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A UNIFIED FORM OF GENERALIZED ELLIPTIC-TYPE INTEGRALS

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ABSTRACT

In this paper, we consider a unified form of yet another generalization of Epstein-Hubble elliptic-type integrals,

$$\Omega_j(x) = \int_0^{\pi} (1-k^2\cos\theta)^{-j-\frac{1}{2}}\, d\theta; \qquad j = 0, 1, 2, \ldots \text{ and } 0 \le k < 1.$$

We examine some of their important properties including asymptotic expansions in the neighbourhood of $k^2 = 1$.

Several results obtained recently by some authors, follow as special cases of our results established here.

1. INTRODUCTION

Recently, Al-Zamel et al [4] introduced a generalized form of a family of elliptic-type integrals to Epstein-Hubbel [8], given by the formula:

$$Z_{(\gamma)}^{(\alpha,\beta)}(k) = \int_0^{\pi} \cos^{2\alpha-1}(\tfrac{\theta}{2}) \sin^{2\beta-1}(\tfrac{\theta}{2}) \prod_{j=1}^{n} (1-k_j^2\cos\theta)^{-\gamma_j}\, d\theta \tag{1.1}$$

where $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$; $|k_j| < 1$, $\alpha, \gamma_j \in \not\subset$, $j = 0, 1, \ldots; n$.

They gave an explicit representation of (1.1) in terms of $F_D^{(n)}$, the Lauriecella hypergeometric function of n- variables [17], and obtained its asymptotic expansion as $k^2 \to 1$.

Also, Al-Zamel [3] gave a different class of generalized elliptic-type integrals to Epstein-Hubble family of integrals, of the form:

$$R_{\mu}^{\alpha}(k;\lambda,\nu)=\int_{0}^{\pi}\frac{\exp(\alpha\sin^{2}\frac{\theta}{2})\cos^{2\lambda-1}(\frac{\theta}{2})\sin^{2\nu-2\lambda-1}(\frac{\theta}{2})}{(1-k^{2}\cos\theta)^{\mu+\frac{1}{2}}}d\theta \tag{1.2}$$

with $0\le k<1$, $\mathrm{Re}(\lambda)>0$, $\mathrm{Re}(\nu-\lambda)>0$ and $\mathrm{Re}(\mu)>-1/2$.

He gave a series expansion of R_{μ}^{α} and established its relationship with Φ_1 [9], the confluent hypergeometric function of two variables, and its asymptotic expansions as $k^2\to 1$.

An another class of generalized elliptic-type integrals has been studied by Ben Nakhi and Kalla [5], in the following form:

$$R_{\gamma}^{\omega}(x,\ell,\beta)=\int_{0}^{\pi}\frac{\cos^{\frac{2\ell}{\omega}-1}(\frac{\theta}{2})\left(1-\cos^{\frac{2}{\omega}}(\frac{\theta}{2})\right)^{\beta-\ell-1}\sin\frac{\theta}{2}}{(1-x^{2}\cos\theta)^{\gamma+\frac{1}{2}}}d\theta \tag{1.3}$$

where $0\le x<1$, $\omega>0$, $\mathrm{Re}(\beta)>\mathrm{Re}(\ell)\ge 0$, $\mathrm{Re}(\gamma)>-\frac{1}{2}$.

This class includes the families of elliptic-type integrals given earlier by Kalla et al. [14], and others.

Garg et al. [11] have considered a family of elliptic-type integrals involving Gauss hypergeometric function in the integrand.

In the present paper, we define and study another general class of elliptic-type integrals $R_{(\gamma)}^{\alpha}(k;\lambda,\nu)$. We investigate their properties regarding their relationship to Φ_1 and $F_D^{(n)}$. We also give some recurrence relations and its asymptotic expansions in the neighbourhood of $k^2=1$, and an approximation formula.

2. NOTATION, DEFINITION AND SPECIAL CASES

Let $R=R_{(\gamma)}^{(\alpha)}(k;\lambda,\nu)$

$$=R_{(\gamma_1,\gamma_2,\ldots,\gamma_n)}^{(\alpha)}(k_1,\ldots,k_n;\lambda,\nu)$$

$$= \int_0^{\pi} \frac{e^{\alpha \sin^2 \frac{\theta}{2}} \cos^{2\lambda-1}(\frac{\theta}{2}) \sin^{2\nu-2\lambda-1}(\frac{\theta}{2})}{\prod_{j=1}^{n} (1-k_j^2 \cos\theta)^{\gamma j}} d\theta \tag{2.1}$$

where $\mathrm{Re}(\lambda) > 0, \quad \mathrm{Re}(\nu - \lambda) > 0, \ |k_j| < 1, \alpha, \ \gamma_j \in \not\subset, j = 1....,n.$

Special Cases:

