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ALGEBRAIC DISCRETE SEMI-DYNAMICAL SYSTEMS

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ABSTRACT

In the present paper we study the structure of discrete semi-dynamical system on a set X without any kind of topological structure.

In the first section we do not impose any structure on the set X. Some results are then obtained relative to the classification of solutions and to the invariance of subsets.

In the second section we assume X is a partially ordered set. This allows us to introduce the weak notions which are adequate for systems without uniqueness. Of particular interest in the characterization of the weak positive invariance presented here.

KEY WORDS: Dynamical systems, weak solution, weak invariance.

0. INTRODUCTION

In recent years multiple applications of the theory of systems have been found, that motivate the wide expansion of the abstract study of systems.

In 1970, G.P. Szegö and G. Treccani [9] have introduced the notion of discrete semi-dynamical system without uniqueness in order to perform an axiomatic study of some algorithms of optimization. Later on new applications of such a theory have been found.

The semi-dynamical systems used in such applications are defined on the family of the non-empty compact subsets of an Euclidean space [9], or of a Hilbert space [4], where several structures are then defined. Many of the results that have been found for such semi-systems do not require so much structure. This is the reason why in this paper we consider a discrete semi-dynamical system on an arbitrary set X. We introduce the necessary algebraic notions and we obtain some results related to them. We then impose on X a partial order and this allows us to introduce the corresponding weak notions. Of particular interest is the study of the weak invariance and the characterization of the weakly positively invariant subsets, that play a so important role in the applications.

1. DISCRETE SEMI-DYNAMICAL SYSTEMS

1.1 Definition and properties

Notation: X denotes an infinite set; I^+ denotes the set of nonnegative integers, I^- denotes the set of nonpositive integers; Π denotes a map from the product set $X \times I^+$ into X; the image $\Pi(x,t)$ of an element (x,t) in $X \times I^+$ will be written simply as xt.

The triplet (X, I^+, Π) is called a discrete semi-dynamical system on X, if the two following conditions hold:

i) $x0=x$, $x \in X$.

ii) $(xt)s=x(t+s)$, $x \in X$ and $t,s \in I^+$.

In the last definition, i) can be replaced by

i') Image Π = X.

Let f be a map from X into X; we define the map Π_f from $X\times I^+$ into X, with $\Pi_f(x,t)=xt=f^t(x)$. Then the triplet (X,I^+,Π_f) is a discrete semi-dynamical system on X, and we say that it is induced by f. Reciprocally: If (X,I^+,Π) is a discrete semi-dynamical system on X, (X,I^+,Π) is the one induced by the map $f_\Pi:X\to X$, with $f_\Pi(x)=x1$.

In the following discussion we suppose that it is given a discrete semi-dynamical system on X, which we denote by (X,I^+,Π).

Definition: 1) (X,I^+,Π), or Π, is said to have negative existence, if for every $x\in X$ there is some $y\in X$ and $t\in I^+$, $t>0$, such that $yt=x$.

2) Π is said to have negative unicity, if for any $x_1,x_2\in X$, $x_1t=x_2t$ if and only if $x_1=x_2$.

3) An element $x\in X$ is said to be a start point, if $x\neq yt$ for any $y\in X$ and $t\in I^+$, $t>0$.

4) An element $x\in X$ is said to be a singular point, if there exist $x_1,x_2\in X$, $x_1\neq x_2$, and $t\in I^+$ with $x_1t=x_2t=x$.

5) An element $x\in X$ is said to be a critical point, if $xt=x$ for every $t\in I^+$.

Notice that Π has: negative existence, if and only if for every $x\in X$ there exists $y\in X$ such that $y1=x$; negative existence, if and only if it has no start points; negative unicity, if and only if it has no singular points.

Consider $M\subset X$, and $A\subset I^+$. As usual, we represent:

$\Pi(M,A)=\{xt:\ x\in M,\ t\in A\}$, $\Pi(x,A)=\{xt:\ t\in A\}$.

Furthermore, the following relations hold:

$$\Pi[(\cup_i M_i), (\cup_j A_j)] = \cup_{i,j}\Pi(M_i, A_j)$$

$$\Pi[(\cap_i M_i), t] \subset \cap_i \Pi(M_i, t)$$

$$\Pi[(M-N), t] \supset \Pi(M,t)-\Pi(N,t).$$

1.2 Solutions and trajectories

Let $x \in X$ be arbitrary: The positive trajectory through x, is the set xI^+, also denoted $C(x)$. The right maximal solution throught x is the map ${}_x\Pi: I^+ \to X$ defined by the assignment ${}_x\Pi(t)=xt$.

