Rev. Acad. Canar. Cienc., XII (Núms. 1-2), 97-105 (2000) - (Publicado en julio de 2001)

# THE METHOD OF ADOMIAN FOR A NONLINEAR BOUNDARY VALUE PROBLEM

Luis Casasús, Waleed Al-Hayani[1]
Departamento de Matemática Aplicada
Escuela Técnica Superior de Ingenieros Industriales
Universidad Politécnica de Madrid
C / José Gutiérrez Abascal, 2
28006 Madrid - ESPAÑA.



**ABSTRACT**- In this paper, the decomposition method is applied to a Nonlinear Boundary-Value Problem (NLBVP) for ordinary differential equations [9]. We compare the convergence of this method with the spline approximation, studying the order of convergence and the applicability to similar NLBVP's.

**KEYWORDS**-Decomposition method, Adomian's polynomials, NLBVP's, Green's function.

## 1. INTRODUCTION

In the beginning of the eighties, Adomian [4-7] proposed a new and fruitful method (so called decomposition method) for solving linear and nonlinear (algebraic, differential, partial differential, integral, etc.) equations. It has been shown that the decomposition method yields a rapid convergence of the solutions series to linear and nonlinear deterministic and stochastic equations.

Consider the NLBVP [9]:

$$\frac{d^2y}{dx^2} = e^y, 0 \leq x \leq 1 \tag{1.1}$$

$$y(0) = y(1) = 0. \tag{1.2}$$

[1]He was a professor in the Departement of Mathematics, College of Mathematics and Computers, Mosul University, Mosul-IRAQ.

## 2. DECOMPOSITION METHOD AND NONLINEAR BVP

In this section we shall describe the main algorithm of Adomian's decomposition method when it is applied to a general nonlinear equation of the form

$$y - N(y) = f, \tag{2.1}$$

where $N$ is a nonlinear operator, $f$ is a known function, and we are seeking the solution $y$ satisfying (2.1). We assume that for every $f$, Eq. (2.1) has one and only one solution.

The Adomian's technique consists of approximating the solution of (2.1) as an infinite series

$$y = \sum_{n=0}^{\infty} y_n, \tag{2.2}$$

and decomposing the nonlinear operator $N$ as

$$N(y) = \sum_{n=0}^{\infty} A_n, \tag{2.3}$$

where $A_n$ are polynomials (called Adomian polynomials) of $y_0, \ldots, y_n$ [4-7], given by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty} \lambda^i y_i\right)\right]_{\lambda=0}, \; n = 0, 1, 2, \ldots.$$

The proofs of the convergence of the series $\sum_{n=0}^{\infty} y_n$ and $\sum_{n=0}^{\infty} A_n$ are given in [1,2,3,6,8,11]. Substituting (2.2) and (2.3) into (2.1) yields

$$\sum_{n=0}^{\infty} y_n - \sum_{n=0}^{\infty} A_n = f.$$

Thus, we can identify

$$\begin{aligned} y_0 &= f, \\ y_1 &= A_0(y_0), \\ y_2 &= A_1(y_0, y_1), \\ &\vdots \\ y_{n+1} &= A_n(y_0, \ldots, y_n). \end{aligned}$$

Thus all components of $y$ can be calculated once the $A_n$ are given for $n = 0, 1, 2, \ldots$. We then define the n-term approximant to the solution $y$ by $\phi_n[y] = \sum_{i=0}^{n-1} y_i$ with $\lim_{n\to\infty} \phi_n[y] = y$.

Applying the decomposition method as in [4-7], Eq. (1.1) can be written as

$$Ly = N(y) \tag{2.4}$$

where $L = \frac{d^2}{dx^2}$ is the linear operator and $N(y) = e^y$ is the nonlinear operator. Since $L = \frac{d^2}{dx^2}$ now, the inverse operator $L^{-1}$ is no longer the simple two-fold integral as in [6], and we must determine the Green's function $G$ for this $L$. $G$, of course, is determinable in a number of ways.

