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A NOTE ON IDEALS IN TOPOLOGICAL QUASI-VECTOR SPACES

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Abstract

In [2] we introduced the concept of quasi-vector space as an associate of a vector space. In this note we introduce the concept of ideal, maximal ideal, minimal ideal in a quasi-vector space and discuss their nature in some particular quasi-vector spaces. Some aspects of ideals in topological quasi-vector spaces are also discussed.

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1 Introduction

In a topological vector space the sum of two compact sets is compact; also scalar multiple of a compact set is compact. So for a topological vector space X, if we consider the collection $C(X)$ of all nonempty compact subsets of X then the aforesaid addition of two sets and multiplication of a set by a scalar become closed operations. Also the following results hold for any two compact subsets A, B of X and any scalar α, β

(i) $A \subseteq B \Rightarrow \alpha A \subseteq \alpha B$

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(ii) $(\alpha+\beta)\cdot A \subseteq \alpha A+\beta A$. These simple facts have induced us to find a structural beauty in the collection $C(X)$ with the help of aforesaid addition, scalar multiplication and inherent set-inclusion order. In doing so we have introduced in [2] a new structure which we call a "quasi-vector space". It has two structures, a semigroup structure and a partial order structure, both being compatible with each other; also there is a scalar multiplication which generalises the concept of vector space in the sense that every vector space can be embedded in a quasi-vector space and every quasi-vector space contains a vector space as its subspace. In this space some properties of a vector space are lacking and some are present; the properties which are absent appear in a different shape with the existing properties and with the inherent order under consideration.

The study was carried out and a topology was introduced in this new space compatible with its existing structures; a new topological algebraic structure was thus created which was named "Topological Quasi-vector Space".

It has been observed that there is some novelty in the structure which is why there is an endeavour to usher in further studies of such space.

In the present paper in §3 we have introduced the concept of 'ideals' in a quasi-vector space, which is totally different from the well-known concept of ideal in a ring. In a ring, ideal is a special type of subring whereas ideal in a topological quasi-vector space can never be a sub-quasi-vector space, if it is to be a proper ideal. We have also defined a maximal and minimal ideal of a quasi-vector space and shown that every quasi-vector space has a unique maximal ideal; although a quasi-vector space may or may not have a minimal ideal. We have also found a necessary and sufficient condition for a minimal ideal. Some simple but useful results relating to ideal have also been obtained.

In §4 ideals in some particular examples have been discussed thoroughly.

In the last article, the topological character of an ideal has been discussed. The concept of ideal has been utilised to find a necessary condition for a topological quasi-vector space to be compact. In this context it should be noted that, a non-trivial topological vector-space can never be compact; but a topological quasi-vector space may or may not be compact.

2 Prerequisites

Definition 2.1 Let X be a nonempty set and '$\leq$' be a partial order in it. Let '$+$' be a binary operation on X and '$\cdot$' : $K \times X \longrightarrow X$ be another composition [K being the field of complex numbers]. If '$\leq$', '$+$' and '$\cdot$' satisfy the following axioms, we call ($X, \leq, +, \cdot$) a

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quasi-vector space [in short, QVS].

A_1 : $(X,+)$ is a commutative semigroup with identity 'θ'.

A_2 : $x \leq y \ (x,y \in X) \Rightarrow x+z \leq y+z$ and $\alpha \cdot x \leq \alpha \cdot y, \ \forall z \in X, \ \forall \alpha \in K$.

A_3 : (i) $\alpha \cdot (x+y) = \alpha \cdot x + \alpha \cdot y$

(ii) $\alpha \cdot (\beta \cdot x) = (\alpha \cdot \beta) \cdot x$

(iii) $(\alpha + \beta) \cdot x \leq \alpha \cdot x + \beta \cdot x$.

(iv) $1 \cdot x = x$, where '1' is the multiplicative identity in K, $\forall x,y \in X, \ \forall \alpha, \beta \in K$

A_4 : $\alpha \cdot x = \theta$ iff $\alpha = 0$ or $x = \theta$

A_5 : $x + (-1) \cdot x = \theta$ iff $x \in X_0 = \{x \in X : \ y \nleq x, \ \forall y \in X \setminus \{x\}\}$

A_6 : For each $x \in X$, $\exists \ y \in X_0$ such that $y \leq x$.

Definition 2.2 An element $x \in X$ is said to be of order '1' if $y \nleq x, \ \forall y \in X \setminus \{x\}$, otherwise x will be said to have order greater than '1'.

Note 2.3 (i) Every qvs contains at least one '1' order element viz. the additive identity θ. (ii) One order elements are the only invertible elements in X. In-fact: $a + b = \theta (a, b \in X) \Longrightarrow a + b - b = -b + \theta = -b$. Again, $a + b - b \geq a + \theta \Longrightarrow -b \geq a \Longrightarrow b \geq -a \Longrightarrow \theta = a + b \geq a - a \geq \theta \Longrightarrow a$ must be of order '1' (by axiom A_5). Similarly b must be of order '1'. It now follows that, $a + b \ (a, b \in X)$ will be of order '1'iff both a and b are of order '1'. (iii) We also observe that for each $m \in X_o$, $\exists \ y \in X$ s.t $y \geq m$ and $y \neq m$ for, if x be an element of order greater than '1' then $x - x + m \geq m$ and $x - x + m \neq m$ [since $x - x \neq \theta$].