It is clear that the set of all right solutions determines Π,

Roughly speaking, right maximal solutions are obtained by fixing x in xt. This also suggest fixing t and allowing x to vary. This is how the translations, whose definitions follows, are obtained.

For every $t \in I^+$, we define the map $\Pi_t: X \to X$ with $\Pi_t(x)=xt$, and call it a translation of Π.

It is clear that the set of all translations of Π determines Π,

Notice that $\Pi_t \circ \Pi_s = \Pi_{t+s}$ and $\Pi_t(x) = {}_x\Pi(t)$.

Definition: Given $x \in X$ and $s,t \in I^+$, with $s<t$,

1) The set $T(x)=\{y \in X: x \in C(y)\}$ is called the negative trajectory funnel trough x.

2) The set $T_s^t(x)=\{y \in X: x \in y[s,t]\}$ is called a section of the negative trajectory funnel through x.

3) The set $T_t^t(x)=\{y \in X: x=yt\}$ is called a cross-section of the negative trajectory funnel through x.

4) The set $T_c(x)=T(x) \cup C(x)$ is called the complete trajectory funnel through x.

Finally, for any subset $M \subset X$ we define

$C(M)=\cup\{C(x): x \in M\}$, ..., $T_c(M)=\cup\{T_c(x): x \in M\}$.

Theorem. Let $x\in X$ and $y\in C(x)$, then $C(x)\supset C(y)$, $T(x)\subset T(y)$ and $T_c(x)\subset T_c(y)$.

The proof is quite inmediate.

Theorem. If $x\in X$, there exists an additive subgroup of the integers, G, and $\lambda\in I^+$, such that $xt_1=xt_2$, $t_1>t_2$, if and only if $t_1-t_2\in G$, $t_2\geq\lambda$. Furthermore, G and λ are determined uniquely by x.

Proof: Let us see that for each $t\in I^+$, the set A_t with

$A_t=\{s\in I^+ : xt=x(t+s)\}$

is the trace on I^+ of a subgroup G_t of the integers.

We only need to show that if $s_1,s_2\in A_t$ with $s_1\geq s_2$, then $s_1-s_2\in A_t$, and we have $xt=x(t+s_1)=x(t+s_1+s_2-s_2)=[x(t+s_2)](s_1-s_2)=(xt)(s_1-s_2)=x(t+s_1-s_2)$.

Let us now show that if $t\in I^+$, $r\in I^+$ with $t\leq r$, then $A_t\subset A_r$.

If $s\in A_t$, $xt=x(t+s)$ and hence $xr=x(t+r-t)=(xt)(r-t)==[x(t+s)](r-t)=x(r+s)$. Then $s\in A_r$ and $A_t\subset A_r$.

Let us see, finally, that if $t\leq r$, and $A_t\neq\{0\}$, then $A_t=A_r$.

We only need to prove that $A_t\supset A_r$. Let $\sigma\in A_r$. By assumption, there is $s>0$, $s\in A_t$, and hence for some n with $n\in I^+$, we have $r<t+ns$. If $s\in A_t$, also $ns\in A_t$, and $xt=x(t+ns)$. Furthermore, if $r<t+ns$ then $\sigma\in A_r\subset A_{t+ns}$, and $x(t+s)=x(t+ns+\sigma)$. So we have proved that if $A_t\neq\{0\}$, and $r\in I^+$, $A_t=A_r$.

The result just established shows that there are two cases:

Either for every t is $G_t=\{0\}$, and then the theorem is fulfilled with $G=\{0\}$ and $\lambda=0$.

Or there exists a $\lambda\in I^+$, and a nontrivial subgroup G of the integers, such that

$$G_t = \begin{cases} \{0\}, & \text{if } 0\leq t<\lambda \\ G, & \text{if } \lambda\leq t \end{cases}$$

The above theorem leads to the following classification of the positive trajectories:

Definition. Let $x\in X$, and G and λ be the same as in the last theorem, if

1) $\lambda=0$, and $G=\{0\}$, $C(x)$ is called a non-self-intersecting trajectory.

2) $\lambda=0$, and G a proper nontrivial subgroup of the integers, $C(x)$ is called a periodic or ciclic positive trajectory (with primitive period the least positive element of G).

