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ON BILATERAL GENERATING FUNCTION OF MODIFIED BESSEL POLYNOMIAL

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Abstract

In this paper the author establishes a theorem on bilateral generating function of modified Bessel polynomial from Lie group view point. A nice application of the theorem on bilateral generating relation of the above-mentioned polynomial is also pointed out.

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1 Introduction

The object of this paper is to derive some bilateral generating relations for modified Bessel polynomial, introduced by H. L. Kralll and O. Frink [1], which is defined by

$$Y_n^{(\alpha)}(x) = {}_2F_0\left[-n, n+\alpha-1; -; -\frac{x}{\beta}\right],$$

by means of suitable interpretation of the index n ($n = 0, 1, 2, ...$) of the polynomial, provided the parameter α is not a negative integer or zero, and the parameter β is not zero. In this connection we like to establish the following theorem.

Theorem 1 *If there exists a generating relation of the form*

$$G(x,w) = \sum_{n=0}^{\infty} a_n Y_n^{(\alpha-2n)}(x) w^n, \tag{1.1}$$

then

$$(1+wx)^{\alpha-2} exp\left(\frac{\beta w}{1+wx}\right) G\left(x(1+wx), \frac{wz}{(1+wx)^2}\right)$$
$$= \sum_{n=0}^{\infty} \sigma_n(z) Y_n^{(\alpha-2n)}(x) w^n \tag{1.2}$$

where

$$\sigma_n(z) = \sum_{k=0}^{n} a_k \frac{\beta^{n-k}}{(n-k)!} z^k$$

The importance of our above mentioned theorem lies in the fact that one can get a large number of bilateral generating relations from (1.2) by attributing different suitable values to a_n in (1.1).

2 Proof of the Theorem

For the Bessel polynomial, we consider the following linear partial differential operator R [2]

$$R = x^2 y \frac{\partial}{\partial x} - 2xy^2 \frac{\partial}{\partial y} + (\alpha - 2)xy + \beta y, \tag{2.1}$$

such that

$$R\left(Y_n^{(\alpha-2n)}(x) y^n\right) = \beta Y_{n+1}^{(\alpha-2n-2)}(x) y^{n+1} \tag{2.2}$$

The extended form of the group generating by R is

$$e^{wR} f(x,y) = (1+wxy)^{\alpha-2} exp\left(\frac{\beta wy}{1+wxy}\right) f\left(x(1+wxy), \frac{y}{(1+wxy)^2}\right) \tag{2.3}$$

Let us consider the following unilateral generating relation

$$G(x,w) = \sum_{n=0}^{\infty} a_n Y_n^{(\alpha-2n)}(x) w^n \tag{2.4}$$

Replacing w by wyz, we obtain

$$G(x, wyz) = \sum_{n=0}^{\infty} a_n (zw)^n \left(Y_n^{(\alpha-2n)}(x) y^n\right) \tag{2.5}$$

Now operating both sides by e^{wR}, we get

$$e^{wR}[G(x, wyz)] = e^{wR}\left[\sum_{n=0}^{\infty} a_n (zw)^n \left(Y_n^{(\alpha-2n)}(x) y^n\right)\right] \tag{2.6}$$

The left hand side of (2.6) becomes

$$(1+wxy)^{\alpha-2} exp\left(\frac{\beta wy}{1+wxy}\right) G\left(x(1+wxy), \frac{wyz}{(1+wxy)^2}\right) \tag{2.7}$$

Meanwhile, the right hand side of (2.6) can be written as

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_n (zw)^n \frac{w^k}{k!} R^k \left(Y_n^{(\alpha-2n)}(x) y^n\right) \tag{2.8}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_n z^n \frac{w^{n+k}}{k!} \left(\beta^k Y_{n+k}^{(\alpha-2n-2k)}(x) y^{n+k}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_n z^n \frac{(wy)^{n+k}}{k!} \beta^k \left(Y_{n+k}^{(\alpha-2n-2k)}(x)\right)$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_{n-k} z^{n-k} (wy)^n \frac{\beta^k}{k!} \left(Y_n^{(\alpha-2n)}(x)\right)$$

$$= \sum_{n=0}^{\infty} (wy)^n Y_n^{(\alpha-2n)}(x) \sum_{k=0}^{n} a_{n-k} \frac{\beta^k}{k!} z^{n-k}$$

Equating (2.7) and (2.8) we get

$$(1+wxy)^{\alpha-2} exp\left(\frac{\beta wy}{1+wxy}\right) G\left(x(1+wxy), \frac{wyz}{(1+wxy)^2}\right)$$

$$= \sum_{n=0}^{\infty} \sigma_n(z) Y_n^{(\alpha-2n)}(x)(wy)^n \tag{2.9}$$

where

$$\sigma_n(z) = \sum_{k=0}^{n} a_k \frac{\beta^{n-k}}{(n-k)!} z^k$$

Setting $y = 1$, we complete the proof of the theorem.

3 Application

As an application of the theorem, we consider the following generating relation [2]

$$(1+xt)^{\alpha-2} exp\left(\frac{\beta t}{1+tx}\right) = \sum_{k=0}^{\infty} \frac{\beta^k}{k!} Y_k^{(\alpha-2k)}(x) t^k$$

In the above mentioned theorem, if we substitute

$$a_n = \frac{\beta^n}{n!}$$

then we obtain

$$G(x, w) = (1+wx)^{\alpha-2} exp\left(\frac{\beta w}{1+wx}\right)$$

Hence, by the application of our theorem, one can obtain the following generalization

$$(1+wx+wxz)^{\alpha-2} exp\left(\frac{\beta w}{1+wx} + \frac{\beta wz}{(1+wx)(1+wx+wxz)}\right)$$

$$= \sum_{n=0}^{\infty} \sigma_n(z) Y_n^{(\alpha-2n)}(x) w^n$$

where

$$\sigma_n(z) = \sum_{k=0}^{n} \frac{\beta^n}{k!(n-k)!} z^k$$

References

[1] **Krall, H. L. and O. Frink**, A new class of orthogonal polynomials: The Bessel polynomials, Trans. Amer. Math. Soc., 65 (1949), 100-115.

[2] **Chongder, A. K.**, Weisner's methodic survey on Bessel polynomials, *Communicated.*