Definition 2.4 [1] A partial order '$\leq$' in a topological space Z is said to be closed if its graph $\{(x, y) \in Z \times Z : x \leq y\}$ is closed in $Z \times Z$, endowed with the product topology.

Theorem 2.5 [1] *A partial order '$\leq$' in a topological space Z will be a closed order iff for any $x, y \in Z$ with $x \nleq y$, $\exists$ open nbds U, V of x, y respectively in Z such that $(\uparrow U) \cap (\downarrow V) = \Phi$, where, $\uparrow U = \{x \in Z : x \geq u$ for some $u \in U\}$ and $\downarrow V = \{x \in Z : x \leq v$ for some $v \in V\}$.*

Definition 2.6 A qvs X is said to be a topological qvs if X has a topological structure with respect to which '+' and '$\cdot$' are continuous and '$\leq$' is a closed order such that for each open set V in X, $\downarrow V = \{y \in X : \ y \leq x$ for some $x \in V\}$ is open in X.

Proposition 2.7 *Every topological qvs is Hausdorff.*

Proof : Let X be a topological qvs. Let $x, y \in X$ with $x \neq y$. Then either $x \nleq y$ or $y \nleq x$. Without loss of generality let, $x \nleq y$. Since the order '$\leq$' is closed, $\exists$ two open nbds U, V of x, y respectively in X such that $(\uparrow U) \cap (\downarrow V) = \Phi$ [by theorem 2.5] $\Rightarrow U \cap V = \Phi$.

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3 Ideals in a QVS

Definition 3.1 A nonempty subset I of a qvs X, is said to be an ideal of X if
i) $X + I \subseteq I$ ii) $\alpha.I \subseteq I, \alpha \neq 0$ iii) $\uparrow I = I$ where $\uparrow I = \{x \in X : x \geq y$ for some $y \in I\}$.

Note 3.2 We observe that if $\theta \in I$ then $I = X$; therefore I will be proper ideal of X iff $\theta \notin I$. It is equivalent to say that I cannot contain any element of order '1'. Thus if X_0 denotes the set of all '1' order elements of X, then I is a proper ideal of X iff $I \cap X_0 = \Phi$.

Theorem 3.3 *$X \setminus X_0$ is a proper ideal of X.*

Proof : First we have $X_0 \neq X$. Let $x \in X$ and $a \in X \setminus X_0$. Then a cannot be of order '1'. We claim that $x+a$ is also not of order '1', for otherwise, $(x+a)-(x+a) = x-x+a-a = \theta$. Again, $x - x \geq \theta \Rightarrow a - a + x - x \geq a - a \Rightarrow \theta \geq a - a \Rightarrow a - a = \theta \Rightarrow a$ is of order '1' which is a contradiction.
Now, let $\alpha \in K, \alpha \neq 0$ and $a \in X \setminus X_0$. We show that $\alpha a \in X \setminus X_0$. For, if $\alpha a \in X_0$ then $\alpha a - \alpha a = \theta \Rightarrow \alpha(a - a) = \theta \Rightarrow a - a = \theta$ $(\alpha \neq 0) \Rightarrow a \in X_0$ which is a contradiction.
Let $x \in \uparrow (X \setminus X_0)$. Then $\exists$ $a \in X \setminus X_0$ s.t $x \geq a$. We show that $x \in X \setminus X_0$. If not, $x \in X_0 \Rightarrow x - x = \theta$. Thus, $x \geq a \Rightarrow x - x \geq a - a \Rightarrow \theta \geq a - a \Rightarrow a - a = \theta \Rightarrow a \in X_0$ which is a contradiction.
Consequently $\uparrow (X \setminus X_0) = X \setminus X_0$ [since $(X \setminus X_0) \subseteq \uparrow (X \setminus X_0)$]. Therefore $X \setminus X_0$ is an ideal of X. Clearly it is a proper ideal since $X_0 \neq \Phi$.

Note 3.4 We note that $X \setminus X_0$ is a maximal ideal of X in the sense that, there is no proper ideal I of X such that $X \setminus X_0 \subset I$. It is also clear that X cannot have any other maximal ideal i.e. $X \setminus X_0$ is the unique maximal ideal.