(i) If we let $n = 1, \ k_1 = k$, and $\gamma_1 = \mu + \frac{1}{2}$ in (2.1), we get

$$R = \int_0^{\pi} \frac{e^{\alpha \sin^2 \frac{\theta}{2}} \cos^{2\lambda-1}(\frac{\theta}{2}) \sin^{2\nu-2\lambda-1}(\frac{\theta}{2})}{(1-k^2 \cos\theta)^{\mu+\frac{1}{2}}} d\theta \tag{2.2}$$

$$= R_{\mu}^{\alpha}(k, \lambda, \nu)$$

This family of elliptic-type integrals are considered recently by Al-Zamel who also established its relation with Φ_1, the confluent hypergeometric function of two variables in the following form;

$$R_{\mu}^{\alpha}(k, \lambda, \nu) = \frac{\Gamma(\lambda)\Gamma(\nu-\lambda)}{\Gamma(\nu)(1+k^2)^{\mu+\frac{1}{2}}} \ e^{\alpha} \phi_1 \left(\lambda, \mu + \tfrac{1}{2}; \nu, \omega, -\alpha\right)$$

Moreover, if we let $\alpha = 0$, we have;

$$R = \int_0^{\pi} \cos^{2\lambda-1}(\tfrac{\theta}{2}) \sin^{2\nu-2\lambda-1}(\tfrac{\theta}{2}) (1-k^2 \cos\theta)^{-\mu-\frac{1}{2}} d\theta \tag{2.3}$$

$$= R_{\mu}(k, \lambda, \nu) .$$

This family of elliptic-type integrals was studied by Kalla et al. [14].

(ii) If we let $\beta = \nu - \lambda, \quad \alpha = 0, \lambda = \alpha$ in (2.1), then

$$R = \int_0^{\pi} \cos^{2\lambda-1}(\tfrac{\theta}{2}) \sin^{2\beta-1}(\tfrac{\theta}{2}) \prod_{j=1}^{n} (1-k_j^2 \cos\theta)^{-r_j} d\theta \tag{2.4}$$

$$= Z_{(\gamma)}^{(\alpha,\beta)}(k)$$

This family of elliptic-type integrals (2.4) is due to Al-Zamel et al. [4], and they have established the following relationship:

$$R = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \prod_{j=1}^{n} (1-k_j^2)^{-\gamma_j} \; F_D^{(n)} \left(\beta ; b_1, \ldots, b_n ; \alpha+\beta ; \frac{2k_1}{k_1^2-1}, \ldots\ldots, \frac{2k_n}{k_n^2-1} \right) \tag{2.5}$$

where $F_D^{(n)}$ is the Lauriecella hypergeometric function of n-variables [17]. If we let n =3 and $\gamma_1 = \mu + \frac{1}{2}$, $\gamma_2 = \gamma_3 = \lambda$, $k_1^2 = k^2$, $k_2^2 = \frac{\rho}{\rho-2}$ and $k_3^2 = \frac{-\delta}{2+\delta}$ in (2.1) we get

$$R = Z_{(\gamma_1,\gamma_2,\gamma_3)}^{(\alpha,\beta)} (k_1, k_2, k_3)$$

$$= \left(1 - \tfrac{\rho}{2}\right)^2 \left(1 + \tfrac{\delta}{2}\right)^{\gamma} \Lambda^{(\alpha,\beta)} (\rho, \delta, k)$$

where $\gamma = \alpha + \beta$ and $\Lambda^{(\alpha,\beta)}$ represents a certain family of elliptic-type integrals of Srivastava and Siddiqi [19].

A number of other families of elliptic-type-integrals, considered by several authors, as mentioned in our survey paper [2] can be easily recovered from our unified form of generalized elliptic-type integrals defined by equations (2.1).

3. REPRESENTATION OF $R_{(\gamma)}^{\alpha}(k;\lambda,\nu)$ IN TERMS OF $F_D^{(n)}$ and $\Phi_1^{(n+1)}$

First we express the family of integrals $R_{(\gamma)}^{\alpha}(k;\lambda,\nu)$ in terms of Lauricella's hypergeometric function $F_D^{(n)}$.

We have,

$$R = \int_0^{\pi} e^{\alpha \sin^2 \frac{\theta}{2}} \cos^{2\lambda-1}\left(\tfrac{\theta}{2}\right) \sin^{2\nu-2\lambda-1}\left(\tfrac{\theta}{2}\right) \prod_{j=1}^{n} (1 - k_j^2 \cos\theta)^{-\gamma_j} \, d\theta \,,$$

let $x = \sin^2 \frac{\theta}{2}$, then $1 - x = \cos^2 \frac{\theta}{2}$, $1 - 2x = \cos\theta$, and $dx = \sin\frac{\theta}{2} \cos\frac{\theta}{2} \, d\theta$.