Using $L$ with the conditions (1.2), we reduce the problem (2.4) by writing $y = y_1^* + y_2^*$, where $y_1^*$satisfies $Ly_1^* = N(y)$ with $y_1^*(0) = y_1^*(1) = 0$ and $y_2^*$ satisfies $Ly_2^* = 0$ with $y_2^*(0) = 0$ and $y_2^*(1) = 0$ as in [5]. For the homogeneous conditions, i.e., for $y_1^*$, we have

$$y_1^* = \int_0^1 G(x,\xi)N(y(\xi))d\xi$$

where $G(x,\xi)$ is the Green's function given by

$$G(x,\xi) = \begin{cases} x(\xi-1), & 0\leq x\leq \xi\leq 1 \\ \xi(x-1), & 0\leq \xi\leq x\leq 1 \end{cases} \tag{2.5}$$

and for the nonhomogeneous conditions, i.e., for $y_2^*$, we get

$$y_2^* = 0$$

Consequently,

$$y = y_1^* + y_2^* = \int_0^1 G(x,\xi)N(y(\xi))d\xi + y_2^*$$

Upon using (2.2) and (2.3) it follows that

$$\sum_{n=0}^{\infty} y_n = \int_0^1 G(x,\xi)\sum_{n=0}^{\infty} A_n d\xi + y_2^* \tag{2.6}$$

From Eq. (2.6) the iterates are then determined in the following recursive way:

$$\begin{aligned} y_0 &= y_2^* = 0 \\ y_1 &= \int_0^1 G(x,\xi)A_0 d\xi \\ y_2 &= \int_0^1 G(x,\xi)A_1 d\xi \\ &\vdots \\ y_{n+1} &= \int_0^1 G(x,\xi)A_n d\xi, \ n = 0,1,2,\ldots. \end{aligned} \tag{2.7}$$

For the nonlinear term $N(y) = e^y = \sum_{n=0}^{\infty} A_n$, the Adomian polynomials are derived as follows:

$$f(y_0) = e^{y_0}, h_\nu(y_0) = \frac{d^\nu}{dy_0^\nu} f(y_0) = e^{y_0}, \nu = 0,1,2,\ldots \tag{2.8}$$

By Eq. (2.8), the Adomian polynomials are:

$$\begin{aligned}
A_0 &= e^{y_0} \\
A_1 &= e^{y_0} y_1 \\
A_2 &= e^{y_0}\left[y_2 + \left(\frac{1}{2!}\right) y_1^2\right] \\
A_3 &= e^{y_0}\left[y_3 + y_1 y_2 + \left(\frac{1}{3!}\right) y_1^3\right] \\
A_4 &= e^{y_0}\left[y_4 + \left(\frac{1}{2!}\right) y_2^2 + y_1 y_3 + \left(\frac{1}{2!}\right) y_1^2 y_2 + \left(\frac{1}{4!}\right) y_1^4\right] \\
A_5 &= e^{y_0}\left[\begin{array}{c} y_5 + y_2 y_3 + y_1 y_4 + y_1\left(\frac{1}{2!}\right) y_2^2 + \left(\frac{1}{2!}\right) y_1^2 y_3 \\ + \left(\frac{1}{3!}\right) y_1^3 y_2 + \left(\frac{1}{5!}\right) y_1^5 \end{array}\right] \\
A_6 &= e^{y_0}\left[\begin{array}{c} y_6 + \left(\frac{1}{2!}\right) y_3^2 + y_2 y_4 + y_1 y_5 + \left(\frac{1}{3!}\right) y_2^3 + y_1 y_2 y_3 + \left(\frac{1}{2!}\right) y_1^2 y_4 \\ + \left(\frac{1}{2!}\right) y_1^2 \left(\frac{1}{2!}\right) y_2^2 + \left(\frac{1}{3!}\right) y_1^3 y_3 + \left(\frac{1}{4!}\right) y_1^4 y_2 + \left(\frac{1}{6!}\right) y_1^6 \end{array}\right] \\
A_7 &= e^{y_0}\left[\begin{array}{c} y_7 + y_3 y_4 + y_2 y_5 + y_1 y_6 + \left(\frac{1}{2!}\right) y_2^2 y_3 + y_1\left(\frac{1}{2!}\right) y_3^2 + y_1 y_2 y_4 \\ + \left(\frac{1}{2!}\right) y_1^2 y_5 + y_1\left(\frac{1}{3!}\right) y_2^3 + \left(\frac{1}{2!}\right) y_1^2 y_2 y_3 + \left(\frac{1}{3!}\right) y_1^3 y_4 \\ + \left(\frac{1}{3!}\right) y_1^3 \left(\frac{1}{2!}\right) y_2^2 + \left(\frac{1}{4!}\right) y_1^4 y_3 + \left(\frac{1}{5!}\right) y_1^5 y_2 + \left(\frac{1}{7!}\right) y_1^7 \end{array}\right] \\
A_8 &= e^{y_0}\left[\begin{array}{c} y_8 + \left(\frac{1}{2!}\right) y_4^2 + y_3 y_5 + y_2 y_6 + y_1 y_7 + y_2\left(\frac{1}{2!}\right) y_3^2 + \left(\frac{1}{2!}\right) y_2^2 y_4 \\ + y_1 y_3 y_4 + y_1 y_2 y_5 + \left(\frac{1}{2!}\right) y_1^2 y_6 + \left(\frac{1}{4!}\right) y_2^4 + y_1\left(\frac{1}{2!}\right) y_2^2 y_3 \\ + \left(\frac{1}{2!}\right) y_1^2 \left(\frac{1}{2!}\right) y_3^2 + \left(\frac{1}{2!}\right) y_1^2 y_2 y_4 + \left(\frac{1}{3!}\right) y_1^3 y_5 \\ + \left(\frac{1}{2!}\right) y_1^2 \left(\frac{1}{3!}\right) y_2^3 + \left(\frac{1}{3!}\right) y_1^3 y_2 y_3 + \left(\frac{1}{4!}\right) y_1^4 y_4 \\ + \left(\frac{1}{4!}\right) y_1^4 \left(\frac{1}{2!}\right) y_2^2 + \left(\frac{1}{5!}\right) y_1^5 y_3 + \left(\frac{1}{6!}\right) y_1^6 y_2 + \left(\frac{1}{8!}\right) y_1^8 \end{array}\right] \\
&\vdots
\end{aligned}$$