Theorem 3.5 *Arbitrary non empty intersection of ideals is an ideal.*

Proof : Let $\{I_\alpha : \alpha \in \Lambda\}$ be an arbitrary family of ideals of X such that $\bigcap_{\alpha \in \Lambda} I_\alpha \neq \Phi$.
Let $I = \bigcap_{\alpha \in \Lambda} I_\alpha$
i) Let $a \in I$ then $a \in I_\alpha$ $\forall \alpha \in \Lambda$
Therefore, $x + a \in I_\alpha$ $\forall \alpha \in \Lambda$ [since I_α is an ideal]$\Rightarrow x + a \in \bigcap_{\alpha \in \Lambda} I_\alpha = I$ $\forall x \in X$.
ii) Let $a \in I$. Then $a \in I_\alpha$ $\forall \alpha \in \Lambda$
Therefore, $\beta a \in I_\alpha$ for all non zero β and $\forall \alpha \in \Lambda$ [since I_α is an ideal]$\Rightarrow \beta a \in \bigcap_{\alpha \in \Lambda} I_\alpha = I$ $\forall \beta \neq 0$
iii) Let $p \in \uparrow I$.

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Then $p \geq q$ for some $q \in I \Rightarrow q \in I_\alpha\ \forall \alpha \in \Lambda \Rightarrow p \in \uparrow I_\alpha\ \forall \alpha \in \Lambda$

$\Rightarrow p \in I_\alpha\ \forall \alpha \in \Lambda$ [since I_α is an ideal of X, $\uparrow I_\alpha = I_\alpha$].

Therefore, $p \in \bigcap_{\alpha \in \Lambda} I_\alpha = I$.

Consequently, $\uparrow I = I$ [since $I \subseteq \uparrow I$]

This completes the proof.

Proposition 3.6 *Arbitrary union of ideals of X is an ideal.*

Proof : Immediate from definition.

Theorem 3.7 *Let $a \in X$, where X is a quasi-vector space. Then $I(a) = \{y \in X : y \geq x + \alpha a,\ x \in X,\ \alpha \in K^*\}$ is an ideal of X containing a, where $K^* = K \setminus \{0\}$.*

Proof : Clearly, $a \in I(a)$ [since $a = \theta + 1.a$ where $\theta \in X,\ 1 \in K^*$].

i) Let $x_1 \in X$ and $y_1 \in I(a)$ Then $\exists\ x_2 \in X$ and $\alpha \in K^*$ such that $y_1 \geq x_2 + \alpha a$

Now, $x_1 + y_1 \geq x_1 + x_2 + \alpha a \Rightarrow x_1 + y_1 \in I(a)$ [since $x_1 + x_2 \in X$]

Therefore $X + I(a) \subseteq I(a)$

ii) Let $\alpha \in K^*$ and $y \in I(a)$. Then $\exists\ x \in X$ and $\beta \in K^*$ such that $y \geq x + \beta a$

$\Rightarrow \alpha y \geq \alpha(x + \beta a) = \alpha x + \alpha\beta a \Rightarrow \alpha y \in I(a)$ [since $\alpha x \in X$ and $\alpha\beta \in K^*$]

$\Rightarrow \alpha I(a) \subseteq I(a)$

iii) Let $p \in \uparrow I(a)$. Then $\exists\ q \in I(a)$ such that $p \geq q$

Now, $q \in I(a) \Rightarrow \exists\ x \in X$ and $\alpha \in K^*$ such that $q \geq x + \alpha a$

Therefore $p \geq q \geq x + \alpha a \Rightarrow p \in I(a) \Rightarrow \uparrow I(a) = I(a)$.

In view of (i), (ii) and (iii) the theorem follows.

Corollary 3.8 *If 'a' be an element of order '1', then $I(a) = X$ and conversely.*

Proof : Let $x \in X$. Then there exists an element y in X of order '1' such that $x \geq y \Rightarrow x \geq y - a + a$ [since $a - a = \theta$ for, a is an element of order '1'].

Now $y - a \in X \Rightarrow x \in I(a)$. Therefore $X = I(a)$.

Conversely, let $I(a) = X$ for some $a \in X$. Then $\theta \in I(a)$. So $\exists\ x \in X$ and $\alpha \in K^*$ such that $\theta \geq x + \alpha a \Rightarrow x + \alpha a = \theta \Rightarrow \alpha^{-1}x + a = \theta \Rightarrow a$ is of order '1' [by note 2.3].

Note 3.9 The ideal $I(a)$ is said to be the ideal generated by 'a'. It is the smallest ideal containing a; for, if J be another ideal containing a such that $J \subseteq I(a)$ then $p \in I(a) \Rightarrow p \geq x + \alpha a$ for some $x \in X,\ \alpha \in K^*$

Now, $x + \alpha a \in J$ [since J is an ideal and $a \in J$] $\Longrightarrow p \in \uparrow J = J$.

Thus, $I(a) \subseteq J \Rightarrow I(a) = J$.

It is also clear from theorem 3.5 that $I(a)$ is the intersection of all ideals containing a.