3) $\lambda=0$, and G the group of the integers, $C(x)$ is called a critical positive trajectory, and we have $C(x)=x$.

4) $\lambda>0$, hence $G\neq\{0\}$, $C(x)$ is said to lead to a cicle, if G is a proper subgroup of the integers; and to lead to a critical point, if G is the group of the integers.

Theorem. Let $x\in X$, if Π has negative unicity

a) Only the cases 1), 2), 3) may occur.

b) If $C(x)$ is periodic, with primitive period τ_x, and $y\in C(x)$, there is a $\rho_y\in I^+$, $\rho_y<\tau_x$, such that the solutions of the equation $y=xt$ in I^+ are $n\tau_x+\rho_y$ with n any element of I^+.

c) If $C(x)$ is periodic, then $C(x)=T_c(x)$.

d) If x is a critical point, then $\{x\}=C(x)=T_c(x)$.

The proof is quite simple.

Theorem. Let $x\in X$,

a) x is a critical point, if and only if $x=x1$, i.e. if x is invariant under the map Π_1.

b) $C(x)$ is periodic, if and only if $x1\neq x$, and there is $t\in I^+$ such that $x=xt$, i.e. if x is invariant under some Π_t, but not under Π_1.

Sometimes, it is said that a critical positive trajectory is periodic with primitive period 1. This would produce a change of notation.

We have defined the right maximal solution through x as the map ${}_x\Pi$, we shall give a more general concept.

An object σ is a solution of Π, if the two following conditions hold:

i) $\sigma: I \to X$, with I an interval of $I^- \cup I^+$.

ii) $\sigma(s+t)=[\sigma(t)]s$, if $t \leq t+s$, $t, t+s \in I$.

Lemma. a) If σ is a solution defined on $[t_1, t_2]$ and k is an integer, σ^* defined by the assignment $\sigma^*(t)=\sigma(t+k)$, is a solution defined on $[t_1-k, t_2-k]$.

b) If $\{\sigma_i\}$ is a monotonous sequence of solutions, I_i being the domain of σ_i (i.e. for any $i>j$, $I_i \supset I_j$ and $\sigma_i(t)=\sigma_j(t)$ for every t in I_j), then $\cup\sigma_i$ is a solution with domain $I=\cup I_i$.

Both assertions are easy to verify.

Definition. Let $x \in X$; a solution σ of Π is called:

1) A left-solution, if $[\text{domain}\sigma] \cap I^+ = \{0\}$. A left-solution through x, if $[\text{domain}\,\sigma] \cap I^+ = \{0\}$ and $\sigma(0)=x$.

2) A right-solution, if $[\text{domain}\,\sigma] \cap I^- = \{0\}$. A right-solution through x, if $[\text{domain}\,\sigma] \cap I^- = \{0\}$ and $\sigma(0)=x$.

3) A left maximal solution, if it is a right-solution, and is maximal with respect to the property of being a left-solution.

4) A right maximal solution, if it is a right-solution, and is maximal with respect to the property of being a right-solution.

5) A maximal solution, if it is a solution, and its restriction to I^- is a left maximal solution; and its restriction to I^+ is a right maximal solution.

Remarks: A solution, maximal relative to the property of being a solution, is called a maximal solution as above defined, only if its domain contains I^+. Let x be a start point of X; the map $\sigma: [5,+\infty) \to X$ with $\sigma(t)=x(t-5)$ is a solution, maximal relative to the property of being a solution, but σ is not a maximal solution according to 5) because its restriction to I^+ is not a right maximal solution.

A right maximal solution through x is precisely "the" right maximal solution through x, ${}_x\Pi$, defined by ${}_x\Pi(t)=xt$.

For any x there exists some maximal solution.

A subset of X is called a <u>negative trajectory</u>, if it is the range of a left maximal solution; and a <u>negative trajectory through x</u> if it is the range of a left maximal solution through x.

A left solution with domain I^- is necessarily a left maximal solution. Such left solutions, and the corresponding trajectories, will be called <u>principal</u>.

Notice that if N is a negative trajectory through x, then $N \subset T(x)$, and that if Π has negative unicity $N=T(x)$, but if x is a singular point, then $N \neq T(x)$.

