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ON A CLASS OF *K*-CONTACT MANIFOLDS

Pradip Majhi, Y. Matsuyama & U. C. De
Department of Pure Mathematics
University of Calcutta
35, Ballygunge Circular Road
Kolkata-700019, India
sjpm12@yahoo.co.in / supriyo88@gmail.com

ABSTRACT. The object of the present paper is to study projective curvature tensor in K-contact manifolds. Projectively birecurrent K-contact manifold have been studied. K-contact manifold satisfying $P.R = 0$, $R(X,\xi).P = P(X,\xi).R$ and $P.S = 0$ are also considered. Finally, we study ϕ-projectively symmetric K-contact manifolds. It is shown that in all the cases the K-contact manifold becomes Sasakian.

1. INTRODUCTION

A complete regular contact metric manifold M^{2n+1} carries a K-contact structure (ϕ, ξ, η, g), defined in terms of the almost Kähler structure (J, G) of the base manifold M^{2n}. Here the K-contact structure (ϕ, ξ, η, g) is Sasakian if and only if the base manifold (M^{2n}, J, G) is Kählerian. If (M^{2n}, J, G) is only almost Kähler, then (ϕ, ξ, η, g) is only K-contact [3]. In a Sasakian manifold the Ricci operator Q commutes with ϕ, that is, $Q\phi = \phi Q$. In [12] it has been shown that there exists K-contact manifolds with $Q\phi = \phi Q$ which are not Sasakian. It is to be noted that a K-contact manifold being intermediate between a contact metric manifold and a Sasakian manifold. K-contact and Sasakian manifolds have been studied by several authors such as ([2], [6], [7], [8], [9], [10], [17], [18], [20], [21], [22]) and many others. It is well known that every Sasakian manifold is K-contact, but the converse ia not true, in general. However a three-dimensional K-contact manifold is Sasakian [11]. The nature of a manifold mostly depends on its curvature tensor. Using the tools of conformal transformation geometers have deduced conformal curvature tensor. In the similar way with the help of projective transformation the notion of projective curvature has been defined [15]. Apart from conformal curvature tensor, the projective curvature tensor is another important tensor from the differential geometric point of view. A Riemannian manifold is

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said to be semisymmetric if its curvature tensor R is satisfies $R(X,Y).R = 0$, where $R(X,Y)$ acts on R as a derivation [13].
The object of the present paper is to enquire under what conditions a K contact manifold will be a Sasakian manifold.
The present paper is organized as follows:
After in brief introduction in Section 2, we discuss about some preliminaries that will be used in the later sections. In section 3, we consider projectively birecurrent K-contact manifolds and prove that a projectively birecurrent K-contact manifold is Sasakian. Section 4 is devoted to study K-contact manifolds satisfying $P.R = 0$. In section 5, we consider K-contact manifolds satisfying $R(X,\xi).P = P(X,\xi).R$. Section 6 deals with K-contact manifolds satisfying $P.S = 0$. Finally, we study ϕ-projectively symmetric K-contact manifolds.

2. Preliminaries

An odd dimensional manifold M^{2n+1} $(n \geq 1)$ is said to admit an almost contact structure, sometimes called a (ϕ,ξ,η)-structure, if it admits a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η satisfying ([3], [4])

$$\phi^2 = -I + \eta \otimes \xi, \;\; \eta(\xi) = 1, \;\; \phi\xi = 0, \;\; \eta \circ \phi = 0. \tag{2.1}$$

The first and one of the remaining three relations in (2.1) imply the other two relations in (2.1). An almost contact structure is said to be normal if the induced almost complex structure J on $M^n \times \mathbb{R}$ defined by

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt}) \tag{2.2}$$

is integrable, where X is tangent to M, t is the coordinate of $\mathbb{R}$ and f is a smooth function on $M^n \times \mathbb{R}$. Let g be a compatible Riemannian metric with (ϕ,ξ,η), structure, that is,