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Theorem 3.10 *Any two proper ideals must intersect.*

Proof : Let I and J be any two proper ideals of X. Now, $I + J \subseteq J$ and $J + I \subseteq I$. Consequently $I + J \subseteq J \cap I$

Theorem 3.11 *For any ideal $J \subseteq X$, $\downarrow J = X$, where $\downarrow J = \{x \in X : x \leq j$ for some $j \in J\}$.*

Proof : Let $x \in X$. Then either $x \in J$ or $x \notin J$.
Now, $x \in J \Rightarrow x \in \downarrow J \Rightarrow X \subseteq \downarrow J \Rightarrow \downarrow J = X$.
Let $x \notin J$. Now $I(x) \cap J \neq \Phi$ [by theorem 3.10] $\Rightarrow \exists\, p \in X$ such that $p \in I(x)$ and $p \in J \Rightarrow \exists\, x_1 \in X$ and $\alpha \in K^*$ such that $p \geq x_1 + \alpha x \Rightarrow p - x_1 \geq x_1 - x_1 + \alpha x \geq \theta + \alpha x = \alpha x$.
or $\alpha^{-1}(p - x_1) \geq x$ [since $\alpha \neq 0$]
Now, $\alpha^{-1}(p - x_1) \in J$ [since J is an ideal of X and $p \in J$]
Therefore there exists an element $\alpha^{-1}(p - x_1) \in J$ such that $\alpha^{-1}(p - x_1) \geq x \Rightarrow x \in \downarrow J \Rightarrow X \subseteq \downarrow J$. Therefore $\downarrow J = X$.

Proposition 3.12 *If $a \leq b$ $(a, b \in X)$ then $I(b) \subseteq I(a)$*

Proof : Straight forward.

Theorem 3.13 *For any ideal $J \subseteq X$, $J = \bigcup_{x \in J} I(x)$.*

Proof : Let $x \in J \Rightarrow x \in I(x) \subseteq \bigcup_{x \in J} I(x)$.
Conversely, let $y \in \bigcup_{x \in J} I(x)$. Then $\exists\, p \in J$ such that $y \in I(p) \Rightarrow \exists\, a \in X$ and $\alpha \in K^*$ such that $y \geq a + \alpha p$.
Now, $a + \alpha p \in J$ [since $p \in J$ and J is an ideal].
Therefore, $y \in J$ [since $\uparrow J = J$] $\Rightarrow \bigcup_{x \in J} I(x) \subseteq J$. Thus $\bigcup_{x \in J} I(x) = J$.

Definition 3.14 An ideal I of a qvs X is said to be a minimal ideal of X if there does not exist any proper ideal J of X such that $J \subseteq I$.

Theorem 3.15 *J is a minimal ideal of X iff $J = I(x)$ $\forall x \in J$.*

Proof : Let $x \in J$. Then $I(x) \subseteq J$ [by note 3.9]. Also, J being minimal $J \subseteq I(x)$.
Conversely, let $J = I(x)$ $\forall x \in J$. If possible, let J_1 be another ideal of X such that $J_1 \subseteq J$.
Let $p \in J_1 \Rightarrow p \in J \Rightarrow J = I(p)$. Now, $I(p)$ is the smallest ideal containing p.
Hence $I(p) \subseteq J_1 \Rightarrow J \subseteq J_1$. Consequently J is minimal.

Note 3.16 Since any two ideal intersect, minimal ideal (if any) of a qvs must be unique.

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Theorem 3.17 *If a qvs X contains a maximal element a, w.r.t the partial order then $I(a)$ is a minimal ideal.*

Proof : Let $x \in I(a)$. Then $I(x) \subseteq I(a)$. Also $x \geq y + \alpha a$ for some $y \in X,\ \alpha \in K^* \Rightarrow x - y \geq \alpha a \Rightarrow \alpha^{-1}(x-y) \geq a \Rightarrow \alpha^{-1}x - \alpha^{-1}y = a$ [since a is a maximal element in X] $\Rightarrow a \in I(x) \Rightarrow I(a) \subseteq I(x)$.

Theorem 3.18 *Let A be a subset of a qvs X. Then* $I(A) = \{y : y \geq x + \sum_{i=1}^{n} \alpha_i a_i,\ \alpha_i \in K^*,\ a_i \in A,\ i = 1, \ldots, n; x \in X, n = 1, 2, 3, \ldots\}$ *is an ideal of X containing A.*

Proof : Let $a \in A$. Then $a = \theta + 1.a \Rightarrow a \in I(A) \Rightarrow A \subseteq I(A)$. The rest of the proof is similar to that of theorem 3.7.

Note 3.19 *Arguing similarly as in note 3.9 we can say $I(A)$ is the smallest ideal containing A.*

Proposition 3.20 $I(A) = \cup_{a \in A} I(a)$.

Proof : Let $b \in I(A)$. Then $\exists\ x \in X$ and $a_1, \ldots, a_n \in A,\ \alpha_1, \ldots, \alpha_n \in K^*$ such that $b \geq x + \sum_{i=1}^{n} \alpha_i a_i = x + \sum_{i=1}^{n-1} \alpha_i a_i + \alpha_n a_n$. Therefore, $b \in I(a_n)$ [since $x + \sum_{i=1}^{n-1} \alpha_i a_i \in X$]$\Rightarrow b \in \cup_{a \in A} I(a)$. Let p be an element in $\cup_{a \in A} I(a)$. Then $\exists\ a \in A$ such that $p \in I(a)$. Therefore $p \geq x + \alpha a$ for some $x \in X$ and $\alpha \in K^* \Rightarrow p \in I(A)$.
Therefore, $I(A) = \cup_{a \in A} I(a)$.

