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HIGHER ORDER LACUNARY INTERPOLATION*

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ABSTRACT - A generalization of a previously analyzed lacunary interpolation problem is considered. We prove the unconditional solvability of the problem, using elements in an appropriate class of "deficient" splines. Also, the error analysis presented here sharpens the estimates obtained in the earlier study. Moreover the convergence rates obtained here are shown to be optimal.

AMS(MOS) Subject Classifications: 41A15, 65D05
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1. Introduction. In this paper we consider a lacunary interpolation problem, which is a generalization of a previous investigation [7], the so-called (0,4) lacunary interpolation problem and improve the results therein. We assume that information is provided on a function and its q-th order derivative at a set of equispaced nodes. Our task is the reconstruction of the function by means of a suitably defined spline. It turns out that in such instance, the spline is "deficient", in the sense that it is impossible to ensure continuity of all derivatives up to the maximum possible order, since there is a constraint relating the number of the conditions, of interpolatory or continuity type, and the number of coefficients in each polynomial arc of the spline. Solution of lacunary interpolation problems is usually obtained using special classes of lacunary splines [1,2,3,4,6].
This study can have application in the solution of boundary value problems governed by ordinary differential equations of higher order. The fourth order case already dealt with has a direct practical relevance, since it corresponds to the so-called cantilever problem [5] and is related to the calculation of the deformations of beams. The method presented here assumes that the underlying differential equations governing the two point boundary value problem is solved by means of a finite difference approximation scheme. At the expense of just one more function evaluation, it is possible to obtain data also on the highest order derivative, in addition to the value of the function. These are the data we assume given for the problem here at hand. The higher order case considered here in our opinion possesses a mathematical interest, in itself worthy of investigation.
The paper is organized as follows: after giving the necessary definitions and stating the problem, we reduce its solution to a linear algebraic system. The investigation of the structure of the matrix is performed in section 3. In section 4, we analyze a solution scheme and show that the system is always solvable. Section 5 contains the error analysis.

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We would like to emphasize that here the resolution scheme is directly used as a model to follow for the derivation of the stability result. In this way the use of the 2- norm as in [7] is avoided. The direct use of the supremum norm indeed allows the recovering of a factor of 1/2, lost in the previous analysis. As a consequence, it turns out that we can recover the optimal order of convergence for the spline, thereby sharpening the results of the former investigation.

2. The problem. We assume to work on the normalized interval $[a, b] \equiv [0, 1]$, partitioned by means of the nodes $x_k = kh$, $k = 0, \dots, n$, where $h = 1/n$. We want to determine the spline function $s(x)$ of degree $q + 2$ in each subinterval, satisfying the following conditions:
i) $s(x) \in S^{(2)}_{n,q+2}$ i.e. its "deficiency" is 2, as emphasized by the upper index;
ii) $s(x) \in C^q[0, 1]$ i.e. it is continuous up to order q included.
Let $s_k(x)$ be the restriction of $s(x)$ to the interval $\Delta_k \equiv [x_{k-1}, x_k]$, $k = 1, \dots, n$.

We assume that the number of subintervals in the partition is larger than the order of the highest known derivative, thus $q < n$. In this situation there are $n(q+3)$ free parameters to be determined, i.e. the coefficients of the polynomials making up the spline. On the other hand the number of interpolatory conditions is $2(n + 1)$.

Explicitly, the latter are

$$\text{(2.1)} \qquad \begin{array}{lll} s_k^{(i)}(x_{k-1}) = f_{k-1}^{(i)} & , \quad i = 0, q & , \quad k = 1, \dots, n \quad , \\ s_n^{(i)}(x_n) = f_n^{(i)}(x_n) & , \quad i = 0, q & , \end{array}$$

where for easeness we use the shorthand notation $f_k^{(j)} = f^{(j)}(x_k)$.