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{2.3}$$

or equivalently,

$$g(X, \phi Y) = -g(\phi X, Y) \tag{2.4}$$

and

$$g(X,\xi) = \eta(X),$$

for all vector fields X, Y tangent to M. Then M becomes an almost contact metric manifold equipped with an almost contact metric structure (ϕ,ξ,η,g).
An almost contact metric structure becomes a contact metric structure if

$$g(X, \phi Y) = d\eta(X,Y), \tag{2.5}$$

for all X, Y tangent to M. The 1-form η is then a contact form and ξ is its characteristic vector field.
If ξ is a Killing vector field, then M^{2n+1} is said to be a K-contact manifold ([3], [14]). A contact metric manifold is Sasakian if and only if

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y. \tag{2.6}$$

Every Sasakian manifold is K-contact, but the converse need not be true, except in dimension three [13]. K-contact manifolds are not too well known, because there is no such a simple expression for the curvature tensor as in the case of Sasakian manifolds. For details we refer to ([1], [3], [14]).
Besides the above relations in K-contact manifold the following relations hold ([1], [3], [14]):

$$\nabla_X \xi = -\phi X. \tag{2.7}$$

$$\tilde{R}(\xi, X, Y, \xi) = \eta(R(\xi, X)Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.8}$$

$$R(\xi, X)\xi = -X + \eta(X)\xi. \tag{2.9}$$

$$S(X, \xi) = 2n\eta(X). \tag{2.10}$$

$$(\nabla_X \phi)Y = R(\xi, X)Y, \tag{2.11}$$

for any vector fields X, Y.
Again a K-contact manifold is called Einstein if the Ricci tensor S is of the form $S = \lambda g$, where λ is a constant and η- Einstein if the Ricci tensor S is of the form $S = ag + b\eta \otimes \eta$, where a, b are smooth functions on M. It is well known [11] that in a K-contact manifold a and b are constants. Also it is known [5] that a compact η-Einstein K-contact manifold is Sasakian provided $a \geq$-2.
Let M be a $(2n+1)$-dimensional Riemannian manifold. If there exists a one-to-one correspondence between each coordinate neighbourhood of M and a domain in Euclidian space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 1$, M is locally projectively flat if and only if the well known projective curvature tensor P vanishes. Here P is defined by [15]

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{2n}[S(Y, Z)X - S(X, Z)Y], \tag{2.12}$$

for all X, Y, $Z \in T(M)$, where R is the curvature tensor and S is the Ricci tensor. In fact M is projectively flat if and only if it is of constant curvature [19]. Thus the projective curvature tensor is the measure of the failure of a Riemannian manifold to be of constant curvature.
A Riemannian manifold M is called projectively recurrent if there exist a 1-form α such that $(\nabla_X P)Y = \alpha(X)PY$ for any X and Y tangent to M. Also a Riemmanian manifold is called projectively birecurrent if there exist a covariant tensor field α of order 2 such that $(\nabla_X \nabla_Y P - \nabla_{\nabla_X Y} P)W = \alpha(X, Y)PW$ for all X, Y, W tangent to M.

Definition 2.1. *A contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is said to be locally ϕ-symmetric in the sense of Takahashi [16] if it satisfies*

$$\phi^2((\nabla_W R)(X, Y)Z) = 0,$$

for all vector fields X, Y, Z, W orthogonal to ξ.

3. Projectively birecurrent K-contact manifolds

In this section we consider projectively birecurrent K-contact manifolds. In [9], De at al proved that a projectively semisymmetric K-contact manifold is Sasakian and a projectively recurrent K-contact manifold is also Sasakian.
To prove the main result of this section we first prove the following:

Lemma 3.1. *A projectively birecurrent Riemannian manifold is projectively semi-symmetric.*

Proof. Suppose that M is a Riemannian manifold with birecurrent projective curvature tensor. We put

$$P(X,Y,W,Z) = g(P(X,Y)W,Z);$$
$$P^2(X,Y,W,Z,U,V) = g((P(X,Y)P)(W,Z)U,V).$$

In order to prove this lemma, we use the equation

$$\nabla_Y P^2 = (\nabla_Y P)P + P(\nabla_Y P). \tag{3.1}$$

and consider the equation

$$\begin{aligned} & (\nabla_X(\nabla_Y P^2))Z - (\nabla_{\nabla_X Y} P^2)Z \\ = \; & \nabla_X((\nabla_Y P^2)Z) - (\nabla_Y P^2)\nabla_X Z - (\nabla_{\nabla_X Y} P^2)Z \end{aligned} \tag{3.2}$$