4 Examples of ideals

4.1 Example

Let X be a topological vector space over the field of complex number K. Let $C(X)$ be the set of all non-empty compact subsets of X. We define addition $(+)$ and scalar multiplication $(\cdot)$ in $C(X)$ as follows:
Let $A, B \in C(X)$ and $\alpha \in K$
$A + B = \{a + b : a \in A, b \in B\},\ \alpha \cdot A = \{\alpha a : a \in A\}$.
Clearly $A + B$ and αA are compact since the addition and scalar multiplication in X are continuous, X being a topological vectorspace and hence $A + B, \alpha \cdot A \in C(X)$. It is easy to check that $C(X)$ with the aforesaid operations and with respect to the usual set-inclusion

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order forms a qvs over K , $\{\theta\}$ being the identity element in $C(X)$, where θ is the identity in X.

Ideals of $C(R)$

Let us consider the topological vector-space R. The closed and bounded subsets of R are the only compact subsets of R. It is easy to observe that the singleton sets $\{a\}, a \in R$ are the only one order elements of $C(R)$. Also $C(R) \setminus \mathcal{S}$ is the maximal ideal of $C(R)$, where $\mathcal{S} = \{\{a\} : a \in R\}$ (by note 3.4).

We now discuss the ideals generated by single elements of $C(R)$.

Proposition 4.1.1 *Let A, B be two two-point sets of R. Then $I(A) = I(B)$.*

Proof : Let $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$. Then $I(A) \subseteq I(B)$ iff $A \in I(B)$ iff $A \supseteq E + \alpha B$ for some $E \in C(R)$ and $\alpha \in R \setminus \{0\} = R^*$ (say)

Since A and B contains same number of elements it follows that E must be singleton and hence $A = E + \alpha B$ ... $(\star)$. Let $E = \{x\}$. Then above relation $(\star)$ holds iff $a_1 = x + \alpha b_1$ and $a_2 = x + \alpha b_2$ (if $a_1 = x + \alpha b_2$ and $a_2 = x + \alpha b_1$ then B can be renamed just by interchanging b_1 and b_2 so that we get the above relations) which being a system of two linear equations in two variables x and α must have a unique solution.

Consequently, $I(A) \subseteq I(B)$ is always true.

Similarly, $I(B) \subseteq I(A)$.

Proposition 4.1.2 *If $A = \{a_1, \ldots, a_n\}$, $B = \{b_1, \ldots, b_n\}$ where $n > 2$, then $I(A) = I(B)$ iff* $\begin{vmatrix} 1 & a_i & b_i \\ 1 & a_j & b_j \\ 1 & a_k & b_k \end{vmatrix} = 0$ *for* $1 \leq i, j, k \leq n, \quad i \neq j \neq k$.

Proof : $I(A) \subseteq I(B)$ iff $A \in I(B)$ iff $A \supseteq E + \alpha B$ for some $E \in C(R)$ and $\alpha \in R^*$. Since A and B contains same number of elements it follows that E must be singleton and hence $A = E + \alpha B$. Let $E = \{x\}$, $x \in R$. Therefore $A = \{x\} + \alpha B$. This is true iff the following system of linear equations $a_i = x + \alpha\ b_i$, $i = 1, \ldots, n$ in x and α has a solution. If $a_i = x + \alpha b_j$, $i, j \in \{1, 2, \ldots, n\}$ instead of the said equations, then B can be renamed in such a way that we get the above equations.

Here order of the coefficient matrix $P = \begin{pmatrix} 1 & b_1 \\ \ldots & \ldots \\ 1 & b_n \end{pmatrix}$ is $n \times 2$ and that of the augmented

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matrix $Q = \begin{pmatrix} 1 & b_1 & a_1 \\ \dots & \dots & \dots \\ 1 & b_n & a_n \end{pmatrix}$ is $n \times 3$. The above system has a solution iff $r(P) = r(Q)$, where $r(P)$ denotes the rank of P. In this case, $r(P) \leq 2$ and $r(Q) \leq 3$. So if $r(P) = r(Q)$ then $r(Q) \neq 3$ and hence every minor of Q of order 3 is zero i.e $\begin{vmatrix} 1 & b_i & a_i \\ 1 & b_j & a_j \\ 1 & b_k & a_k \end{vmatrix} = 0$ for

$1 \leq i, j, k \leq n$; $i \neq j \neq k \implies \begin{vmatrix} 1 & a_i & b_i \\ 1 & a_j & b_j \\ 1 & a_k & b_k \end{vmatrix} = 0$ for $1 \leq i, j, k \leq n, \quad i \neq j \neq k$..

Conversely, if these determinants be all zero then $r(Q) \neq 3$. Also $\begin{vmatrix} 1 & b_i \\ 1 & b_j \end{vmatrix} \neq 0 \ \forall i, j (i \neq j)$ [since $b_i \neq b_j$, $\forall i, j \ (i \neq j)$]. So $r(P) = r(Q) = 2$. Consequently, the above system of n linear equations in two variables x and α has unique solution. If we interchange A and B in above arguments then, under same condition, $I(B) \subseteq I(A)$. This completes the proof.