We need also to satisfy the following continuity conditions

$$\text{(2.2)} \qquad s_k^{(i)}(x_k) = s_{k+1}^{(i)}(x_k) \quad , \quad i = 0, \dots, q \quad , \quad k = 1, \dots, n-1 \; .$$

Their number is then $(q+1)(n-1)$. The free parameters that are still undetermined are then $n(q+3) - 2(n+1) - (q+1)(n-1) = q - 1$ and equal the number of extra (boundary) conditions that need to be specified to guarantee a unique solution for the problem. For easeness but without loss of generality, in the sense that specification of conditions at the other endpoint would result only in marginal modifications of the matrix of the system and of the relative solution scheme, we choose them to be "initial" conditions, in the form

$$\text{(2.3)} \qquad s_1^{(i)}(x_0) = f_0^{(i)} \quad , \quad i = 1, \dots, q-1.$$

We introduce now the "unknowns" of the problem, namely the moments of $s(x)$ and of its derivatives at the breakpoints:

$$
\begin{aligned}
M_{k-1}^{(i)} &= s^{(i)}(x_{k-1}) \quad , \quad i = 0, \dots , q \ , \\
M_{k-1}^{(i)} &= s_k^{(i)}(x_{k-1}) \quad , \quad i = q+1, q+2 \ , \\
M_n^{(i)} &= s_n^{(i)}(x_n) \quad , \quad i = 0, q \ .
\end{aligned}
$$

Then the restriction of the spline to the k-th subinterval can be expressed as follows

$$
s_k(x) = \sum_{j=0}^{q+2} M_{k-1}^{(j)} \frac{(x - x_{k-1})^j}{j!} \quad , \quad k = 1, \dots , n \quad , \quad x \in \Delta_k .
$$

Let us recall that

$$
\begin{aligned}
M_{k-1}^{(i)} &= f_{k-1}^{(i)} \quad , \quad i = 0, q \quad , \quad k = 2, \dots , n+1 \ , \\
M_0^{(i)} &= f_0^{(i)} \quad , \quad i = 0, \dots , q \quad .
\end{aligned}
$$

Differentiating,

$$
s_k^{(i)}(x) = \sum_{j=0}^{q+2-i} M_{k-1}^{(i+j)} \frac{(x - x_{k-1})^j}{j!} \quad , \quad i = 0, \dots , q \ .
$$

If we now impose the interpolation conditions that are not yet implicitly satisfied, i.e. (2.1), and the continuity conditions (2.2) we obtain a square linear algebraic system of $(n-1)(q+1)+2$ equations in as many unknowns. Explicitly, the latter are

$$
\begin{aligned}
\mathbf{m}^T = \big[& M_0^{(q+2)}, M_0^{(q+1)}, M_1^{(q+2)}, M_1^{(q+1)}, M_1^{(q-1)}, \dots , M_1^{(1)}, \dots \\
& \dots , M_{n-1}^{(q+2)}, M_{n-1}^{(q+1)}, M_{n-1}^{(q-1)}, \dots , M_{n-1}^{(1)} \big] .
\end{aligned}
$$

Notice that they are written in reverse order, with respect to the derivative order. This will enhance the resulting structure of the matrix for the investigation of the next section. Let us write explicitly the i-th equation of the system. We need to distinguish three different cases:

for $i = q$ it is

$$
\sum_{j=1}^{2} M_{k-1}^{(q+j)} \frac{h^j}{j!} = M_k^{(q)} - M_{k-1}^{(q)} \quad , \quad k = 1, \dots , n \ ,
$$

while for $i = q - 1(-1)1$ it is

$$
\sum_{j=1}^{2} M_0^{(q+j)} \frac{h^{q+j-i}}{(q+j-i)!} - M_1^{(i)} = - \sum_{j=0}^{q-i} M_0^{(i+j)} \frac{h^j}{j!} \ ,
$$

$$\sum_{\substack{j=0\\ j\neq q-i}}^{q+2-i} M_{k-1}^{(i+j)} \frac{h^j}{j!} - M_k^{(i)} = -M_{k-1}^{(q)} \frac{h^{q-i}}{(q-i)!} \quad , \quad k = 2, \dots , n-1 ,$$

and finally for $i = 0$ it is given by

$$\sum_{j=1}^{2} M_0^{(q+j)} \frac{h^{q+j}}{(q+j)!} = M_1^{(0)} - \sum_{j=0}^{q} M_0^{(j)} \frac{h^j}{j!} ,$$

$$\sum_{\substack{j=1\\ j\neq q}}^{q+2} M_{k-1}^{(j)} \frac{h^j}{j!} = M_k^{(0)} - M_{k-1}^{(0)} - M_{k-1}^{(q)} \frac{h^q}{q!} \quad , \quad k = 2, \dots , n .$$

