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A MEASURE OF WEAK NONCOMPACTNESS IN $L^1(\Omega,\Sigma,\lambda)$

I.J. Cabrera and K.B. Sadarangani
Departamento de Matemáticas, Universidad de Las Palmas de G.C.
Campus de Tafira Baja, 35017 LAS PALMAS.

We introduce a measure of weak noncompactness in the space $L^1(\Omega,\Sigma,\lambda)$ and we prove that this measure is less or equal to the classical De Blasi measure of weak noncompactness of $L^1(\Omega,\Sigma,\lambda)$. Moreover, we give a sufficient condition related with λ and Σ for that both measures are equal.

1. INTRODUCTION

The theory of measures of weak noncompactness was initiated by De Blasi in the paper [5]. This measure was applied successfully to nonlinear functional analysis, to operator theory and to the theory of differential and integral equations (see [1,2,3,7,8], for axample).

In the paper [4], a formula is given to express the De Blasi measure of weak noncompactness in the Lebesgue space $L^1(0,1)$. Following the same ideas of [4], in this paper we give a measure of weak noncompactness in the space $L^1(\Omega,\Sigma,\lambda)$, where λ is a nonnegative measure on a set Ω with Σ a σ-field of subsets of Ω, which is less or equal to the classical De Blasi measure of weak non-compactness of $L^1(\Omega,\Sigma,\lambda)$. Moreover, we give a sufficient condition for that both measure are equal.

2. NOTATION AND DEFINITIONS

Let E be a n infinite dimensional real Banach space with norm $\|\cdot\|$ and zero element Θ. Denote by B_E the closed unit ball of the space E.

Next, let M_E be the family of all nonempty and bounded subsets of E and let N_E^w be its subfamily consisting of all relatively weakly compact sets.

The so-called De Blasi measure of weak noncompactness $\beta : M_E \to [0,\infty)$ is defined in the following way:

$$\beta(X) = \inf\{\varepsilon > 0 : \exists Y \in N_E^w \,/\, X \subset Y + \varepsilon B_E\}$$

Let us observe that in the case when E is a reflexive Banach space then $\beta(X)=0$, for every $X \in M_E$. Hence it is only of interest to consider the function β in a nonreflexive Banach space. Recall that $\beta(B_E)=1$ when E is nonreflexive. For other properties of the De Blasi measure of noncompactness we refer to [5,8], for example.

In what follows, suppose Ω is a set , Σ is a σ-field of Ω and λ a nonnegative measure on (Ω,Σ). By $L^1(\Omega,\Sigma,\lambda)$ we denote the following set:

$$L^1(\Omega,\Sigma,\lambda)=\left\{f:\Omega\to C,\text{ measurable with }\int_\Omega |f|\,d\lambda<\infty\right\}$$

Now, based on the paper [4], we introduce the function H defined on the family $M_{L^1(\Omega,\Sigma,\lambda)}$ by the formula:

$$H(X)=\lim_{\lambda(A)\to 0} H_A(X)$$

where $A\in\Sigma$ and $H_A(X)=\sup_{f\in X}\left(\int_A |f|\,d\lambda\right)$.

The properties of the function H will be investigated in the next section. In particular, we show that H provides a lower estimate for $\beta_{L^1(\Omega,\Sigma,\lambda)}$

3. RESULTS

We start with the following result which proves that H is a measure of weak noncompactness.

THEOREM 1. The function H satisfies the following conditions:

(a) $H(X)=0 \Leftrightarrow X\in N_E^w$

(b) $X\subset Y \Rightarrow H(X)\le H(Y)$

(c) $H(cX)=|c|\,H(X),\ \forall c\in R$

(d) $H(X+Y)\le H(X)+H(Y)$

(e) $H(\text{conv}X)=H(X)$

(f) $H(B_{L^1(\Omega,\Sigma,\lambda)})\le 1$

Proof.

(a) is consequence of a result of [6, pag 93], while (b) follows inmediately from the definition of the function H.

In order to prove (c) let us take a set $X \in M_{L^1(\Omega,\Sigma,\lambda)}$ and $A \in \Sigma$, and assume that $c \in R$ and $c \neq 0$, then we have

$$H_A(cX) = \sup_{f\in X}\left(\int_A |cf|\,d\lambda\right) = \sup_{f\in X}\left(|c|\int_A |f|\,d\lambda\right) = |c|\sup_{f\in X}\left(\int_A |f|\,d\lambda\right) = |c|\,H_A(X)$$

and, consequently, $H(cX) = |c|\,H(X)$.