for any X, Y, Z tangent to M^n.
Using (3.1) in (3.2) we have

$$\begin{aligned} & (\nabla_X(\nabla_Y P^2))Z - (\nabla_{\nabla_X Y} P^2)Z \\ = \; & \nabla_X((\nabla_Y P)PZ + P(\nabla_Y P)Z) - ((\nabla_Y P)P)\nabla_X Z - (P\nabla_Y P)\nabla_X Z \\ & -(\nabla_{\nabla_X Y} P)PZ - P(\nabla_{\nabla_X Y} P)Z \\ = \; & (\nabla_X \nabla_Y P)PZ + (\nabla_Y P)(\nabla_X P)Z + (\nabla_Y P)P\nabla_X Z \\ & +(\nabla_X P)(\nabla_Y P)Z + P(\nabla_X\nabla_Y P)Z + P(\nabla_Y P)\nabla_X Z \\ & -(\nabla_Y P)P\nabla_X Z - (P\nabla_Y P)\nabla_X Z - (\nabla_{\nabla_X Y} P)PZ - P(\nabla_{\nabla_X Y} P)Z \\ = \; & (\nabla_X\nabla_Y P - \nabla_{\nabla_X Y} P)PZ + P(\nabla_X \nabla_Y P - \nabla_{\nabla_X Y} P)Z \\ & +(\nabla_Y P)(\nabla_X P)Z + (\nabla_X P)(\nabla_Y P)Z. \end{aligned} \tag{3.3}$$

Now, using the assumption that P is the birecurrent Ricci tensor, we see that

$$(\nabla_X \nabla_Y P - \nabla_{\nabla_X Y} P)W = \alpha(X,Y)PW \tag{3.4}$$

Hence it follows that

$$\begin{aligned} & (\nabla_X(\nabla_Y P^2))Z - (\nabla_{\nabla_X Y} P^2)Z \\ = \; & (\alpha(X,Y)P)PZ + P(\alpha(X,Y)P)Z + (\nabla_Y P)(\nabla_X P)Z + (\nabla_X P)(\nabla_Y P)Z \\ = \; & 2\alpha(X,Y)P^2 Z + (\nabla_Y P)(\nabla_X P)Z + (\nabla_X P)(\nabla_Y P)Z. \end{aligned} \tag{3.5}$$

Similar calculation shows that

$$\begin{aligned} & (\nabla_Y(\nabla_X P^2))Z - (\nabla_{\nabla_Y X} P^2)Z \\ = \; & 2\alpha(Y,X)P^2 Z + (\nabla_X P)(\nabla_Y P)Z + (\nabla_Y P)(\nabla_X P)Z. \end{aligned} \tag{3.6}$$

Hence we obtain

$$\begin{aligned}
& (\nabla_X(\nabla_Y P^2) - \nabla_{\nabla_X Y} P^2)Z - (\nabla_Y(\nabla_X P^2) - \nabla_{\nabla_Y X} P^2)Z \\
= \; & (\nabla_X \nabla_Y P^2 - \nabla_Y \nabla_X P^2 - \nabla_{[X,Y]} P^2)Z \\
= \; & 2(\alpha(X,Y) - \alpha(Y,X))P^2 Z. \qquad (3.7)
\end{aligned}$$

From this equation and the commutativity of the contraction and the derivation, we have

$$(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})(P^2)^{kj}{}_{j}{}^{i}{}_{ik} = 2(\alpha(X,Y) - \alpha(Y,X))(P^2)^{kj}{}_{j}{}^{i}{}_{ik}, \qquad (3.8)$$