The exact solution of (1.1), (1.2) is given by

$$y^*(x) = 2\ln\left\{c\sec\left[\frac{c}{2}\left(x-\frac{1}{2}\right)\right]\right\} - \ln 2$$

where $c$ is the unique solution of.

$$c = \sqrt{2}\cos\left(\frac{c}{4}\right).$$

The numerical results demonstrating Theorem 8.1 and the estimated orders of convergence (EOC) are calculated in the usual way are given in Table 1, taken from [9], where $n$ represents the number of nodes.

**Table 1**

| $n$ | $\|y_h - y^*\|_\infty$ | $EOC$ | $\|D(y_h - y^*)\|_\infty$ | $EOC$ | $\|D^2(y_h - y^*)\|_\infty$ | $EOC$ |
|---|---|---|---|---|---|---|
| 5 | $2.0378E-06$ | 3.7966 | $5.8804E-05$ | 2.8363 | $4.5894E-03$ | 2.0438 |
| 10 | $1.4665E-07$ | 3.8799 | $8.2334E-06$ | 2.8831 | $1.1130E-03$ | 2.0569 |

The Adomian results are given in Table 2, where $n$ represents the number of iterations. Note that Adomian method, unlike the method of [9] gives comparable errors for $y(x), y'(x)$ and $y''(x)$.

**Table 2**

| $n$ | $\|\phi_n(x) - y^*(x)\|_\infty$ | $\|D(\phi_n(x) - y^*(x))\|_\infty$ | $\|D^2(\phi_n(x) - y^*(x))\|_\infty$ |
|---|---|---|---|
| 5 | $0.63752E-04$ | $0.19013E-03$ | $0.68213E-03$ |
| 6 | $0.13725E-04$ | $0.40661E-04$ | $0.14872E-03$ |
| 7 | $0.30861E-05$ | $0.91018E-05$ | $0.33743E-04$ |
| 8 | $0.7166E-06$ | $0.21058E-05$ | $0.78851E-05$ |
| 9 | $0.1703E-06$ | $0.4998E-06$ | $0.18853E-05$ |
| 10 | $0.415E-07$ | $0.1208E-06$ | $0.45871E-06$ |

The estimated orders of convergence (EOC) are calculated at the points $x = 0.1, 0.3, 0.49$ in Table 3.

**Table 3**

| $x$ | EOC of $y(x)$ | EOC of $y'(x)$ | EOC of $y''(x)$ |
|---|---|---|---|
| 0.1 | 0.9579765778 | 0.9598054006 | 0.9400877961 |
| 0.3 | 0.9592072387 | 0.9641110898 | 0.9600753758 |
| 0.49 | 0.9633065411 | 0.9661222712 | 0.9658450243 |

Now we give two examples with different nonlinearities.