3. Matrix structure. Let the system be written in compact form as $A\mathbf{m} = \mathbf{b}$. Here the matrix $A \equiv [A_{i,j}]$, $i, j = 1, \dots , q+1$ denotes a block Hessemberg matrix, where however each block has dimensions that may differ from those of other blocks.
More precisely,

$$\begin{aligned}
A_{i,j}^{(n,n)} &\quad , \quad i = 1, q+1 , \quad j = 1, 2 , \\
A_{i,j}^{(n-1,n)} &\quad , \quad i = 2, \dots , q, \quad j = 1, 2 , \\
A_{i,j}^{(n-1,n-1)} &\quad , \quad i = 2, \dots , q, \quad j = 3, \dots , q+1 , \\
A_{i,j}^{(n,n-1)} &\quad , \quad i = q+1 \quad , \quad j = 3, \dots , q+1 .
\end{aligned}$$

Moreover, let us observe that $A_{i,j} \equiv 0$ for $j > i+1$, $i = 1, \dots , q-1$, while all the other blocks are either diagonal, subdiagonal or bidiagonal as explicitly stated below

$$A_{ij} = \frac{h^{i-j+2}}{(i-j+2)!} I_n \quad , \quad i = 1, q+1 \quad , \quad j = 1, 2 ,$$

$$A_{ij} = \frac{h^{i-j+2}}{(i-j+2)!} E \quad , \quad i = 2, \dots , q \quad , \quad j = 1, 2 ,$$

and for the cases $j = 3, \dots , q+1$

$$A_{ij} = \begin{cases} F - I_{n-1} & , \quad i = j-1 \quad , \\ \frac{h^{i-j+1}}{(i-j+1)!} F & , \quad i = j, \dots , q \quad , \\ \frac{h^{i-j+1}}{(i-j+1)!} D & , \quad i = q+1 \quad , \end{cases}$$

where, by denoting by $\delta_{i,j}$ the Kronecker symbol, we write

$$\begin{aligned}
E &= (\delta_{i,j})_{n-1,n} , \\
F &= (\delta_{i-1,j})_{n-1,n-1} , \\
D &= (\delta_{i-1,j})_{n,n-1} .
\end{aligned}$$

4. The algoritm. It is possible to easily solve the system by "almost" direct forward substitution. Let us assume to be at the $k-$th stage of the procedure, $k = 1, \ldots, n-1$. Then we proceed as follows:
1) We solve the 2 by 2 system obtained from the $k-$th equations of the first and last horizontal blocks, getting the unknowns $M_{k-1}^{(q+2)}, M_{k-1}^{(q+1)}$. Then the $k-$th columns in the first two vertical blocks can be eliminated;

2) from the $k-$th equations of the $j-$th horizontal blocks, $j = 2, \ldots, q$, we can immediately determine the unknowns $M_k^{(i)}, i = q-1(-1)1$, since $M_s^{(i)}$, $s = 1, \ldots, k-1$, $i = q-1(-1)1$ are already known from the previous steps;

3) at the end of the $(n-1)-$th step of the procedure, it is sufficient to solve the last equations of the first and last horizontal blocks, thus determining the remaining unknowns $M_{n-1}^{(q+2)}$, $M_{n-1}^{(q+1)}$.