Hence,

$$R = \int_0^1 e^{\alpha x} x^{\nu-\lambda-1} (1-x)^{\lambda-1} \prod_{j=1}^{n} (1 - k_j^2 + 2k_j^2 x)^{-\gamma_j} \, dx \,,$$

$$= \int_0^1 \sum_{r=0}^{\infty} \frac{(\alpha)_r}{r!} \; x^{\nu-\lambda+r-1} (1-x)^{\lambda-1} \prod_{j=1}^{n} (1-k_j^2)^{-\gamma_j} (1 - \omega_j x)^{-\gamma_j} \, dx \quad ,$$

(where $\omega_j = \frac{2k_j^2}{k_j^2 - 1}$, $j = 1, \ldots\ldots, n$).

$$= \prod_{j=1}^{n} (1-k_j^2)^{-\gamma_j} \sum_{r=0}^{\infty} \frac{(\alpha)_r}{r!} . \frac{\Gamma(\nu-\lambda+r)\Gamma(\lambda)}{\Gamma(\nu+r)} . \frac{\Gamma(\nu+r)}{\Gamma(\nu-\lambda+r)\Gamma(\lambda)} .$$

$$\int_0^1 \lambda^{\nu-\lambda+r-1}(1-x)^{\lambda-1}\left(\prod_{j=1}^{n}(1-\omega_j\ x)^{-\gamma_j}\right) dx ,$$

$$= \prod_{j=1}^{n}(1-k_j^2)^{-\gamma_j} \sum_{r=0}^{\infty} \frac{(\alpha)_r}{r!} \frac{\Gamma(\nu-\lambda+\gamma)\Gamma(\lambda)}{\Gamma(\nu+r)} F_D^{(n)}(\nu-\lambda+r,\ \gamma_1,\dots,\gamma_n;\nu+r,\ \omega_1,\dots,\omega_n).$$

where $F_D^{(n)}$ is the Lauricella hypergeometric function of n-variables.

Now, if in (3.1), we let

(i) $\beta=\nu-\lambda,\ \alpha=\theta$, we have

$$R = \prod_{j=1}^{n}(1-k_j^2)^{-\gamma_j} \frac{\Gamma(\beta)\Gamma(\lambda)}{\Gamma(\beta+\lambda)} F_D^{(n)}\left(\beta;\ \gamma_1,\dots,\gamma_n;\beta+\lambda;\frac{2k_1^2}{k_1^2-1},\dots\dots,\frac{2k_n^2}{k_n^2-1}\right) \quad (3.2)$$

The result (3.2) is in agreement with the expression given recently by Al-Zamel et al. [4, P. 12].

(ii) $n=1$, $k_1=k$, $\gamma_1=\mu+\frac{1}{2}$, then we obtain the result given by Al-Zamel [3].

(iii) Further, for n = 3, the result (3.1) reduces to the formula given by Kalla and Tuan [15, P. 51] which itself is a generalization of several other results given by Srivastava and Siddiqi [19], Kalla et al. [14].

Now we express $R_\gamma^\alpha(\mathbf{k};\lambda,\nu)$ In terms of $\Phi_1^{(n+1)}$:

We have;

$$R = \prod_{j=1}^{n}(1-k_j^2)^{-\gamma_j} \int_0^1 e^{\alpha x} x^{\nu-\lambda-1}(1-x)^{\lambda-1} \prod_{j=1}^{n}(1-\omega_j x)^{-\gamma_j}\ dx ,$$

$$= \prod_{j=1}^{n}(1-k_j^2)^{-\gamma_j} \sum_{\ell_1=0}^{\infty}\cdots\sum_{\ell_n=0}^{\infty}\sum_{\ell=0}^{\infty} \int_0^1 x^{\nu-\lambda-1}(1-x)^{\lambda-1} \frac{\alpha^\ell x^\ell}{\ell!} \frac{(\gamma_1)_{\ell_1}}{\ell_1!}(\omega_1)^{\ell_1} x^{\ell_1} \dots\dots \frac{(\gamma_n)_{\ell_n}}{\ell_n!}(\omega_n)^{\ell_n} x^{\ell_n}$$

$$= \sum_{\ell_1=0}^{\infty}\cdots\cdot\sum_{\ell_n=o}^{\infty}\sum_{\ell=0}^{\infty}\prod_{j=1}^{n}(1-k_j^2)^{-\gamma_j} \frac{(\gamma_j)_{\ell_j}}{\ell_j!}(\omega_j)^{\ell_j} \frac{\alpha^\ell}{\ell!}\int_0^1 (1-x)^{\lambda-1} x^{\nu-\lambda+\ell+\ell_1+\dots+\ell_n-1}\ dx ,$$