In the case when $c = 0$, we have that cX is zero function and, thus, $H_A(cX) = 0$ and in this case the equality (c) is also satisfied.

In order to prove (d), let us take a set $A \in \Sigma$, $f \in X$ and $g \in Y$. Then we have

$$\int_A |f+g|\,d\lambda \le \int_A |f|\,d\lambda + \int_A |g|\,d\lambda \le H_A(X) + H_A(Y)$$

and, thus

$$H_A(X+Y) \le H_A(X) + H_A(Y)$$

consequently, $H(X+Y) \le H(X) + H(Y)$

For the proof of (e) it is sufficient to prove the inequality $H(\mathrm{conv}X) \le H(X)$. In fact, we take $f = \sum_i \lambda_i f_i$ with $\sum_i \lambda_i = 1$, $\lambda_i \ge 0$, $f_i \in X$ and, moreover, $A \in \Sigma$.

Then we can obtain

$$\int \left|\sum_i \lambda_i f_i\right| d\lambda \le \sum_i \lambda_i \int |f_i|\,d\lambda \le \sum_i \lambda_i H_A(X)$$

consequently, $H_A(\mathrm{conv}X) \le H_A(X)$ and, thus, $H(\mathrm{conv}X) \le H(X)$.

Finally, as $H_A(X) \le \|X\|$, where $\|X\| = \sup_{f\in X}\left(\int_\Omega |f|\,d\lambda\right)$, we have that

$$H_A(B_{L^1(\Omega,\Sigma,\lambda)}) \le \left\|B_{L^1(\Omega,\Sigma,\lambda)}\right\| \le 1$$

and, thus, $H(B_{L^1(\Omega,\Sigma,\lambda)}) \le 1$.

This following result proves that the function H is a lower estimate for the classical De Blasi measure of weak noncompactness β in $L^1(\Omega,\Sigma,\lambda)$.

THEOREM 2. $H(X) \leq \beta(X)$ for any $X \in M_{L^1(\Omega,\Sigma,\lambda)}$

Proof.

Suppose that $\beta(X) = r$. Then for any $\varepsilon > 0$ we can find a set $Y \in N^w_{L^1(\Omega,\Sigma,\lambda)}$ such that $X \subset Y + (r+\varepsilon)B_{L^1(\Omega,\Sigma,\lambda)}$ (see section 2). Hence, using theorem 1, we obtain

$$H(X) \leq H(Y) + (r+\varepsilon)\,H(B_{L^1(\Omega,\Sigma,\lambda)}) \leq r+\varepsilon$$

in virtue of the arbitrariness of ε, this implies $H(X) \leq \beta(X)$.

Our last result will be to give a sufficient condition related with the σ-field Σ and the measure λ for that both measure H and β are equal. This condition has been taken following the same ideas of the paper [4].

THEOREM 3. Suppose that exists a sequence (A_n) of elements of Σ such that $\lambda(A_n)$ tends to zero when $n \to \infty$, and, moreover, for any bounded subset X of $L^1(\Omega,\Sigma,\lambda)$, $X_{\Omega-A_n} = \left\{ f\chi_{\Omega-A_n} : f \in X \right\}$ is relatively weakly compact in $L^1(\Omega,\Sigma,\lambda)$ for every n. In these conditions $H = \beta$.

Proof.

It´s sufficient prove that $\beta \leq H$, in virtue of theorem 2. In fact, every function $f \in L^1(\Omega,\Sigma,\lambda)$ may be written in the form:

$$f = f\chi_{A_n} + f\chi_{\Omega-A_n} \quad (1)$$

where χ_A denotes the characteristic function of the set A.

Using the representation (1) it may be easily show that

$$X \subset X_{\Omega-A_n} + H_{A_n}(X)\,B_{L^1(\Omega,\Sigma,\lambda)}$$

Hence, in view of the properties of the De Blasi measure of weak noncompactness β, we get

$$\beta(X) \leq \beta(X_{\Omega-A_n}) + H_{A_n}(X)\,\beta(B_{L^1(\Omega,\Sigma,\lambda)})$$

In virtue of the hipothesis, $\beta(X_{\Omega-A_n}) = 0$ and as we consider the case of $L^1(\Omega,\Sigma,\lambda)$ a nonreflexive space, because the other case is trivial, we obtain

$$\beta(X) \leq H_{A_n}(X)\,\beta(B_{L^1(\Omega,\Sigma,\lambda)}) = H_{A_n}(X)$$

and thus $\beta(X) \leq H(X)$.

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