Since $(P^2)^{kj}{}_{j}{}^{i}{}_{ik}$ is a differentiable function on M^n, the left hand side of this equation equals to zero. Therefore we get $\alpha(X,Y) = \alpha(Y,X)$ or $(P^2)^{kj}{}_{j}{}^{i}{}_{ik} = 0$. If $(P^2)^{kj}{}_{j}{}^{i}{}_{ik} = 0$, then we deduce that $P = 0$. Hence we can see that $(R(X,Y)P)(W,Z)U = 0$. If $\alpha(X,Y) = \alpha(Y,X)$, then we have

$$\begin{aligned}
(R(X,Y)P)(W,Z)U &= (\nabla_X \nabla_Y P - \nabla_Y \nabla_X P - \nabla_{[X,Y]} P)(W,Z)U \\
&= (\nabla_X \nabla_Y P - \nabla_Y \nabla_X P - \nabla_{\nabla_X Y} P + \nabla_{\nabla_Y X} P)(W,Z)U \\
&= (\alpha(X,Y) - \alpha(Y,X))P(W,Z)U \\
&= 0. \qquad (3.9)
\end{aligned}$$

Therefore we conclude that $(R(X,Y)P)(W,Z)U = 0$ for any vector fields X, Y, Z, W and U tangent to M.
Hence by virtue of the result of De et al [9] we can state the following:

Theorem 3.1. *A projectively birecurrent K-contact manifold is Sasakian*

4. K-contact manifolds satisfying $P.R = 0$

In view of (2.12) the projective curvature tensor of a $(2n+1)$-dimensional K-contact manifold is given by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y]. \qquad (4.1)$$

Now from the above equation with the help of (2.1) and (2.9) we get

$$P(\xi,V)\xi = 0 = P(V,\xi)\xi. \qquad (4.2)$$

In this section we study K-contact manifolds satisfying

$$(P(X,Y).R)(U,V)W = 0. \qquad (4.3)$$

Substituting $Y = \xi$ in (4.3) we have

$$\begin{aligned}
(P(X,\xi).R)(U,V)W &= P(X,\xi)R(U,V)W - R(P(X,\xi)U,V)W \\
&\quad - R(U,P(X,\xi)V)W - R(U,V)P(X,\xi)W \qquad (4.4)
\end{aligned}$$

Putting $U = W = \xi$ in (4.4), we get

$$\begin{aligned}
(P(X,\xi).R)(\xi,V)\xi &= P(X,\xi)R(\xi,V)\xi - R(P(X,\xi)\xi,V)\xi \\
&\quad - R(\xi,P(X,\xi)V)\xi - R(\xi,V)P(X,\xi)\xi \qquad (4.5)
\end{aligned}$$

Now,

$$\begin{aligned} P(X,\xi)R(\xi,V)\xi &= P(X,\xi)(-V+\eta(V)\xi) \\ &= -P(X,\xi)V+\eta(V)P(X,\xi)\xi \\ &= -P(X,\xi)V. \end{aligned} \tag{4.6}$$

$$R(P(X,\xi)\xi,V)\xi = 0. \tag{4.7}$$

$$\begin{aligned} R(\xi,P(X,\xi)V)\xi &= -P(X,\xi)V+g(P(X,\xi)V,\xi)\xi \\ &= -P(X,\xi)V-g(X,V)\xi+\frac{1}{2n}S(X,V)\xi. \end{aligned} \tag{4.8}$$

$$R(\xi,V)P(X,\xi)\xi = 0. \tag{4.9}$$

Using (4.6), (4.7), (4.8) and (4.9) in (4.5) we have

$$-P(X,\xi)V+P(X,\xi)V+g(X,V)\xi-\frac{1}{2n}S(X,V)\xi = 0. \tag{4.10}$$

Taking inner product of (4.10) by ξ we obtain

$$S(X,V) = 2ng(X,V). \tag{4.11}$$

Therefore the manifold is an Einstein manifold. Thus we can state the following:

Theorem 4.1. *A K-contact manifold satisfying $P.R = 0$ is an Einstein manifold.*

It is know that [5] a compact K-contact Einstein manifold is Sasakian. Thus we get the following:

Corollary 4.1. *A compact K-contact manifold satisfying $P.R = 0$ is Sasakian.*

5. K-CONTACT MANIFOLDS SATISFYING $R(X,\xi).P = P(X,\xi).R$

In this section we study K-contact manifolds satisfying

$$R(X,\xi).P = P(X,\xi).R \tag{5.1}$$

Now,

$$\begin{aligned} (R(X,\xi).P)(U,V)W &= R(X,\xi)P(U,V)W-P(R(X,\xi)U,V)W \\ &\quad -P(U,R(X,\xi)V)W-P(U,V)R(X,\xi)W. \end{aligned} \tag{5.2}$$