<u>Theorem</u>. Let σ be a left maximal solution, and $N=\text{range}\,\sigma$ the corresponding negative trajectory, then one and only one of the following alternatives holds:

a) σ and N are principal.

b) Domain $\sigma = [\alpha, 0]$, with $-\infty < \alpha \leq 0$. Then $\sigma(\alpha)$ is a start point. We say that σ and N lead from the start point $\sigma(\alpha)$.

The proof is quite simple.

Notice that if Π has negative existence, the only alternative is a).

It should be observed that different left maximal solutions can define the same negative trajectory.

If N is a negative trajectory through x, the set $N \cup C(x)$ is called a <u>complete trajectory through x</u>. $N \cap C(x) \neq \{x\}$ is not excluded. If N is principal, the complete trajectory $N \cup C(x)$ is also called principal.

Let $x \in X$, and σ be a left maximal solution through x, we write:

$\alpha_x = -\sup \{t \in I^+ : x \in \Pi(x,t)\}$

$\alpha_\sigma = \inf \{\text{domain } \sigma\}$

and call α_x the <u>negative escape time of x</u>, and α_σ the <u>negative escape time of σ</u>. It is clear that $-\infty \leqslant \alpha_x \leqslant \alpha_\sigma \leqslant 0$. It may well happen that $\alpha_x < \alpha_\sigma$ for all left maximal solutions σ through x.

1.3 Invariance

Let $M \subset X$: M is called <u>positively invariant</u>, if $C(M) \subset M$; M is called <u>negatively invariant</u>, if $T(M) \subset M$; M is called <u>invariant</u> if $T_c(M) \subset M$, i.e. if M is both, negatively and positively invariant.

<u>Lemma</u>. The following assertions are equivalent:

a) M is positively invariant.

b) $xt \in M$, for any $x \in M$, $t \in I^+$.

c) $\Pi(M,1) \subset M$.

d) $M = C(M)$.

e) $x1 \in M$, for any $x \in M$.

The proof is quite simple.

<u>Theorem</u>. $M \subset X$ is positively invariant, if and only if $X-M$ is negatively invariant.

Proof: Let M be positively invariant. If $x \in X-M$, then we must show that $T(x) \subset X-M$. Suppose not. Then there is $y \in T(x)$, with $y \notin X-M$, but then there is $t \in I^+$ with $yt = x$, and by positive invariance of M,

$x \in M$. This contradiction shows that $X-M$ is negatively invariant. The proof of the converse is entirely similar.

Theorem. A set $M \subset X$ is negatively invariant, if and only if $x1 \in M$ implies $x \in M$.

The proof is quite simple.

Theorem. a) X and $\emptyset$ are both invariant.

b) A subset consisting of one element is positively invariant, if and only if it is a critical point.

c) If Π has negative unicity, a subset consisting of one element is negatively invariant, if and only if it is either critical or a start point.

d) If $M_i \subset X$ are all positively invariant (or all negatively invariant, invariant), then so are $\cap M_i$, and $\cup M_i$.

e) The complement of a positively invariant (negatively invariant, invariant) is negatively invariant (positively invariant, invariant).

f) The least positively invariant subset containing a given $M \subset X$ is $C(M)$.

g) The least negatively invariant subset containing a given $M \subset X$ is $T(M) = \{y \in X: C(y) \cap M \neq \emptyset\}$.

h) A set $M \subset X$ is negatively invariant, if and only if for any $x \in M$, each negative trajectory N through x verifies $N \subset M$.

The negative trajectories allow to define yet another kind of invariance, which is called quasi-invariance.

Let $M \subset X$; M is called: Negatively quasi-invariant, if for every $x \in M$, there exists some negative trajectory through x which is contained in M; and Quasi-invariant, if it is both positively invariant and negatively quasi-invariant.

<u>Lemma</u>. If M is negatively invariant, then M is negatively quasi-invariant. If each M_i is negatively quasi-invariant, then so is $\cup M_i$. If M_1 is negatively invariant, and M_2 is negatively quasi-invariant, then $M_1 \cap M_2$ is negatively quasi-invariant. A set $M \subset X$ is negatively quasi-invariant, if for every non-start point $x \in M$, there is $y \in M$ with $y1=x$.

2. DISCRETE SEMI-DYNAMICAL SYSTEMS ON A PARTIALLY ORDERED SET

2.1 Definition and properties

In this section we suppose that <u>X is an infinite partially ordered set</u>, i.e. X is an infinite set, and there is a relation R defined in X such that:

i) xRx for every $x \in X$.

ii) xRy, yRz implies xRz, for every $x,y,z \in X$

iii) xRy and yRx implies $x=y$.