$$= \prod_{j=1}^{n}(1-k_j^2)^{-\gamma_j} \sum_{\ell_1=0}^{\infty}\cdots\cdot\sum_{\ell_n=o}^{\infty}\sum_{\ell=0}^{\infty}\prod_{j=1}^{n} \frac{(\gamma_j)_{\ell j}}{\ell_j!}(\omega_j)^{\ell_j}\frac{\alpha^\ell}{\ell!}\frac{\Gamma(\nu-\lambda+\ell+\ell_1+\dots+\ell_n)\Gamma(\lambda)}{\Gamma(\nu+\ell+\ell_1+\dots\cdot+\ell_n)} ,$$

$$= \frac{\Gamma(\nu-\lambda)\Gamma(\lambda)}{\Gamma(\nu)} \sum_{\ell_1=0}^{\infty}\cdots\sum_{\ell_n=0}^{\infty}\sum_{\ell=0}^{\infty} \frac{(\nu-\lambda)_{\ell+\ell_1+\dots+\ell_n}}{(\nu)_{\ell+\ell_1+\dots+\ell_n}} . \prod_{j=1}^{n}(1-k_j^2)^{-\gamma_j} \frac{(\gamma_1)_{\ell_1}\cdots(\gamma_n)_{\ell_n}\,\alpha^\ell}{\ell_1!\dots\dots\ell_n!\dots\dots\ell!}(\omega_1)^{\ell_1}\dots(\omega_n)^{\ell_n},$$

$$= \frac{\Gamma(\nu-\lambda)\Gamma(\lambda)}{\Gamma(\nu)} \prod_{j=1}^{n}(1-k_j^2)^{-\gamma_j} \Phi_1^{(n+1)}(\nu-\lambda;\gamma_1,...,\gamma_n;\nu;\omega_1,...\omega_n,\alpha) \quad (3.3)$$

where $\Phi_1^{(n+1)}$ is the confluent hypergeometric function of n + 1 variables.

Now if we let, $j=1$, $k_1 = k$, $\gamma_1 = \mu+\frac{1}{2}$ in (3.3), we get

$$R_\mu^\alpha(k,\lambda,\nu)= \frac{\Gamma(\nu-\lambda)\Gamma(\lambda)}{\Gamma(\nu)} (1-k_j^2)^{-\mu-\frac{1}{2}}. \quad \Phi_1\left(\nu-\lambda,\mu+\tfrac{1}{2};\nu;\omega;\alpha\right).$$

where $\omega = \dfrac{2k^2}{k^2-1}$.

This result coincides with the formula given by Al-Zamel [3] by virtue of the identity due to Tuan and Kalla [20], namely

$$\Phi_1(a,b,c;\omega,z) = (1-\omega)^{-b} e^z \Phi_1\left(c-a,b,c,\tfrac{\omega}{\omega-1}-,-z\right)$$

4. RECURRENCE RELATIONS

Here we establish some recurrence relations, using simple trigonometric identities and other formulas.

As,

$$R_{(\gamma)}^\alpha(k;\lambda,\nu) = \int_0^\pi \frac{e^{\alpha\sin^2\frac{\theta}{2}} \cos^{2\lambda-1}(\frac{\theta}{2}) \sin^{2\nu-2\lambda-1}(\frac{\theta}{2})}{\prod_{j=1}^{n}(1-k_j^2\cos\theta)^{\gamma_j}} d\theta, \quad (4.1)$$

Let $x=\sin^2\frac{\theta}{2}$, then

$$R_\gamma^\alpha(k;\lambda,\omega) = \prod_{j=1}^{n}(1-k_j^2)^{-\gamma_j} \int_0^1 e^{\alpha x} x^{\nu-\lambda-1}(1-x)^{\lambda-1} \prod_{j=1}^{n}(1-\omega_j x)^{-\gamma_j} dx \quad (4.2)$$

where $\omega_j = \dfrac{2k_j^2}{k_j^2-1}$ ' $j=1,2,...,n$.

Now let $x=\cos^2\frac{\theta}{2}$, then

$$R_\gamma^\alpha(k;\lambda,\nu) = e^\alpha \prod_{j=1}^{n}(1+k_j^2)^{-\gamma_j} \int_0^1 e^{-\alpha x} x^{\lambda-1}(1-x)^{\nu-\lambda-1} \prod_{j=1}^{n}(1-\tau_j x)^{-\gamma_j} dx \quad (4.3)$$

where $\tau_j = \dfrac{2k_j^2}{1+k_j^2}$, $j=1,\ldots,n.$

$$= e^{\alpha}\, R_{\gamma}^{-\alpha}\,(ik\,;\,\nu-\lambda,\nu)\,.$$

Hence from (4.2) and (4.3) we have the recurrence relation

$$R_{\gamma}^{\alpha}\,(k;\lambda,\nu) = e^{\alpha}\, R_{(\gamma)}^{-\alpha}\,(i\,k;\ \nu-\lambda,\ \nu)\,. \tag{4.4}$$