Note 4.1.3 It is easy to note that if A contains n elements and B contains n+1 elements of R, then $I(A)$ cannot be contained in I(B). Also $I(B)$ is not necessarily contained in I(A).

Proposition 4.1.4 *If $A = \{a_1, \dots, a_n\}$, $B = \{b_1, \dots, b_{n+1}\}$ where $n > 2$, then $I(B) \subset I(A)$ iff $\begin{vmatrix} 1 & a_i & b_r \\ 1 & a_j & b_s \\ 1 & a_k & b_t \end{vmatrix} = 0$ for $1 \leq i, j, k \leq n$; $1 \leq r, s, t \leq n+1$; $i \neq j \neq k$ and $r \neq s \neq t$.*

Proof : Similar as proposition 4.1.2

Note 4.1.5 (i) We observe that any ideal generated by a countably infinite compact subset of R is contained in some ideal generated by a finite subset of R. Also, any two ideals generated by two countably infinite compact subsets of R need not be same. For example, $I(\{1/p : p \text{ is prime}\} \cup \{0\})$ and $I(\{1/n : n = 1, 2, 3, \dots\} \cup \{0\})$.

(ii) Each ideal generated by countably infinite compact subset of R contains an ideal generated by some uncountable compact subsets of R. Also any two ideals generated by two uncountable compact subsets of R need not be same. For example, $I(\{\sum_{i=1}^{\infty} \frac{a_i}{3^i} : a_i = 0, 2\})$ and $I(\{\sum_{i=1}^{\infty} \frac{a_i}{5^i} : a_i = 0, 2, 4\})$.

Minimal Ideal Of $C(R)$.

The closed interval $[0, 1]$ is an uncountable compact subset of R. We claim that $I([0, 1])$ is contained in any ideal of $C(R)$. In fact, we show $I([0, 1]) \subseteq I(A)$ for any compact subset

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A of R. For this we are only to show that $[0,1] \in I(A)$. Since A is a compact subset of R, $\exists\, a, b \in R$ such that $A \subseteq [a,b] \Rightarrow \{\frac{-a}{b-a}\} + \frac{1}{b-a}A \subseteq \{\frac{-a}{b-a}\} + \frac{1}{b-a}[a,b] = [0,1] \Rightarrow [0,1] \in I(A)$. We now prove that $I([0,1])$ is the minimal ideal of $C(R)$. Let $B \in I([0,1])$. Then obviously $I(B) \subseteq I([0,1])$. Also by above discussion, $I([0,1]) \subseteq I(B)$ since B is a compact subset of R. Thus $I([0,1]) = I(B)\ \forall B \in I([0,1])$. Therefore by theorem 3.15, $I([0,1])$ is the minimal ideal of $C(R)$.

4.2 Example

Let G be the set of all 2×2 matrices A over R such that $|A| \geq 1$, where $|A|$ denotes the determinant of A. We define a relation ρ on G as follows :

$A\rho B$ holds iff $|A| = |B|$. Then clearly ρ is an equivalence relation on G. This determines a partition on G. Let $[A]$ denotes the equivalence class containing A.

Let $X = \{[A] : A \in G\}$. We define addition $(+)$, scalar multiplication $(\cdot)$ and a partial order '$\leq$' on X as follows :

(i) $[A] + [B] = [AB]$ (ii) $\alpha[A] = [\alpha A], \alpha \neq 0$ and $0[A] = [I]$ where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(iii) $[A] \leq [B]$ iff $|A| \leq |B|$. It is easy to verify that all operations defined above are well defined.

Then $(X, \leq, +, \cdot)$ is a qvs over the field R.

<u>**Ideal of X**</u>

Here $[I]$ is the only one order element, i,e. $X_0 = \{[I]\}$. So, $X \setminus \{[I]\}$ is the maximal ideal of X.

Let $[A] \in X \setminus X_0$. Then, $I([A]) = \{[B] \in X : |B| \geq |A|\}$. Hence, $I([A]) = I([B])$ iff $|A| = |B|$. Also, for any two elements $[A], [B] \in X \setminus X_0$ either $I([A]) \subseteq I([B])$ or $I([B]) \subseteq I([A])$. Thus the collection of all ideals of X forms a chain of ideals. We claim that X has no minimal ideal.

If possible let, J is the minimal ideal of X. Then $\exists\, [A] \in X \setminus X_0$ such that $J = I([A])$. Let B be a 2×2 square matrix over R such that $|B| > |A|$. Then $[B] \in I([A]) = J \Rightarrow I([B]) \subset J$ but $I([B]) \neq J$ [since $|B| \neq |A|$]. This contradicts the minimality of J.