Let m be an element of X, we say that m is a <u>minimal element of X</u> if $m \in X$ and yRm implies $m=y$.

Let M be a subset of X, we define:

$M_F = \{y \in X\text{: there exists } x \in M\text{, with } yRx\}$

$M_P = \{y \in X\text{: } y \text{ is a minimal element of } X,\ y \in M_F\}$.

The triplet (X, I^+, Π) is called a <u>discrete semi-dinamical system on the partially ordered set X</u>, if the following conditions hold:

i) $x0=x$, $x \in X$.

ii) $(xt)s=x(t+s)$, $x \in X$ and $t,s \in I^+$.

iii) xRy implies $xtRyt$, $t \in I^+$ and $x,y \in X$.

Notice that a discrete semi-dynamical system on a partially ordered set X is a discrete semi-dynamical system on X.

In the rest of this section we suppose that a discrete semi-dynamical system on the partially ordered set X, (X, I^+, Π), is given.

2.2 Weak concepts

An object Σ is called a <u>weak solution of Π</u>, if the two following conditions hold:

i) $\Sigma: I \rightarrow X$, I being an interval of $I^- \cup I^+$.

ii) $\Sigma(s+t) R \Sigma(t)s$, if $t \leq t+s$; $t, t+s \in I$.

Notice that a solution is a weak solution.

<u>Lemma</u>. If Σ is a weak solution of Π defined on $[t_1, t_2]$, k is an integer and $\Sigma^*(t)=\Sigma(t+k)$, then Σ^* is a weak solution of Π, defined on $[t_1-k, t_2-k]$.

If $\{\Sigma_i\}$ is a monotonous sequence of weak solutions of Π with $[\text{domain } \Sigma_i]=I_i$, then $\cup\Sigma_i$ is a weak solution of Π defined in $\cup I_i$.

Definition. Let $x \in X$. A weak solution Σ of Π is said to be:

1) A <u>weak left solution</u> if $[\text{domain } \Sigma] \cap I^+=\{0\}$. A <u>weak left solution through x</u>, if $[\text{domain } \Sigma] \cap I^+=\{0\}$ and $\Sigma(0)=x$.

2) A <u>weak right solution</u>, if $[\text{domain } \Sigma] \cap I^-=\{0\}$. A <u>weak right solutiong through x</u>, if $[\text{domain } \Sigma] \cap I^-=\{0\}$ and $\Sigma(0)=x$.

3) A <u>weak left maximal solution</u>, if it is a weak left solution and is maximal with respect to the property of being a weak left solution.

4) A <u>weak right maximal solution</u>, if it is a weak right solution and is maximal with respect to the property of being a weak right solution.

5) A <u>weak maximal solution</u>, if it is a weak solution, its restriction to I^- is a weak left maximal solution, and the one to I^+ is a weak right maximal solution.

Notice that not every weak solution, maximal with respect to the property of being a weak solution, is a weak maximal solution. Notice that for every $x \in X$ there is some weak maximal solution through x.

Definition. A subset of X is said to be:

1) A weak negative trajectory through x, if it is the range of a weak left maximal solution through x. A weak left solution, defined on I^-, is necessarily a weak left maximal solution. Such solutions and the corresponding trajectories are calle principal.

2) A weak positive trajectory through x, if it is the range of a weak right maximal solution through x.

3) A weak complete trajectory, if it is the range of a weak maximal solution.

Definition. Let $M \subset X$; M is called:

1) weakly negatively invariant, if for every $x \in M$ there is some weak negative trajectory through x, contained in M.

2) weakly positively invariant, if for every $x \in M$ there is some weak positive trajectory through x contained in M.

3) weakly invariant, if for every $x \in M$ there exists some weak complete trajectory through x contained in M.

Theorem. Let $M \subset X$. M is a weakly positively invariant set, if and only if for every $x \in M$, there is some $y \in M$ with $y\mathcal{R}x1$.

Proof: If M is a weakly positively invariant set, then the condition holds. Since if $x \in M$ there is a Σ, weak right maximal solution through x, with $\Sigma(I^+) \subset M$, and therefore, if we take $y=\Sigma(1)$, it is clear that $y\mathcal{R}x1$.