By using the identity $\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} = 1$ in (4.1) we obtain the following formulas:

$$R_{(\gamma)}^{\alpha}\,(k;\lambda,\nu) = R_{(\gamma)}^{\alpha}\,(k;\lambda,\nu-1) - R_{(\gamma)}^{\alpha}(k;\lambda+1,\nu) \tag{4.5}$$

and $$R_{(\gamma)}^{\alpha}\,(k;\lambda,\nu) = R_{(\gamma)}^{\alpha}\,(k;\lambda-1,\nu-1) - R_{(\gamma)}^{\alpha}(k,\lambda-1,\nu) \tag{4.6}$$

We can also obtain the following recurrence relations simply by multiplying the integrand of

definition (4.1) by $\dfrac{1-k_j^2\cos\theta}{1-k_j^2\cos\theta}$;

$$R_{(\gamma_{1,\ldots,}\gamma_n)}^{\alpha}\,(k;\lambda,\nu) = (1+k_j^2)\,R_{(\gamma_1,\ldots,1+\gamma_j,\ldots,\gamma_n)}^{\alpha}(k;\,\lambda,\nu)\ \ -2k_j^2\,R_{(\gamma_1,\ldots,1+\gamma_j,\ldots,\gamma_n)}^{\alpha}(k,\lambda+1;\nu),$$

for $j=1,\ldots,n.$ (4.7)

and

$$R_{(\gamma_1,\ldots,\gamma_j,\ldots,\gamma_n)}^{\alpha}\ (k;\lambda,\nu) = (1-k_j^2)\ R_{(\gamma_1,\ldots,\gamma_j+1,\ldots,\gamma_n)}^{\alpha}\ (k;\lambda,\nu) + 2k_j^2\ R_{(\gamma_1,\ldots,\gamma_j+1,\ldots,\gamma_n)}^{\alpha}(k;\lambda,\nu+1$$

for $j=1,\ldots,n$. (4.8)

Moreover, by multiplying the integrand of (4.1) by $\dfrac{1-\cos^2\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}$, we get these relations

$$R_{(\gamma)}^{\alpha}\,(k;\lambda,\nu) = \int_0^{\pi} \frac{e^{\alpha\sin^2\frac{\theta}{2}}\ \sin^{2(\nu-1)-2\lambda-1}\left(\frac{\theta}{2}\right)\ \cos^{2\lambda-1}\left(\frac{\theta}{2}\right)}{\prod_{j=1}^{n}(1-k_j^2\cos\theta)^{\gamma j}}\,d\theta$$

$$-\int_0^{\pi} \frac{e^{\alpha\sin^2\frac{\theta}{2}}\ \sin^{2\nu-2(\lambda+1)-1}\left(\frac{\theta}{2}\right)\ \cos^{2(\lambda+1)-1}\left(\frac{\theta}{2}\right)}{\prod_{j=1}^{n}(1-k_j^2)^{\gamma j}}\,d\theta$$

which yields the recurrence relation

$$R_{(\gamma)}^{\alpha}\,(k;\lambda,\nu) = R_{(\gamma)}^{\alpha}\,(k;\lambda,\nu-1) - R_{(\gamma)}^{\alpha}(k;\lambda+1,\nu) \tag{4.9}$$

Multiply the integrand again by $\dfrac{1-\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}$ to obtain,

$$R^{\alpha}_{(\gamma)} = \int_0^{\pi} \frac{e^{\alpha\sin^2\frac{\theta}{2}} \sin^{2\nu-2\lambda-1}(\tfrac{\theta}{2}) \cos^{2\lambda-3}(\tfrac{\theta}{2})}{\prod_{j=1}^{n}(1-k_j^2\cos\theta)^{\gamma_j}} \, d\theta$$

$$- \int_0^{\pi} e^{\alpha\sin^2\frac{\theta}{2}} \sin^{2\nu-2\lambda+1}(\tfrac{\theta}{2}) \cos^{2\lambda-3}(\tfrac{\theta}{2}) \, d\theta ,$$

$$= \int_0^{\pi} \frac{e^{\alpha\sin^2\frac{\theta}{2}} \sin^{2(\nu-1)-2(\lambda-1)-1}(\tfrac{\theta}{2}) \cos^{2(\lambda-1)-1}(\tfrac{\theta}{2})}{\prod_{j=1}^{n}(1-k_j^2\cos\theta)^{\gamma_j}} \, d\theta$$

$$- \int_0^{\pi} e^{\alpha\sin^2\frac{\theta}{2}} \sin^{2\nu-2(\lambda-1)-1}(\tfrac{\theta}{2}) \cos^{2(\lambda-1)-1}(\tfrac{\theta}{2}) \, d\theta ,$$

Hence we have the recurrence relation,

$$R^{\alpha}_{(\gamma)}(k;\lambda,\nu) = R^{\alpha}_{(\gamma)}(k;\lambda-1,\nu-1) - R^{\alpha}_{(\gamma)}(k;\lambda-1,\nu) . \qquad (4.10)$$

5. SINGLE-TERM APPROXIMATION TO R

In this section, we establish a reasonable single term approximation to R.

