit as $i \to \infty$ in (2) one get $d(u,v) = \infty$.

If g is any member of Γ, we have by continuity of g ,

$$\lim_{i\to\infty} d\left(g\ f_{n_i}(x),\ g\, f_{n_i}(y)\right) = d\left(g(u),\ g(v)\right).$$

Since $d\left(g\ f_{n_i}(x),\ g\, f_{n_i}(y)\right) \ge \mu_{n_i}$ always, we have,

$d\left(g\left(u\right),\ \ g\left(v\right)\right) \geq \alpha$, a contradiction that Γ is proximally contractive. The proof is now complete.

Theorem 3. Let π be separated quasi-proximity on X and the bitopological space $(X, \tau_\pi, \tau_{\pi^*})$ is compact and pairwise T_2. Let Γ denote a commutative semigroup of quasi proximal mapping on (X, π) such that Γ is proximally contractive. Then Γ has a unique common fixed point.

Proof: Let $f \in \Gamma$ and $x \in X$. By Theorem 2, x and $f(x)$ are proximal, and Theorem 1 applies. If I denotes the identity mapping belonging to Γ .The subfamily (I, f) now possesses a common fixed point. We show that every finite subfamily of Γ has a common fixed point.

Let $\{f_1,\ f_2,\dots, f_m\} \subset \Gamma$.

Since z and $f(z)$ are proximal, there is a net $\{g_{k_i}\}$, $g_{k_i} \in \Gamma$ as can be seen in the proof of Theorem 1, such that $\lim_i g_{k_i}(z) = u(\text{say})$.

For $1 \leq j \leq m$, we have

$$\begin{aligned} f_j(u) &= \lim_i f_i g_{k_i}(z) \\ &= \lim_i g_{k_i} f_i(z) \\ &= \lim_i g_{k_i}(z) \\ &= u. \end{aligned}$$

Hence u is a common fixed point of $\{f_1,\ f_2,\dots, f_m\}$. Now suppose $f \in \Gamma$ and let $\Phi(f)$ denote the set of fixed points of f in (X, π). If x is an accumulation point of $\Phi(f)$, let $\{x_k\}$ be a net in $\Phi(f)$ such that $\lim_k x_k = x$.

By continuity of f, we have

$$\lim f(x_k) = f(x)$$

i.e., $\lim_k x_k = f(x)$, since $x_k \in \Phi(f)$

i.e., $x = f(x)$, since (X, π, π^*) is pairwise T_2. Hence $\Phi(f)$ is τ_π - closed.

Finally, let $\Phi = \{\Phi(f) : f \in \Gamma\}$

By argument above Φ has finite intersection property, by compactness of (X, τ_π), the set

Φ={ I $\Phi(f): f \in \Gamma\} \neq \phi$

Let $x, y \in \Phi$ with $x \neq y$. Since (X, π) is separated we have

$$d(x, y) = r > 0 \quad \text{.........} (3)$$

for some gauge d for (X, π). Since Γ is proximally contractive we have

$d(g(x), g(y)) < r$ for some member $g \in \Gamma$,

i.e., $d(x, y) < r$. This is the desired contradiction in view of (3). Hence Φ is a singleton i.e. the family possesses a unique common fixed point in X .

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