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ON BILATERAL GENERATING FUNCTIONS OF MODIFIED LAGUERRE POLYNOMIALS FROM THE LIE GROUP VIEW POINT

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Abstract : In this note a theorem in connection with bilateral generating functions of modified Laguerre polynomials has been established by group-theoretic method. Some special case as well as applications are also given here.

Key words and Phrases : Generating functions, Laguerre polynomials.

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1. Introduction

The modified Laguerre polynomials as defined by G.K. Goyal [1] is

$$L_{a,b,m,n}(x) = \frac{b^n (m)_n}{n!} \; {}_1F_1\left(-n; m; \frac{ax}{b}\right) \tag{1.1}$$

In this paper we have derived some bilateral generating functions of modified Laguerre polynomials from Lie group view point. The main result of this note is given below :

Theorem : If

$$G(x,w) = \sum_{n=o}^{\infty} a_n L_{a,b,m,n}(x) w^n , \tag{1.2}$$

then

$$(1+bt)^{m-1} \exp(-axt) G(x(1+bt),ty) = \sum_{n=o}^{\infty} t^n \sigma_n(x,y) \tag{1.3}$$

where

$$\sigma_n(x,y) = \sum_{k=o}^{n} a_k \frac{(k+1)_{n-k}}{(n-k)!} \times L_{a,b,m-(n-k),n}(x) y^k . \tag{1.4}$$

The importance of the above theorem lies on the fact that when one knows a generating relation of the form (1.2), the corresponding bilateral generating relation can at once be written down from (1.3). Thus one can get a large number of generating relations from (1.3) by attributing different values to a_n in (1.2).

We now proceed to prove the above mentioned theorem :

2. Proof of the Theorem

Let us first consider the following linear partial differential operator [2] :

$$R = bxy^{-1}z \frac{\partial}{\partial x} + zb \frac{\partial}{\partial y} - y^{-1}z\,(ax+b)$$

such that

$$R(L_{a,b,m,n}(x)\, y^{m}\, z^{n}) = (n+1)\, L_{a,b,m-1,n+1}(x)\; y^{m-1}\, z^{n+1}$$

The extended form of the group, generated by R is

$$e^{wR} f(x,y,z) = (1 + \frac{bwz}{y})^{-1} \exp(- \frac{axwz}{y})\; f(x\xi,\, y\xi,\, z)$$

where

$$\xi = (1 + \frac{bwz}{y}) \quad .$$

We now consider the formula :

$$(2.1) \qquad G(x,w) = \sum_{n=o}^{\infty} a_n\, L_{a,b,m,n}(x)\, w^{n} \quad .$$

Now, replacing w by wz on both sides of (2.1), we get,

$$G(x,wz) = \sum_{n=o}^{\infty} a_n\, L_{a,b,m,n}(x)\, w^{n}\, z^{n} \quad .$$

Now, multiplying both sides by y^{m} and then operating on both sides by e^{wR}, we get,

$$(2.2) \quad e^{wR}\left[y^{m}\, G(x,wz)\right]$$

$$= e^{wR}\left[\sum_{n=o}^{\infty} a_n\, L_{a,b,m,n}(x)\, w^{n}\, y^{m}\, z^{n}\right] \quad .$$

Now, the left hand side of (2.2) is

$$(2.3) \qquad (1+\frac{bwz}{y})^{m-1} \exp(-\frac{axwz}{y})\, y^m \; G(x(1+\frac{bwz}{y}), zw)$$

and the right hand side of (2.2) is

$$(2.4) \qquad \sum_{n=o}^{\infty}\sum_{k=o}^{\infty} a_n \frac{w^{k+n}}{k!} R^k (L_{a,b,m,n}(x)\, y^m z^n)$$

$$= \sum_{n=o}^{\infty}\sum_{k=o}^{\infty} a_n \frac{w^{n+k}}{k!} (n+1)_k L_{a,b,m-k,n+k}(x)\, y^{m-k} z^{n+k} .$$

$$= y^m \sum_{n=o}^{\infty}\sum_{k=o}^{\infty} a_n (\frac{zw}{y})^{n+k} \frac{(n+1)_k}{k!} L_{a,b,m-k,n+k}(x)\, y^n$$

$$= y^m \sum_{n=o}^{\infty}\sum_{k=o}^{n} a_{n-k} (\frac{zw}{y})^{n} \frac{(n-k+1)_k}{k!} L_{a,b,m-k,n}(x)\, y^{n-k}$$

$$= y^m \sum_{n=o}^{\infty} (\frac{zw}{y})^{n} \sum_{k=o}^{n} a_{n-k} \frac{(n-k+1)_k}{k!} L_{a,b,m-k,n}(x)\, y^{n-k}$$