It is fundamental to note that in the implementation of the above procedure, the structure of the matrix is left unchanged. In particular after the deletion at each stage of the $(q+1)$ columns relative to the unknowns determined at that step we find an analogous matrix to the one present before the elimination took place, but of smaller dimensions.
The 2 by 2 matrix B of the linear system that needs to be solved in step 1) of each stage of the procedure to obtain the unknowns $M_j^{(q+2)} M_j^{(q+1)}, j = 0, \ldots, n-1$, and finally in step 3) has the form

$$B = \begin{bmatrix} \frac{h^2}{2!} & h \\ \frac{h^{q+2}}{(q+2)!} & \frac{h^{q+1}}{(q+1)!} \end{bmatrix} .$$

Easily, $\det B \neq 0$, so that in view of this and the fact that the procedure never breaks down, we have the following important result.

THEOREM 1. The linear algebraic system for the calculation of the moments of the spline function is always nonsingular.

5. Convergence results and stability estimates. Let us assume that $f(x) \in C^{(q+3)}$ over $[0,1]$ and let us denote by $T(x)$ its truncated Taylor expansion, i.e. the Taylor polynomial of degree $q+2$

$$T(x) = \sum_{j=0}^{q+2} f^{(j)}(x_{k-1}) \frac{(x - x_{k-1})^j}{j!} \quad , \quad x \in \Delta_k .$$

Then

$$f(x) - T(x) = K h^{q+3} .$$

Here and in what follows, K and $K_{k,i}$ denote suitable constants. Let us define the errors we need to estimate:

$$e^{(p)}(x) = f^{(p)}(x) - s^{(p)}(x) \quad , \quad p = 0, \ldots, q+2 ,$$

$$e_k^{(p)} = f_k^{(p)} - M_k^{(p)} \quad , \quad p = 1, \ldots, q-1, q+1, q+2 \; . \tag{5.1}$$

If $x \in \Delta_k$, denoting by $e_k(x)$ the restriction of $e(x)$ to Δ_k, we have

$$e_k(x) = f(x) - T(x) + T(x) - s_k(x). \tag{5.2}$$

Since

$$T(x) - s_k(x) = \sum_{\substack{j=1 \\ j \neq q}}^{q+2} e_{k-1}^{(j)} \frac{(x - x_{k-1})^j}{j!},$$

by imposing the continuity at the point x_k, we obtain

$$\begin{aligned} \sum_{j=q+1}^{q+2} e_0^{(j)} \frac{h^j}{j!} &= K_{1,0} h^{q+3} \quad , \quad k = 1 \; , \\ \sum_{\substack{j=1 \\ j \neq q}}^{q+2} e_{k-1}^{(j)} \frac{h^j}{j!} &= K_{k,0} h^{q+3} \quad , \quad k = 2, \ldots, n \; . \end{aligned} \tag{5.3}$$

Proceeding in a similar fashion, by imposing the continuity of the $i-$th derivative, $i = 1, \ldots, q-1$ at the breakpoints, we get

$$\begin{aligned} \sum_{j=q+1}^{q+2} e_0^{(j)} \frac{h^{j-i}}{(j-i)!} - e_1^{(i)} &= K_{1,i} h^{q+3-i} \quad , \quad k = 1 \; , \\ \sum_{\substack{j=i \\ j \neq q}}^{q+2} e_{k-1}^{(j)} \frac{h^{j-i}}{(j-i)!} - e_k^{(i)} &= K_{k,i} h^{q+3-i} \quad , \quad k = 2, \ldots, n-1 \; . \end{aligned} \tag{5.4}$$

But, in view of the interpolation conditions,

$$e_k^{(q)} = 0 \quad , \quad k = 1, \ldots, n \; ,$$

so that

$$\sum_{j=q+1}^{q+2} e_{k-1}^{(j)} \frac{h^{j-q}}{(j-q)!} = K_{k,q} h^3 \quad . \tag{5.5}$$