4.3 Example

Let $(S, \leq)$ be a lattice with the least element θ. We define $* : S \times S \to S$ by $a * b = lub(a,b)$ where $lub(a,b)$ denotes the least upper bound of a, b. Also We define '$\cdot$':$K \times S \to S$ by

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$$\alpha.a = \begin{cases} a, & \text{for } \alpha \neq 0 \\ \theta, & \text{for } \alpha = 0 \end{cases}$$ for all a. Then $(S, *, \cdot, \leq)$ is a qvs over K (K being a field).

Ideal Of $\underline{S}$

Here θ is the only one order element of S.

Proposition 4.3.1 *Let $\theta \neq a \in S$. Then $I(a) = \uparrow a$*

Proof : Let $p \in \uparrow a \Rightarrow p \geq a \Rightarrow p \geq \theta * \alpha a\ \forall \alpha \neq 0 \Rightarrow p \in I(a)$
Conversely let, $q \in I(a) \Rightarrow q \geq x * \alpha a$ for some $x \in S$ and $\alpha \in K \setminus \{0\} \Rightarrow q \geq x * a$ [since $\alpha a = a$ for non zero α]. Let $l = x * a$ then $l \geq a$. Therefore $q \geq l \geq a \Rightarrow q \in \uparrow a$.

Proposition 4.3.2 *$I(a) = I(b)$ iff $a = b$*

Proof : Follows immediately from proposition 4.3.1

Proposition 4.3.3 *S has a minimal ideal iff it has a maximal element.*

Proof : Let a be a maximal element of S. Then $I(a) = \uparrow a = \{a\} \Rightarrow I(a)$ is a minimal ideal.
Conversely, let J be a minimal ideal of S. Then $J = I(a)$ for some $a \in S$.
Now, $x \geq a (x \in S) \Rightarrow x \in \uparrow a = J \Rightarrow J = I(x)$ [since J is the minimal ideal]$\Rightarrow I(x) = I(a) \Rightarrow x = a$ (by proposition 4.3.2). So a is a maximal element of S.

5 Topological property of ideal in a topological qvs

Theorem 5.1 *Let I be an ideal of a topological qvs X. Then $\bar{I}$ is also an ideal of X.*

Proof : i) Let $x \in X$ and $p \in \bar{I}$. Let V be any open nbd of $x + p$. Then $\exists$ open nbds W_1 and W_2 of x and p respectively in X such that $W_1 + W_2 \subseteq V$.
Now, W_2 is an open nbd of p and $p \in \bar{I} \Rightarrow W_2 \cap I \neq \Phi$.
Let $q \in W_2 \cap I$. Then $x + q \in V$ [since $q \in W_2$]
Also, $x + q \in X + I \subseteq I$ [since $q \in I$ and I is an ideal]$\Rightarrow x + q \in V \cap I \Rightarrow V \cap I \neq \Phi$.
Hence, $x + p \in \bar{I}$. Thus, $x + p \in X + \bar{I} \Rightarrow x + p \in \bar{I}$. Therefore, $X + \bar{I} \subseteq \bar{I}$.
ii) Let $\alpha \in K^*, p \in \bar{I}$ and V be any open nbd of αp. Then $\exists$ open nbds N of α in K and U of p in X such that $NU \subseteq V$.
Therefore, $U \cap I \neq \Phi$ [since $p \in \bar{I}$ and U is an open nbd of p in X]. Let $t \in U \cap I$. Then $\alpha t \in I$ [since $t \in I$ and I is an ideal]
Also, $\alpha t \in V$ [since $\alpha \in N, t \in U$ and $NU \subseteq V$]$\Rightarrow V \cap I \neq \Phi \Rightarrow \alpha p \in \bar{I}$

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iii) Let, $q \in \uparrow \bar{I}$. Therefore, $\exists\, p \in \bar{I}$ such that $q \geq p$.
Let W be any open nbd of q. So, $p \in \downarrow W$ [since $q \geq p$]. Now $\downarrow W$ being open in X, $\downarrow W \cap I \neq \Phi$. Let $z \in \downarrow W \cap I$. Therefore, $z \leq w$ for some $w \in W$ and $z \in I \Rightarrow w \in \uparrow I = I$ [since I is an ideal of X]$\Rightarrow W \cap I \neq \Phi \Rightarrow q \in \bar{I} \Rightarrow \uparrow \bar{I} \subseteq \bar{I}$. Consequently $\uparrow \bar{I} = \bar{I}$ [since $\bar{I} \subseteq \uparrow \bar{I}$].
In view of (i), (ii) and (iii) the theorem follows.

Definition 5.2 An ideal I is said to be a closed minimal ideal if I is closed and it contains no closed ideal properly.

Proposition 5.3 *If J be the minimal ideal of a topological qvs X, then $\bar{J}$ will be its closed minimal ideal.*

Proof : If possible let I be a closed ideal such that $I \subsetneq \bar{J}$. Since J is the minimal ideal, $J \subseteq I \subseteq \bar{J} \Rightarrow \bar{J} = \bar{I} = I$. So $\bar{J}$ is a closed minimal ideal.