Conversely, assume the condition and take $x_0 \in M$. Now, using the hypothesis, define $x_1 \in M$ with $x_1\mathcal{R}x_0 1$; and by recurrence, for $n \in I^+$, $n>1$ define $x_n \in M$ with $x_n\mathcal{R}x_{n-1}1$. Now, consider the map $\Sigma: I^+ \to X$, defined by the assignment $\Sigma(n)=x_n$ for every n in I^+. Let $t,s\in I^+$. By definition we have $x_{t+1}Rx_t1$, and therefore, $\Sigma(t+1)\ R\,\Sigma(t)1$. Now we have $\Sigma(t+1)1\ R\ [\Sigma(t)]1$, and therefore $\Sigma(t+1)1\ R\ \Sigma(t)2$. Now, since $x_{t+2}Rx_{t+1}1$, we have that $\Sigma(t+2)\ R\,\Sigma(t+1)1$, and by transitive property of R it holds that $\Sigma(t+2)\ R\,\Sigma(t)2$.

Now we have $\Sigma(t+2)1\ R\,\Sigma(t)3$, and since $\Sigma(t+3)\ R\,\Sigma(t+2)1$, we have $\Sigma(t+3)\ R\,\Sigma(t)3$.

Proceeding in this way, we have $\Sigma(t+s)\ R\ \Sigma(t)s$, and therefore, Σ is a weak right maximal solution through x_o, with $\Sigma(I^+)\subset M$. This shows that M is a weakly positively invariant set.

<u>Theorem</u>. The union of weakly positively invariant sets is also a weakly positively invariant set.

<u>Theorem</u>. Let $M\subset X$ be a positively invariant set, then M_F is also a positively invariant set.

Proof: Let $x\in M_F$, then there is some $y\in M_F$ with xRy, and so $x1Ry1$. Now, since M is a positively invariant set, $y1\in M$, and therefore $x1\in M_F$, and finally M_F is a positively invariant set.

Definition. Let $x\in X$; if xRy holds only if $y=x$ and $t=0$, x is called a <u>weak start point</u>.

Notice that if x is a weak start point, then x is a start point, but x can be a start point without being a weak start point.

<u>Theorem</u>. Let $M\subset X$. If M has no weak start points, then it is weakly negatively invariant, if and only if for every $x\in M$ there exists $y\in M$, such that $xRy1$.

<u>Corollary</u>. Let $M\subset X$. M has no weak start points and is weakly negatively invariant, if and only if for every $x\in M$, there exists $y\in M$, such that $xRy1$.

Theorem. Let $x,y\in X$, and $t\in I^+$ with $t>0$. If $y\mathcal{R}xt$ holds, then there exists some Σ, weak maximal solution through x, with $\Sigma(t)=y$.

Proof: Let Σ^* be the map from $[0,t]$ into X defined by $\Sigma^*(s)=xs$ if $s\in I^+$, with $s<t$, and $\Sigma^*(s)=y$ if $s=t$. It is clear that Σ^* is a weak right solution through x, and the theorem is proved, if we take a weak maximal solution Σ with $\Sigma^*\subset\Sigma$.

Let $x\in X$; x is called a <u>weak critical point</u>, if $x\mathcal{R}xt$ for every $t\in I^+$.

Notice that if x is a critical point, then x is a weak critical point, but x can be a weak critical point without being a critical point.

Theorem. Let $x\in X$. The following assertions are equivalent:

a) x is a weak critical point.

b) $x\mathcal{R}x1$.

c) The map $\Sigma: I^+\rightarrow X$ with $\Sigma(t)=x$ for every t in I^+ is a weak right maximal solution through x.

A weak <u>solution of Π</u> is said to be <u>of minimal type</u>, if all its elements are minimal elements of the partially ordered set X.

Theorem. Let X be a partially ordered set, such that if $x\in X$, there is some $y\in X$ with y a minimal element of X, and $y\mathcal{R}x$. Let M be a subset of X positively invariant, then M_p is a weakly positively invariant set, and, evidently, all weak trajectories contained in M_p correspond with weak solutions of minimal type.

Proof: Let $y\in M_p$, then there exists some $x\in M$ with $y\mathcal{R}x$, and $y1\mathcal{R}x1$. Let z be a minimal element of X, with $z\mathcal{R}y1$, then $z\mathcal{R}x1$, and therefore $z\in M_p$. Since $z\mathcal{R}y1$, M_p is a weak positively invariant set.

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