The "error equation" $A\mathbf{e} = \epsilon$ is given by the system whose equations are (5.5), (5.4), (5.3). Here, $\mathbf{e}$ represents the error vector, A is the same matrix discussed in section 3, ϵ denotes the consistency error, in this case the interpolation error. The latter has components of order $\epsilon_k^{(i)} = O(h^{q+3-i})$ in the equation corresponding to the i−th derivative, where the notation for the consistency error mimics the one used for the discretization error (5.1). Once again, we denote the unknown vector using a reverse order:

$$\mathbf{e}^T = \left[e_0^{(q+2)}, e_0^{(q+1)}, e_1^{(q+2)}, e_1^{(q+1)}, e_1^{(q-1)}, \ldots, e_1^{(1)}, \ldots\right.$$
$$\left.\ldots, e_{n-1}^{(q+2)}, e_{n-1}^{(q+1)}, e_{n-1}^{(q-1)}, \ldots, e_{n-1}^{(1)}\right]$$

so that the matrix of the system is the same as the one for the moments, analyzed in the previous section. To determine the errors $e_k^{(p)}$, the same earlier considerations hold. Step 1) of the algorithm applied to the error equation leads then to the solution of the following system, initially in the unknowns $e_0^{(q+2)}$, $e_0^{(q+1)}$

(5.6)
$$\begin{bmatrix} \frac{h^2}{2!} & h \\ \frac{h^{q+2}}{(q+2)!} & \frac{h^{q+1}}{(q+1)!} \end{bmatrix} \begin{bmatrix} e_0^{(q+2)} \\ e_0^{(q+1)} \end{bmatrix} = \begin{bmatrix} K_{1,q} h^3 \\ K_{1,0} h^{q+3} \end{bmatrix} .$$

Easily, then we have the following important stability result

(5.7)
$$\|B^{-1}\| = O(h^{-2}) .$$

Direct solution however allows us to obtain a sharper estimate for $e_0^{(q+1)}$ than the one that can be obtained by a crude application of (5.7) to (5.6):

(5.8)
$$e_0^{(q+2)} = O(h) \quad , \quad e_0^{(q+1)} = O(h^2) .$$

Iteration of the calculations as prescribed by the algorithm leads to the convergence result. Let us denote by R the matrix corresponding to all the elementary operations described by the algorithm. The solution of the error equation is equivalent then to the solution of the equation $\mathbf{e} = R\epsilon$, with $I = RA$, the identity matrix.

In view of the above remarks, it is indeed sufficient to note that $A^{-1} = R$, and that the algorithm when considering the equations related to the i−th derivative, consists in forward substitution, i.e. in moving to the right hand side at most q terms of the same order as the one of the right hand side, i.e. $O(h^{q+3-i})$. The previous statement is easily established by induction. Since $q < n$, no loss of accuracy ensues for the unknown being determined at this stage, the right hand side remaining of order $O(h^{q+3-i})$. Finally, for step 3) of the algorithm, the analysis done earlier for the system (5.6) applies, the only change being in the name of the unknowns. It thus leads to the same conclusion, i.e. estimate (5.8) for the unknowns $e_1^{(2)}, e_1^{(1)}$.
The convergence of the algorithm is thus ensured, and the rate equals the one of the consistency error. In summary

THEOREM 2. For the error in the calculation of the moments, on top of the above estimations for $e_0^{(q+2)}$ and $e_0^{(q+1)}$ we have

$$e_1^{(i)} = O(h^{q+3-i}) \quad , \quad i = q-1(-1)1 ,$$

and, iterating the procedure,

$$e_k^{(i)} = O(h^{q+3-i}) \quad , \quad k = 1, \ldots, n-1 \quad , \quad i = 1, \ldots, q-1, q+1, q+2 \; .$$

As a consequence, since the problem is linear and we use a consistent method, in view of estimate (5.7) we have also the stability result

THEOREM 3. The norm of the inverse matrix of the system satisfies the following estimate

$$\|A^{-1}\| = O(h^{-2}) \; .$$

On using the triangular inequality on the error representation (5.2), we have also the error for the spline at arbitrary points in [0,1].

THEOREM 4. For the error of the spline function and its derivatives the following estimate holds

$$\|f^{(p)}(x) - s^{(p)}(x)\| = O(h^{q+3-p}) \quad , \quad x \in [0,1] \quad , \quad p = 0, \ldots, q+2 \; .$$

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