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FACTORIZATION OF UNBOUNDED WEAKLY COMPACT OPERATORS

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Abstract. In this paper the factorisation result of Davis, Figiel, Johnson and Pelczynski [DFJP] from bounded operators is generalise for unbounded operators .

Resumen. En esta nota generalizamos el resultado de factorizacion de Davis, Figiel, Johnson y Pelczynski [DFJP] para operadores acotados debilmente compactos actuando entre espacios de Banach al caso general de operadores lineales debilmente compactos no necesariamente continuos y actuando entre espacios normados arbitrarios.

Keywords: weakly compact operator, reflexive space, factorization of unbounded operator.

1. INTRODUCTION

Let X and Y be normed spaces and T is a linear operator with domain a linear subspace D(T) of X and range R(T) contained in Y. The class of all such operators T will be denoted by L(X,Y).

We shall denote by X_T the vector space D(T) with the norm $\|x\|_T = \|x\| + \|Tx\|$ and the operator G_T (or simply G) denotes the canonical injection from X_T into D(T). Continuous operators T with D(T) = X will be referred to as bounded; TG is thus a bounded operator. The adjoint of T is the operator defined by $T' : D(T') \subset Y' \to D(T)'$ where $D(T') = \{y' \in Y' : y'\,T$ is continuous on $D(T)\}$ and $T'y'(x) = y'(Tx)$ for $x \in D(T)$ and $y' \in D(T')$ [11; II.2.2].

We shall write $T_o = J_Y T$ where J_Y is the natural injection of Y into its completion $Y^{\sim}$, and J denotes the canonical isometry of a given normed space into its second dual.

If T is continuous, then $T^{\sim}$ is the closure of T regarded as an element of $L(X^{\sim},Y^{\sim})$. In particular $(TG)^{\sim}$is a bounded operator in $L((X_T)^{\sim},Y^{\sim})$.

If M is a linear subspace of X we write T/M for the restriction of T to M; by the usual convention $T/M = T/M \cap D(T)$.

The operator T is called closable if T has an extension $\overline{T}$ whose graph is $\overline{G(T)}$. If $T = \overline{T}$ then T is said to be closed. A closed operator T is called a $\emptyset_+$-operator ($\emptyset_-$-operator, Fredholm operator) if R(T) is closed and its null space is finite dimensional (if R(T) is closed and of finite codimension,if $T \in \varnothing_+ \cap \varnothing_-$). We call T a F_+- operator (partially continuous) if there exists a linear subspace M of X of finite codimension such that $(T/M)^{-1}$ exists and is continuous (T/M is continuous) [4],and T is an F- -operator if $T' \in \varnothing_+$) [7].

I will be recalled (see,e.g [3], [8], [9]) some facts about bounded operators. Let T be bounded and X and Y be Banach spaces. Then T is called weakly compact if TB_X is relatively $\sigma(Y,Y')$-compact (B_X denotes the closed unit ball of X) and the following properties are equivalent:

(1) T is weakly compact.

(2) T' is $\sigma(Y',Y)$-$\sigma(X',X'')$ continuous.

(3) $T''X'' \subset JY$.

(4) T' is weakly compact.

(5) T factors through a reflexive space.

In the general case of $T \in L(X,Y)$ we observe that the second adjoint of T presents as an operator $T'' : D(T'') \subset D(T)'' \rightarrow D(T')'$ and thus $R(T'') \subset IQY''$ where Q denotes the quotient map of Y'' onto $Y''/D(T')^{\perp}$ and I is the canonical isometry mapping of QY'' onto $D(T')'$.

The operator $T \in L(X,Y)$ is said to be weakly compact if $TB_{D(T)}$ is relatively $\sigma(Y, D(T'))$-compact [1]. Some characterizations analogous to (1)-(2) and (3) in the general case of an arbitrary operator T are given in [1] where also it is investigated the connection between weak compactness of T and that of T'.

In the present paper we will analyze the equivalence of properties (1) and (5) in the general situation.

Throughout the remainder of the paper $T \in L(X,Y)$ where X and Y are normed spaces.

2.FACTORIZATION OF AN UNBOUNDED WEAKLY COMPACT OPERATOR

Given an operator $T \in L(X,Y)$ we shall obtain a factorisation $T_o = jA$ through a Banach space Z_p ($1 \leq p \leq \infty$) where j is a tauberian injection and AG_T is a bounded operator. We factorise T_o instead of T in order to retain the character of the [DFJP] factorisation. The space Z_p is constructed by setting $W := \{T_o x : \|x\|_T \leq 1, x \in D(T)\}$ and

then proceeding in the usual way from the gauges of the set $U_n = 2^nW + 2^{-n}B_{Y\sim}$.
We have the bounded operators

$$B : (X_T)^\sim \to Z_p, \; j: Z_p \to Y^\sim , \; A = BJ_{XT}G^{-1} : X \to Z_p ,$$

and $(TG)^\sim = jB$ is the usual factorisation. Hence $T_o = jA$ where $A = BJ_{XT}G$-1 so that AG is bounded . Since T_oGB_{XT} is contained in $(TGB_{XT})^\sim$ the gauges of the sets obtained from $W = T_oGB_{XT}$ coincide with those used in the [DFJP] construction, based on $(TG)^\sim B_{(XT)^\sim}$. Hence the following theorem:

2.1 Theorem.([DFJP] factorization of T_O). *Let $T \in L(X,Y)$ be given .Then corresponding to each $1 \leq p \leq \infty$ there is a Banach space Z_p and a factorisation*

$$A : D(T) \to Z_p, \qquad j_p: Z_p \to Y^\sim, \qquad T_o = j_pA$$

in which j_p is a bounded Tauberian injection, AG is bounded, and $(TG)^\sim = j_p(AG)^\sim$. Moreover if $p>1$ then the second adjoint of j_p is injective. The map j_p coincides with the Tauberian injection in the [DFJP] factorisation of $(TG)^\sim$ corresponding to p.