Putting $U = W = \xi$ in (5.2) we have

$$\begin{aligned} (R(X,\xi).P)(\xi,V)\xi &= R(X,\xi)P(\xi,V)\xi-P(R(X,\xi)\xi,V)\xi \\ &\quad -P(\xi,R(X,\xi)V)\xi-P(\xi,V)R(X,\xi)\xi. \end{aligned} \tag{5.3}$$

From (4.2) we obtain

$$R(X,\xi)P(\xi,V)\xi = 0 = P(\xi,R(X,\xi)V)\xi. \tag{5.4}$$

Again

$$\begin{aligned} P(R(X,\xi)\xi,V)\xi &= P(X-\eta(X)\xi,V)\xi \\ &= P(X,V)\xi-\eta(X)P(\xi,V)\xi \\ &= P(X,V)\xi. \end{aligned} \tag{5.5}$$

and

$$\begin{aligned} P(\xi,V)R(X,\xi)\xi &= P(\xi,V)(X-\eta(X)\xi) \\ &= P(\xi,V)X-\eta(X)P(\xi,V)\xi \\ &= P(\xi,V)X. \end{aligned} \tag{5.6}$$

Using (5.4), (5.5), (5.6) in (5.3) we have

$$(R(X,\xi).P)(\xi,V)\xi=-P(X,V)\xi-P(\xi,V)X. \tag{5.7}$$

On the other hand

$$\begin{aligned} (P(X,\xi).R)(U,V)W &= P(X,\xi)R(U,V)W-R(P(X,\xi)U,V)W \\ &\quad -R(U,P(X,\xi)V)W-R(U,V)P(X,\xi)W. \end{aligned} \tag{5.8}$$

Putting $U=W=\xi$ in (5.8), we get

$$\begin{aligned} (P(X,\xi).R)(\xi,V)\xi &= P(X,\xi)R(\xi,V)\xi-R(P(X,\xi)\xi,V)\xi \\ &\quad -R(\xi,P(X,\xi)V)\xi-R(\xi,V)P(X,\xi)\xi. \end{aligned} \tag{5.9}$$

Now,

$$\begin{aligned} P(X,\xi)R(\xi,V)\xi &= P(X,\xi)(-V+\eta(V)\xi) \\ &= -P(X,\xi)V+\eta(V)P(X,\xi)\xi \\ &= -P(X,\xi)V. \end{aligned} \tag{5.10}$$

$$R(P(X,\xi)\xi,V)\xi=0. \tag{5.11}$$

$$\begin{aligned} R(\xi,P(X,\xi)V)\xi &= -P(X,\xi)V+g(P(X,\xi)V,\xi)\xi \\ &= -P(X,\xi)V-g(X,V)\xi+\frac{1}{2n}S(X,V)\xi. \end{aligned} \tag{5.12}$$

$$R(\xi,V)P(X,\xi)\xi=0. \tag{5.13}$$

Using (5.10), (5.11), (5.12), (5.13) in (5.9) we have

$$(P(X,\xi).R)(\xi,V)\xi=[g(X,V)-\frac{1}{2n}S(X,V)]\xi. \tag{5.14}$$

Therefore using (5.7) and (5.14) in (5.1) we have

$$-P(X,V)\xi-P(\xi,V)X=g(X,V)\xi-\frac{1}{2n}S(X,V)\xi. \tag{5.15}$$

With the help of (2.12), (5.15) becomes

$$R(X,V)\xi + R(\xi,V)X + 2\eta(X)V - \eta(V)X = \frac{1}{n}S(X,V)\xi - g(X,V)\xi. \quad (5.16)$$

Interchanging X and V in (5.16) we get

$$R(V,X)\xi + R(\xi,X)V + 2\eta(V)X - \eta(X)V = \frac{1}{n}S(V,X)\xi - g(V,X)\xi. \quad (5.17)$$