We have,

$$R = \int_0^1 \lambda^{\nu-\lambda-1}(1-x)^{\lambda-1} e^{\alpha x} \prod_{j=1}^{n} (1-k_j^2+2k_j^2 x)^{-\gamma_j} \, dx ,$$

where $|k_j| < 1$, $j = 1, \ldots, n$.

$$= \prod_{j=1}^{1}(1-k_j^2)^{-\gamma_j} \int_0^1 x^{\nu-\lambda-1} (1-x)^{\lambda-1} e^{\alpha x} \prod_{j=1}^{n} (1+\tfrac{x}{\omega_j})^{-\gamma_j} \, dx ,$$

$$= m \int_0^1 x^{\nu-\lambda-1}(1-x)^{\lambda-1} e^{\alpha x} \prod_{j=1}^{n} (1+\tfrac{x}{\omega_j})^{-\gamma_j} \, dx ,$$

where m and ω_j , $j = 1, \ldots, n$ are positive real numbers such that

$$m = \prod_{j=1}^{n} (1-k_j^2)^{-\gamma_j} \text{ and } \omega_j = \frac{1-k_j^2}{2k_j^2} , \; j = 1, \ldots, n.$$

Now since $\left(1+\frac{x}{\omega_j}\right)^{-\gamma_j} \cong (1-x)^{\gamma_j/w_j}$, $j=1,\ldots,n$ and $e^{\alpha x} \cong 1+\alpha x$ for small values of x, then

$$R \cong m\int_0^1 x^{\nu-\lambda-1}(1-x)^{\lambda-1}\ (1+\alpha x)\ \prod_{j=1}^{n}(1-x)^{\gamma_j}\ dx,$$

where $r_j = \dfrac{\gamma_j}{\omega_j}$, $j=1,\ldots,n$.

$$R \cong m\int_0^1 x^{\nu-\lambda-1}\ (1-x)^{\lambda+r-1}\ dx\ +\alpha m\int_0^1 x^{\nu-\lambda+1-1}(1-x)^{\lambda+r-1}\ dx,$$

$$= m\beta(\nu-\lambda,\ \lambda+r)\ +\ \alpha\ m\beta(\nu-\lambda+1,\ \lambda+r)$$

where $r=\sum_{j=1}^{n} r_j$, $(\nu-\lambda)>0$, $\lambda>0$.

$$= m\ \frac{\Gamma(\nu-\lambda)\Gamma(\lambda+r)}{\Gamma(\lambda+r)} + \frac{\alpha m\ \Gamma(\nu-\lambda+1)\Gamma(\lambda+r)}{\Gamma(\nu+r+1)}$$

$$= m\ \frac{\Gamma(\nu+r)\Gamma(\nu-\lambda)\Gamma(\lambda+r)}{\Gamma(\nu+r+1)} + \frac{\alpha m\Gamma(\nu-\lambda)\Gamma(\nu-\lambda)\Gamma(\lambda+r)}{\Gamma(\nu+r+1)}$$

$$= m\ \frac{\Gamma(\nu-\lambda)\Gamma(\lambda+\gamma)}{\Gamma(\nu+r+1)}\big(\nu+r+\alpha(\nu-\lambda)\big)$$

This formula gives a reasonable approximation to R for small values of x and r, and may be used to determine numerical values of some physical problems involving this generalized elliptic-type integrals.

6. ASYMPTOTIC EXPANSION AS $k^2 \to 1$

Consider the transformation [1] formula for Gauss hypergeometric function, normally

$$_2F_1\ (a,b;c,z)=\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}\ (-z)^{-a}\ \ _2F_1\ \left(a,1+a-c;\ 1+a-b;\tfrac{1}{z}\right) \tag{6.1}$$

$$+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}\ (-z)^{-b}\ \ _2F_1\ \left(b,1+a-c,\ 1+b-a;\tfrac{1}{z}\right)$$

with $|\arg(-z)|<\pi$

As,

$$R=\int_0^{\pi}\frac{e^{\alpha\sin^2\frac{\theta}{2}}\ \sin^{2\nu-2\lambda-1}\left(\frac{\theta}{2}\right)\cos^{2\lambda-1}\left(\frac{\theta}{2}\right)}{\prod_{i=1}^{n}(1-k_j^2\ \cos\theta)^{\gamma_j}}\ d\theta$$