Proof. (see [2])

2.2. Proposition. *Let T be densely defined and partially continuous. Then T' is weakly compact if and only if T_O factors through a reflexive space.*

Proof. Suppose that T' is weakly compact . By 2.1 Theorem it is enough to show that $(TG)^\sim$factors through a reflexive space and for this it suffices, by the bounded case, to prove that $((TG)^\sim)' = (TG)'$ is weakly compact. But $T' = (G^{-1})'(TG)'$ [11; lemma 2.4] and since T is partially continuous, so G^{-1} and hence $(G^{-1})'$ is continuous [5; corol.5]. Moreover $(G^{-1})' = (G')^{-1}$ (as D(T) is dense) and consequently R(G') is closed [10; II.3.1]. Therefore $G'T' = I_{R(G')}(TG)'$ and the hypothesis on T' now gives G'T' is weakly compact [1;prop. 2.3] and by [12; 4.6.9] we conclude that (TG)' is weakly compact as required.

Conversely, assume that T_O factors through a reflexive space. Then since the maps j_p in the [DFJP] factorisation of T_O and $(TG)^\sim$coincide by 2.1 Theorem it follows that

(TG)' is weakly compact and an argument similar to the preceding implication shows that $T' = (G^{-1})'(TG)'$ is weakly compact.

2.3. Theorem. *Let T be densely defined and partially continuous. Consider the following properties:*

(i) T is weakly compact.

(ii) T_O factors through a reflexive space.

Then (i) $\Rightarrow$ (ii), and if Y is complete then the two properties (i) and (ii) are equivalent.

Proof. (i) $\Rightarrow$ (ii). It follows readily from the above proposition and that if T is weakly compact then so is T' [1; corol. 4.4].

(ii)$\Rightarrow$ (i). Suppose that Y is Banach. Then by [1; Th. 2.7] it is sufficient to show that T' is $\sigma(D(T'),Y)$-$\sigma(D(T)', D(T)'')$ continuous (as $D(T)'' = D(T'')$ by the continuity of T'). But if T satisfies (ii) then by 2.1 Theorem and the bounded result (4) $\Leftrightarrow$(5) we deduce that (TG)' is $\sigma(Y',Y)$-$\sigma((X_T)', (X_T)'')$ continuous (*) and since $(G^{-1})'$ is continuous we have $(G^{-1})'$ is $\sigma(D(G^{-1})', (D(G^{-1})')')$- $\sigma(D(T)', D(T)'')$ continuous (**) [13; p. 234. ex.9].

Let $(y'\alpha)$ be a net in D(T') which is $\sigma(D(T'),Y)$-convergent to some point $y'\in Y'$. Since Y is complete and T' is continuous is D(T') $\sigma(Y',Y)$-closed [6; Th. 5.8] and so, $y'\in D(T')$. Since $T' = (G^{-1})'(TG)'$ [11; lemma 2.4] it follows from (*) that $\sigma((X_T)', (X_T)'')\text{-}\lim(TG)'y'\alpha = \sigma((X_T)', (X_T)'')\text{-}\lim\ i(TG)'y'\alpha = (TG)'y' = i(TG)'y'$ where i is the inclusion of $D(G^{-1})'$ in $(X_T)'$ and since i' is surjective [10; IV.1.2] we deduce that $\sigma(D(G^{-1})', (D(G^{-1})')')\text{-}\lim(TG)'y'\alpha = (TG)'y'$ and by (**) we conclude that $\sigma(D(T)',D(T)'')\text{-}\lim\ (G^{-1})(TG)'y'\alpha = \sigma(D(T)', D(T)'')\text{-}\lim T'y'\alpha = T'y'$ as required.

The following examples show that the properties " T weakly compact " and " T factors through a reflexive space " are not equivalent for an arbitrary operator T.

There is a weakly compact operator T acting between Banach spaces such that T has not a factorization through a reflexive space.

Let $X= Y = L_1[0,1]$ and define T by $D(T) = \{f\in L_1[0,1]$: f' exists almost everywhere and $f' \in L_1[0,1]\}$. Then T is densely defined and $D(T') = \{0\}$ [7; ex. 2.1].

Hence T is not partially continuos [5; Th.3]. Now, let M be dense subspace of D(T) consisting of the absolutely continuous functions. Then T/M is a surjective closed Fredholm operator with one-dimensional null space [10; VI.3.1] and so by [7; Th. 4.3] $(T/M)G_{T/M}$ is a F_--operator. Hence $((T/M)G_{T/M})'$ is a F_+-operator with non reflexive domain and by [1; prop. 3.5] is not weakly compact.

Suppose that T factors through a reflexive space, then by 2.1 Theorem and the bounded case it follows that $((T/M)G_{T/M})'$ is weakly compact; a contradiction.

There exists an operator T which factors through a reflexive space but is not weakly compact.

Let T be an unbounded closable operator in L(X,Y) with Y reflexive. Then D(T') is a dense proper subspace of Y' [6; Remark 5.12] and hence T is not partially continuous since adjoints of partially continuos are continuous [5; Cor. 5]. Moreover it is clear that T factors through a reflexive space .However T is not weakly compact since if Y is complete and T is closable then T is weakly compact if and only if T' is weakly compact (and then T is continuous) [1; cor. 4.3]. Let X= $L_1[0,1]$, Y= $L_2[0,1]$ and define T and M as the previous example . Then T/M is surjective closed Fredholm operator with one-dimensional null space [11; VI.3.1]. In particular T/M is an unbounded closable operator.

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