Theorem 5.4 *$I(A)$ is path-connected for each A ($\subseteq X \setminus X_0$).*

Proof : $I(A) = \bigcup_{a \in A} I(a)$ [by proposition 3.20]. We first show that $I(a)$ is path connected for each $a \in A$. Let $y_1, y_2 \in I(a)$. Then $\exists\, x_1, x_2 \in X$ and $\alpha_1, \alpha_2 \in K^*$ such that $y_1 \geq x_1 + \alpha_1 a$ and $y_2 \geq x_2 + \alpha_2 a$. We define a function $f : [0,1] \to X$ by $f(t) = (1-t)y_1 + ty_2$. Clearly f is continuous. Also, $f(t) = (1-t)y_1 + ty_2 \geq (1-t)x_1 + tx_2 + (1-t)\alpha_1 a + t\alpha_2 a \Rightarrow f(t) \in I(a)\ \forall t \in$ [0,1]. Thus f is a continuous path in $I(a)$ such that $f(0) = y_1$ and $f(1) = y_2$. Consequently $I(a)$ is path-connected for each $a \in A$. Since any two ideals intersect it follows that $I(A)$ is also path-connected.

Theorem 5.5 *If X be a compact topological qvs, then it has a minimal ideal which is also compact.*

Proof : We first show that X has a closed minimal ideal. Let Γ be the collection of all closed ideals of X. Since closure of an ideal is also an ideal (by theorem 5.1) Γ is non-empty. Let $J = \bigcap_{F \in \Gamma} F$. Since any two ideals intersect (by theorem 3.10) Γ has finite intersection property. Since X is compact, $J \neq \Phi$. So J must be a closed ideal (by theorem 3.5). Also from the construction it follows that J must be the closed minimal ideal.
We now prove that J is minimal. For this we have to prove $J = I(a)\ \forall a \in J$ (by theorem 3.15).
Let $a \in J$. Then $I(a) \subseteq J \Rightarrow \overline{I(a)} = J$ [since J is closed minimal ideal]. If we can show that $I(a)$ is closed the proof is done.

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Let $\{y_n\}_{n\in D}$ be a net in $I(a)$ converging to y, D being a directed set. So for each $n \in D \;\exists\; x_n \in X$ and $\alpha_n \in K^*$ such that $y_n \geq x_n + \alpha_n a$ ——— $\star$ Since X is compact, the net $\{x_n\}_{n\in D}$ in X has a convergent subnet $\{x_m\}_{m\in E}$(say), E being a directed set. Let $x_m \to x$. From $\star$ we have $y_m \geq x_m + \alpha_m a$, $m \in E$. Here $\{\alpha_m\}_{m\in E}$ is a net in K^*. Now two cases may arise.

Case-I : $\{\alpha_m\}_{m\in E}$ has a bounded subnet $\{\alpha_p\}_{p\in E'}$ (say), E' being a directed set. Then $\{\alpha_p\}_{p\in E'}$ has a convergent subnet $\{\alpha_q\}_{q\in D'}$ (say), D' being a directed set. Let $\alpha_q \to \alpha \Rightarrow \alpha_q a \to \alpha a$ [since '$\cdot$' is continuous]. Again J is closed and $\alpha_q a \in J \;\; \forall q \in D' \Rightarrow \alpha a \in J$ i.e. $\alpha \neq 0$ [since $\theta \neq J$]

Now we have $y_q \geq x_q + \alpha_q a \Rightarrow y \geq x + \alpha a$ [since the partial order '$\leq$' is closed and any subnet of a convergent net is convergent and converges to the same limit, limit being unique for, X is Hausdorff] where $\alpha \neq 0 \Rightarrow y \in I(a)$.

Case-II : $\{\alpha_m\}_{m\in E}$ has no bounded subnet. Then $\{{\alpha_m}^{-1}\}_{m\in E}$ is a bounded net in K^* and hence it has a convergent subnet $\{{\alpha_p}^{-1}\}_{p\in E_1}$ (say), E_1 being a directed set. Let ${\alpha_p}^{-1} \to \beta$. Now $\{\alpha_p a\}_{p\in E_1}$ is a net in J and J is compact [since J is closed subset of X which is compact]. So $\{\alpha_p a\}_{p\in E_1}$ has a convergent subnet $\{\alpha_q a\}_{q\in E_2}$ (say), E_2 being a directed set. Let $\alpha_q a \to z$. Then $z \in J$. Again, ${\alpha_q}^{-1} \to \beta$. So ${\alpha_q}^{-1}\alpha_q a \to \beta z \Rightarrow a = \beta z$ [since limit of a convergent net is unique $] \Rightarrow \beta \neq 0$ [since $a \neq \theta$ $] \Rightarrow z = \beta^{-1}a$. Thus $\alpha_q a \to \beta^{-1}a$ where $\beta^{-1} \neq 0$. Again we have $y_q \geq x_q + \alpha_q a \; \forall q \in E_2 \Rightarrow y \geq x + \beta^{-1}a$ [by same logic as in case-I] $\Rightarrow y \in I(a)$.

Consequently, $I(a)$ is closed.

Now J being a closed ideal it must be compact, since X is compact.

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