If we let $x = \sin^2 \frac{\theta}{2}$, we get

$$R = \prod_{j=1}^{n} (1-k_j^2)^{-\gamma_j} \int_0^1 e^{\alpha z} \; x^{\nu-\lambda-1} \; (1-x)^{\lambda-1} \; \prod_{j=1}^{n} (1-\omega_j x)^{-\gamma_j} \; dx ,$$

where $\omega_j = \dfrac{2k_j^2}{k_j^2 - 1}$, $j = 1, \ldots, n$.

$$= m \sum_{\gamma=0}^{\infty} \frac{\alpha^r}{r!} \sum_{m_1=0}^{\infty} \ldots \sum_{m_n=0}^{\infty} \frac{(\gamma_1)_{m_1} \cdots (\gamma_n)_{m_n}}{m_1! \;\ldots\; m_n!} (\omega_1)^{m_1} \ldots\ldots (\omega_n)^{m_n} \int_0^1 x^{\nu+r+m_1\ldots+m_n-\lambda-1} (1-x)^{\lambda-1} \; dx ,$$

$$= \frac{\Gamma(\nu-\lambda)\Gamma(\lambda)}{\Gamma(\nu)} m \sum_{r=0}^{\infty} \sum_{m_1=0}^{\infty} \ldots\ldots \sum_{m_n=0}^{\infty} \frac{(\nu-\lambda)_{r+m_1+\ldots+m_n} \cdot (\gamma_1)_{m_1} \cdots (\gamma_n)_{m_n} \cdots}{(\nu)_{r+m_1+\ldots+m_n} m_1! \ldots} .$$

$$\frac{\ldots(\gamma_n)_{m_n}}{m_1! \ldots m_n! \, \gamma!} \quad \alpha^r \; (\omega_1)^{m_1} \ldots (\omega_n)^{m_n} .$$

$$= \frac{m\Gamma(\nu-\lambda)\Gamma(\lambda)}{\Gamma(\nu)} \sum_{r=0}^{\infty} \sum_{m_1=0}^{\infty} \ldots \sum_{m_{n-1}=0}^{\infty} \frac{(\nu-\lambda)_{r+m_1+\ldots+m_{n-1}}}{(\nu)_{r+m_1+\ldots+m_{n-1}}} \frac{(\gamma_1)_{m_1} \ldots\ldots (\gamma_{n-1})_{m_{n-1}}}{m_1! \ldots\ldots m_{n-1}! \, \gamma!} .$$

$$\alpha^\gamma \; (\omega_1)^{m_1} \ldots (\omega_{n-1})^{m_{n-1}} . \; {}_2F_1 \; (\nu-\lambda+\gamma+m_1+\ldots+m_n, \gamma_n \; ; \nu+\gamma+m_1+m_2+\ldots+m_{n-1)} \; ; \ldots$$

Now using the transformation formula (6.1), we get

$${}_2F_1 \left(\nu-\lambda+r+m_1+\ldots+m_{n-1}, \gamma_n , \nu+r+m_1+\ldots+m_{n-1} \; ; \omega_n\right) =$$

$$\frac{\Gamma(\nu+\gamma+m_1+\ldots+m_{n-1})\,\Gamma(\gamma_n-\nu+\lambda-r-m_1-\ldots-m_{n-1})}{\Gamma(\lambda)\;\Gamma(\nu-\gamma_n+r+m_1+\ldots+m_{n-1})} \left(\frac{1}{\omega_n}\right)^{\nu+\gamma+m_1+\ldots+m_{n-1}} .$$

$${}_2F_1 \left(\nu-\lambda+r+m_1+\ldots+m_{n-1}, 1-\lambda \; ; 1+\nu-\lambda-\gamma_n+r+m_1+\ldots+m_{n-1)} \; ; \omega_n\right)$$

$$+ \frac{\Gamma(\nu+r+m_1+\ldots+m_{n-1})\Gamma(\nu-\lambda-\gamma_n+\gamma+m_1+\ldots+m_{n-1})}{\Gamma(\nu-\lambda+r+m_1+\ldots+m_{n-1})\;\Gamma(\nu-\gamma_n+\gamma+m_1+\ldots+m_{n-1})} \left(\frac{1}{\omega_n}\right)^{\gamma_n} .$$

$${}_2F_1 \left(\gamma_n , 1+\gamma_n-\nu-r-m_1-\ldots-m_{n-1} ; \; 1+\gamma_n-\nu+\lambda-r-m_1-\ldots-m_{n-1} ; \; \frac{1}{\omega_n}\right)$$