**Example 1** ([10]) *Take*

$$-y'' = 10y^2 + 1,\ y(0) = y(1) = 0. \tag{2.9}$$

Now, we have $L = -\frac{d^2}{dx^2}$ and $N(y) = y^2$. Then

$$Ly = 10N(y) + 1$$

Applying the above mentioned steps on $L$, we get

$$\begin{aligned}
y_0 &= -\frac{1}{2}x^2 + \frac{1}{2}x \\
y_1 &= 10\int_0^1 G(x,\xi)A_0 d\xi \\
y_2 &= 10\int_0^1 G(x,\xi)A_1 d\xi \\
&\vdots \\
y_{n+1} &= 10\int_0^1 G(x,\xi)A_n d\xi,\ n = 0,1,2,\ldots.
\end{aligned}$$

where $G(x,\xi)$ is the Green's function given by

$$G(x,\xi) = \begin{cases} x(1-\xi), & 0 \le x \le \xi \le 1 \\ \xi(1-x), & 0 \le \xi \le x \le 1 \end{cases}$$

and $A_n$ are given as:

$$\begin{aligned}
A_0 &= y_0^2 \\
A_1 &= 2y_0y_1 \\
A_2 &= y_1^2 + 2y_0y_2 \\
A_3 &= 2y_1y_2 + 2y_0y_3 \\
A_4 &= y_2^2 + 2y_1y_3 + 2y_0y_4 \\
A_5 &= 2y_2y_3 + 2y_1y_4 + 2y_0y_5 \\
A_6 &= y_3^2 + 2y_2y_4 + 2y_1y_5 + 2y_0y_6 \\
&\vdots
\end{aligned}$$

In Fig. 1 we represent $\phi_2, \phi_8, \phi_9$.

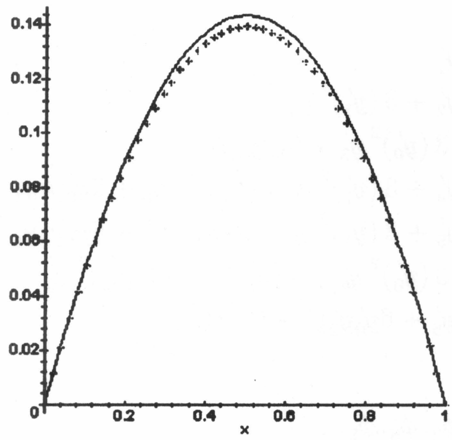

Fig. 1 $\times : \phi_2$ $- : \phi_8$ $- : \phi_9$

**Example 2** ([10]) *Take*

$$-y'' = (y')^3 + 1, \; y(0) = y(1) = 0. \tag{2.10}$$

We have $L = -\frac{d^2}{dx^2}$ and $N(y) = (y')^3$ . Then

$$Ly = N(y) + 1$$

Using the above mentioned steps on $L$, we get

$$\begin{aligned} y_0 &= -\frac{1}{2}x^2 + \frac{1}{2}x \\ y_1 &= \int_0^1 G(x,\xi)A_0 d\xi \\ y_2 &= \int_0^1 G(x,\xi)A_1 d\xi \\ &\vdots \\ y_{n+1} &= \int_0^1 G(x,\xi)A_n d\xi, \; n = 0,1,2,\ldots. \end{aligned}$$

where $G(x,\xi)$ is the Green's function given by

$$G(x,\xi) = \begin{cases} x(1-\xi), & 0 \le x \le \xi \le 1 \\ \xi(1-x), & 0 \le \xi \le x \le 1 \end{cases}$$

and $A_n$ are given as:

$$\begin{aligned}
A_0 &= (y_0')^3 \\
A_1 &= 3(y_0')^2 y_1' \\
A_2 &= 3(y_0')^2 y_2' + 3(y_1')^2 y_0' \\
A_3 &= (y_1')^3 + 3(y_0')^2 y_3' + 6y_0'y_1'y_2' \\
A_4 &= 3(y_0')^2 y_4' + 3(y_1')^2 y_2' + 3(y_2')^2 y_0' + 6y_0'y_1'y_3' \\
A_5 &= 3(y_0')^2 y_5' + 3(y_1')^2 y_3' + 3(y_2')^2 y_1' + 6y_0'y_1'y_4' + 6y_0'y_2'y_3' \\
A_6 &= (y_2')^3 + 3(y_0')^2 y_6' + 3(y_1')^2 y_4' + 3(y_3')^2 y_0' \\
&\quad +6y_0'y_1'y_5' + 6y_0'y_2'y_4' + 6y_1'y_2'y_3' \\
&\;\vdots
\end{aligned}$$

In Fig. 2 we represent $\phi_2, \phi_8, \phi_9$.