Equating (2.3) and (2.4) we get,

$$(2.5) \qquad (1+\frac{bzw}{y})^{m-1} \exp(-\frac{axzw}{y}) \; G(x(1+\frac{bzw}{y}), zw)$$

$$= \sum_{n=o}^{\infty} (\frac{zw}{y})^{n} \sigma_n(x,y)$$

where

$$\sigma_n(x,y) = \sum_{k=o}^{n} a_{n-k} \frac{(n-k+1)_k}{k!} L_{a,b,m-k,n}(x)\, y^{n-k} .$$

Finally, putting $zw/y = t$, we get,

$$(1+bt)^{m-1} \exp(-axt)\, G(x(1+bt),ty) = \sum_{n=o}^{\infty} t^n \sigma_n(x,y)$$

where

$$\sigma_n(x,y) = \sum_{k=o}^{n} a_k \frac{(k+1)_{n-k}}{(n-k)!} L_{a,b,m-(n-k),n}(x)\, y^k .$$

<u>Special Case</u> : Putting $a=b=1$, $m = 1+\alpha$, we get the following result on bilateral generating relations involving Laguerre polynomials [3] :

<u>**Result**</u> : If

$$G(x,w) = \sum_{n=o}^{\infty} a_n L_n^{(\alpha)}(x)\, w^n ,$$

then

$$(1+t)^{\alpha} \exp(-xt)\, G(x(1+t),ty) = \sum_{n=o}^{\infty} t^n \sigma_n(x,y) ,$$

where

$$\sigma_n(x,y) = \sum_{k=o}^{n} a_k \frac{(k+1)_{n-k}}{(n-k)!} L_n^{\alpha-(n-k)}(x)\, y^k ,$$

which is found derived in [3].

3. Applications

As applications of our theorem, we first consider the following generating relation [4]:

$$(3.1) \qquad \sum_{n=0}^{\infty} L_{a,b,m,n}(x)\, w^n = (1-bw)^{-m} \exp\left(\frac{-axw}{1-bw}\right) .$$

If we take $a_n = 1$ in our theorem, then

$$G(x,w) = (1-bw)^{-m} \exp\left(\frac{-axw}{1-bw}\right) .$$

So by applying our theorem we get,

$$(3.2) \qquad (1-bt)^{m-1} (1-bty)^{-m} \exp\left[\frac{-axt(1+y)}{1-bty}\right] = \sum_{n=0}^{\infty} t^n \sigma_n(x,y)$$

where

$$\sigma_n(x,y) = \sum_{k=0}^{n} \frac{(k+1)_{n-k}}{(n-k)!} L_{a,b,m-(n-k),n}(x)\, y^k .$$

Next we consider the following generating relation [4]:

$$(3.3) \qquad \sum_{n=0}^{\infty} \frac{L_{a,b,m,n}(x)}{(m)_n} y^n = \exp(by)\, {}_0F_1(-;\, m;\, -axy)$$

If we take $a_n = 1/(m)_n$ in our theorem, then

$$G(x,y) = \exp(by)\ {}_0F_1(-;\ m;\ -axy)$$

So by applying our theorem we get,

$$(3.4) \qquad (1-bt)^{m-1} \exp(-t(ax-by))\ {}_0F_1(-;\ m;\ -atxy(1+bt))$$

$$= \sum_{n=0}^{\infty} t^n\ \sigma_n(x,y)\ .$$

where

$$\sigma_n(x,y) = \sum_{k=0}^{n} \frac{1}{(m)_k} \frac{(k+1)_{n-k}}{(n-k)!} L_{a,b,m-(n-k),n}(x)\ y^k\ .$$

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