Subtracting (5.17) from (5.16) we have

$$R(X,V)\xi + R(\xi,V)X - R(V,X)\xi - R(\xi,X)V + 3\eta(X)V - 3\eta(V)X = 0. (5.18)$$

Using Bianchi identity we get from the above equation

$$R(X,V)\xi - R(V,X)\xi - R(V,X)\xi + 3\eta(X)V - 3\eta(V)X = 0. \quad (5.19)$$

It follows that

$$R(X,V)\xi = \eta(V)X - \eta(X)V. \quad (5.20)$$

Thus we can state the following:

Theorem 5.1. *A K-contact manifold satisfying $R(X,\xi).P = P(X,\xi).R$ is a Sasakian manifold.*

6. K-CONTACT MANIFOLDS SATISFYING $P.S = 0$

In this section we study K-contact manifold satisfying $P.S = 0$. Therefore

$$(P(X,Y).S)(U,V) = 0. \quad (6.1)$$

This implies

$$S(P(X,Y)U,V) + S(U,P(X,Y)V) = 0. \quad (6.2)$$

Putting $Y = U = \xi$ in (6.2) we obtain

$$S(P(X,\xi)\xi,V) + S(\xi,P(X,\xi)V) = 0. \quad (6.3)$$

Using (4.2) in (6.3), we have

$$S(\xi,P(X,\xi)V) = 0. \quad (6.4)$$

This implies

$$2ng(R(X,\xi)V - \frac{1}{2n}[S(\xi,V)X - S(X,V)\xi],\xi) = 0. \quad (6.5)$$

It follows that

$$g(R(X,\xi)V,\xi) - \frac{1}{2n}[2n\eta(V)\eta(X) - S(X,V)] = 0. \quad (6.6)$$

Therefore

$$S(X,V) = 2ng(X,V). \quad (6.7)$$

Hence the manifold is an Einstein manifold.
Conversely, the manifold is an Einstein manifold, that is, $S(X,V) = 2ng(X,V)$.

$$\begin{aligned}(P(X,Y).S)(U,V) &= S(P(X,Y)U,V) + S(U,P(X,Y)V) \\ &= 2n[g(P(X,Y)U,V) + g(U,P(X,Y)V]. \qquad (6.8)\end{aligned}$$

Since

$$g(P(X,Y)U,V) = -g(P(X,Y)V,U). \qquad (6.9)$$

Using (6.9) in (6.8) we have

$$(P(X,Y).S)(U,V) = 0. \qquad (6.10)$$

Thus we can state the following:

Theorem 6.1. *A K-contact manifold satisfying $P.S = 0$ if and only if it is an Einstein manifold.*

It is know that [5] a compact K-contact Einstein manifold is Sasakian. Thus we get the following:

Corollary 6.1. *A compact K-contact manifold satisfying $P.S = 0$ is Sasakian.*

7. ϕ-projectively symmetric K-contact manifolds

In this section firstly we give the following:

Definition 7.1. *A contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is said to be ϕ-projectively symmetric if it satisfies*

$$\phi^2((\nabla_W P)(X,Y)Z) = 0,$$

for all vector fields X, Y, Z, W.
If the vector fields X, Y, Z, W are orthogonal to ξ, then the contact metric manifold is said to be locally ϕ-projectively symmetric.
Now we investigate ϕ-projectively symmetric K-contact metric manifolds. Therefore

$$\phi^2((\nabla_W P)(X,Y)Z) = 0, \qquad (7.1)$$

for all vector fields X, Y, Z, W. From (2.12) we have

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y], \qquad (7.2)$$

Now differentiating (7.2) covariantly with respect to W we get

$$(\nabla_W P)(X,Y)Z = (\nabla_W R)(X,Y)Z - \frac{1}{2n}[(\nabla_W S)(Y,Z)X - (\nabla_W S)(X,Z)Y].(7.3)$$

Applying ϕ both sides of (7.3) we have

$$\begin{aligned}\phi^2(\nabla_W P)(X,Y)Z &= \phi^2[(\nabla_W R)(X,Y)Z - \frac{1}{2n}[(\nabla_W S)(Y,Z)X \\ &\quad -(\nabla_W S)(X,Z)Y]]. \qquad (7.4)\end{aligned}$$