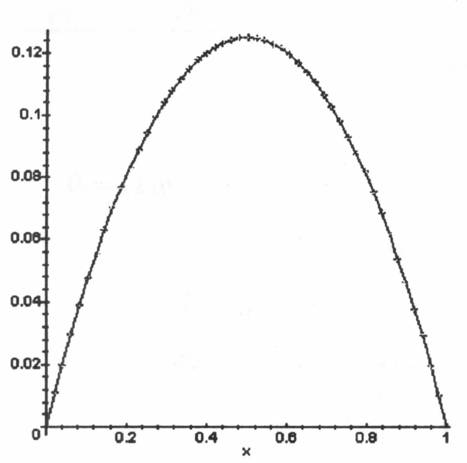


Fig. 2 $\times : \phi_2$ $- : \phi_8$ $- : \phi_9$

The nonlinear problems (1.1), (2.9) and (2.10) satisfy the following theorem.

**Theorem 3** ([2]) *With the following hypotheses,*

1. *$N$ is $C^{(\infty)}$ in a neghbourhood of $y_0$ and $\left\|N^{(n)}(y_0)\right\| \leq M'$, for any $n$ (the derivatives of $N$ at $y_0$ are bounded in norm);*

2. *$\|y_i\| \leq M < 1, i = 1, 2, \ldots$ , where $\|\cdot\|$ is the norm in the Hilbert space $H$; the series $\sum_{n=0}^{\infty} A_n$ is absolutely convergent and, furthermore,*

$$\|A_n\| \leq \left(\exp\left(\pi\sqrt{\frac{2}{3}n}\right)\right) M'M.$$

## 3. CONCLUSIONS

1. If we take $L = \frac{d^2}{dx^2}$ in (2.4) with the inverse operator $L^{-1}[\cdot] = \int\limits_0^x \int\limits_0^x [\cdot]\, dx dx$, the resulting convergence of the Adomian method is much worse.

2. The decomposition method in (1.1) gives a more accurate approximation of $y'(x)$ and $y''(x)$ than the method using splines of [9].

## REFERENCES


[1] ABBAOUI, K. and CHERRUAULT, Y. "Convergence of Adomian's Method Applied to Differential Equations". Mathematical and Computer Modelling 28(5), 103-109, (1994).

[2] ABBAOUI, K. and CHERRUAULT, Y. "New Ideas for Proving Convergence of Decomposition Methods". Computers Math. Applic., Vol. 29, No. 7, pp. 103-108, (1995).

[3] ABBAOUI, K. and CHERRUAULT, Y. "Convergence of Adomian's Method Applied to Nonlinear Equations". Mathematical and Computer Modelling 20(9), 60-73, (1994).

[4] ADOMIAN, G. "Stochastic Systems". Academic Press, (1983).

[5] ADOMIAN, G. "Nonlinear Stochastic Operator Equations". Academic Press, (1986).

[6] ADOMIAN, G. "Nonlinear Stochastic Systems Theory and Applications to Physics". Kluwer, (1989).

[7] ADOMIAN, G. "Solving Frontier Problems of Physics: The Decomposition Method". Kluwer, (1994).

[8] CHERRUAULT, Y. and ADOMIAN, G."Decomposition Methods: A New Proof of Convergence". Mathematical and Computer Modelling 18(12), 103-106, (1993).

[9] GANESH, M. and SLOAN, I. H., "Optimal Order Spline Methods for Nonlinear Differential and Integro-Differential Equations". Applied Numerical Mathematics 29, 445-478, (1999).

[10] GILBERT CHUDURY and PHILIP KORMAN, "Computation of Solutions of Nonlinear Boundary Value Problems". Computers Math. Applic. Vol. 22, No. 8, pp. 49-55, (1991).

[11] GUELLAL, S. and CHERRUAULT, Y. "Practical Formula for Calculation of Adomian's Polynomials and Application to the Convergence of the Decomposition Method". International Journal of Bio-Medical Computing 36, 223-228, (1994).