$$= \frac{\Gamma(\nu)\Gamma(\gamma_1-\nu+\lambda)}{\Gamma(\lambda)\Gamma(\nu-\gamma)_n} \left(\frac{1}{\omega_n}\right)^{\nu-\lambda} \left(-\frac{1}{\omega_n}\right)^{\gamma+m_1+\ldots+m_{n-1}} .$$

$$\sum_{m_n=0}^{\infty} \frac{(\nu)_{\gamma+m_1+\ldots+m_{n-1}} \; (\nu-\lambda+r+m_1+m_{n-1})_{m_n} \; (1-\lambda)_{m_n}}{(\nu-\gamma)_{r+m_1+\ldots+m_{n-1}} \; (1-\gamma_n+\nu-\lambda)_{r+m_1+\ldots+m_{n-1}} \; m_n!} .$$

$$\frac{1}{(1+\nu-\lambda-\gamma_n+r+m_1+\ldots+m_{n-1})_{m_n}}\left(\tfrac{1}{\omega_n}\right)^{m_n}+\frac{\Gamma(\nu)\Gamma(\nu-\lambda-\gamma_n)}{\Gamma(\nu-\lambda)\Gamma(\nu-\gamma_n)}\left(\tfrac{1}{\omega_n}\right)^{\gamma_n}.$$

$$\sum_{m_n=0}^{\infty}\frac{(\nu)_{r+m_1+\ldots+m_{n-1}}}{(\nu-\lambda)_{r+m_1+\ldots+m}}\cdot\frac{(\nu-\lambda-\gamma_n)_{\gamma+m_1+\ldots+m_{n-1}}\quad(\gamma_n)_{m_n}\quad(\nu-\lambda-\gamma_n+\gamma+m_1+\ldots+m_{n-1})_{m_n}\left(\frac{1}{\omega_n}\right)^{m_n}}{(\nu-\gamma_n+r+m_1+\ldots+m_{n-1})_{m_n}\quad m_n!}$$

Substitute ${}_2F_1$ in R to get

$$R=\frac{\Gamma(\nu-\lambda)\Gamma(\gamma_n-\nu+\lambda)}{\Gamma(\nu-\gamma_n)}(2k_n^2)^{\lambda-\nu-\gamma_n}\prod_{j=1}^{n-1}(1-k_j^2)^{-\gamma j}.$$

$$\sum_{r=o}^{\infty}\sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty}\frac{(\nu-\lambda)_{\gamma+m_1+\ldots+m_n}\;(\gamma_1)_{m_1}\cdots(\gamma_{n-1})_{m_{n-1}}\;(1-\lambda)_{m_n}}{(1+\nu-\lambda-\gamma_n)_{\gamma+m_1+\ldots+m_n}\;(\nu-\gamma_n)_{r+m_1+,,,+m_{n-1}}}.$$

$$\frac{1}{r!m_1!\ldots m_n!}\left(\alpha\frac{1}{\omega_n}\right)^{\gamma}\left(\frac{\omega_1}{\omega_n}\right)^{m_1}\cdots\left(\frac{\omega_{n-1}}{\omega_n}\right)^{m_{n-1}}\left(\frac{1}{\omega_n}\right)^{m_n}$$

$$+\Gamma(\lambda)\Gamma(\nu-\lambda-\gamma_n)(2k_n^2)^{-\gamma_n}\prod_{j=1}^{n-1}(1-k_j^2)^{-\gamma_j}\sum_{r=0}^{\infty}\sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty}\frac{(\nu-\lambda-\gamma_n)_{\gamma+m_1+\ldots+m_{n-1}-m_n}}{(\nu-\gamma_n)_{\gamma+m_1+\ldots+m_{n-1}-m_n}}.$$

$$\frac{(\gamma_1)_{m_1}\cdots(\gamma_1)_{m_n}}{r!m_1!\ldots.m_n!}\cdot\alpha^{\gamma}(\omega_1)^{m_1}\ldots(\omega_n)^{m_n}$$

Hence

$$R=\frac{\Gamma(\nu-\lambda)\Gamma(\lambda-\nu+\gamma_n)}{\Gamma(\nu-\gamma_n)}(2k_n^2)^{\lambda-\nu}\prod_{j=1}^{n-1}(1-k_j^2)^{-\gamma j}.$$

$$(1-k_n^2)^{\nu-\lambda-\gamma_n}(1+0(k_n^2-1))$$

$$+\frac{\Gamma(\lambda)\Gamma(\nu-\lambda-\gamma_n)}{\Gamma(\nu-\gamma_n)}(2k_n^2)^{-\gamma_n}\prod_{j=1}^{n-1}(1-k_j^2)^{-\gamma j}(1+0(k-1)).$$

Which gives the required asymptotic expansion of R.

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