Using (7.1) in (7.4) we obtain

$$\phi^2((\nabla_W R)(X,Y)Z) = \frac{1}{2n}[(\nabla_W S)(Y,Z)\phi^2 X - (\nabla_W S)(X,Z)\phi^2 Y]. \tag{7.5}$$

Taking inner product of (7.5) with respect to U we get

$$\begin{aligned} g(\phi^2((\nabla_W R)(X,Y)Z),U) &= \frac{1}{2n}[(\nabla_W S)(Y,Z)g(\phi^2 X,U) \\ &\quad -(\nabla_W S)(X,Z)g(\phi^2 Y,U)]. \end{aligned} \tag{7.6}$$

Putting $Z = \xi$ in (7.6) we have

$$\begin{aligned} g(\phi^2((\nabla_W R)(X,Y)\xi),U) &= \frac{1}{2n}[(\nabla_W S)(Y,\xi)g(\phi^2 X,U) \\ &\quad -(\nabla_W S)(X,\xi)g(\phi^2 Y,U)]. \end{aligned} \tag{7.7}$$

Let $\{e_i\}$, $i = 1,2,...,(2n+1)$ be an orthonormal basis of the tangent space at each point of the manifold. Then putting $X = U = e_i$ in (7.7) and taking summation over $1 \leq i \leq (2n+1)$, we obtain

$$-(\nabla_W S)(Y,\xi) = -(\nabla_W S)(Y,\xi) + \frac{1}{2n}(\nabla_W S)(Y,\xi) - \frac{1}{2n}(\nabla_W S)(\xi,\xi)\eta(Y). \tag{7.8}$$

Therefore from (7.8) we obtain

$$(\nabla_W S)(Y,\xi) = (\nabla_W S)(\xi,\xi)\eta(Y). \tag{7.9}$$

Now,

$$\begin{aligned} (\nabla_W S)(Y,\xi) &= \nabla_W S(Y,\xi) - S(\nabla_W Y,\xi) - S(Y,\nabla_W \xi) \\ &= 2n\nabla_W \eta(Y) - 2n\eta(\nabla_W Y) - S(Y,-\phi W) \\ &= 2n(\nabla_W \eta)(Y) + S(Y,\phi W). \end{aligned} \tag{7.10}$$

Also

$$\begin{aligned} (\nabla_W \eta)(Y) &= \nabla_W \eta(Y) - \eta(\nabla_W Y) \\ &= \nabla_W g(Y,\xi) - g(\nabla_W Y,\xi) - g(Y,\nabla_W \xi) + g(Y,\nabla_W \xi). \\ &= (\nabla_W g)(Y,\xi) + g(Y,\nabla_W \xi) \\ &= -g(Y,\phi W). \end{aligned} \tag{7.11}$$

Using (7.10), (7.11) and (7.9) we obtain

$$S(Y,\phi W) = 2ng(Y,\phi W). \tag{7.12}$$

Putting $W = \phi W$ in (7.12) we have

$$S(Y,W) = 2ng(Y,W). \tag{7.13}$$

Hence the manifold is an Einstein manifold. Thus we can state the following:

Theorem 7.1. *A ϕ-projectively symmetric K-contact manifold is an Einstein manifold.*

It is know that [5] a compact K-contact Einstein manifold is Sasakian. Thus we get the following:

Corollary 7.1. *A compact ϕ-projectively symmetric K-contact manifold is Sasakian.*

It is known that every ϕ-projectively symmetric manifold is locally ϕ-projectively symmetric. Also every locally ϕ-symmetric manifold is locally ϕ-projectively symmetric. Thus in view of Theorem 6.1. we can state the following:

Theorem 7.2. *A locally ϕ-symmetric K-contact manifold is an Einstein manifold.*
Again in [16] Takahashi proved that every locally ϕ-symmetric Sasakian manifold is a manifold of constant curvature 1 and hence an Einstein manifold. Since every Sasakian manifold is K-contact, therefore we obtain the following:

Theorem 7.3. *Every locally ϕ-symmetric Sasakian manifold is an Einstein manifold.*
This Theorem was proved by T. Takahashi [16] in an another way.

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