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REGULAR AND SINGULAR ELEMENTS OF A COMPLETE METRIC I-PARA-ALGEBRA AND A METRIC I-PARA-SEMIALGEBRA

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ABSTRACT : The paper introduces the notions of a metric para-algebra and a metric para-semialgebra. It studies regular and singular elements of an unital metric para-algebra and those of a metric para-semialgebra with S-identity. It also investigates the nature of radical of a metric para-algebra and a metric para-semialgebra.

1. INTRODUCTION : Properties of regular and singular elements of an unital Banach algebra were studied by C.E. Rickart, 1960; G.F. Simmons, 1963 and many others. They also investigated the nature of radical of this algebra. D. K. Bhattacharya and A. K. Maity, 1992 studied regular and singular elements of a nonunital Banach algebra with central idempotents. Also the radical of such algebra was studied by D. K. Bhattacharya and T. Roy, 1996.

In the present paper, it is shown that the above discussions may be carried out for an unital algebra by dropping the property of homogeneity of the norm or even without assuming the existence of additive inverses of elements of the algebra.

The whole discussion is limited to four articles - the first one deals with definitions and examples of different types of metric para-algebras and metric para-semialgebras; the second one deals with regular and singular elements of an unital metric para-algebra; the third one studies similar elements for a metric para-semialgebrs with S-identity; the last article investigates the nature of radical of the above algebras.

2. SOME DEFINITIONS AND EXAMPLES :

Definition 2.1. A semilinear space X over $R^+ \cup \{0\}$ is an additively commutative semigroup with identity θ,

where for all $x, y \in X$, $\alpha, \beta \in R^+ \cup \{0\}$, $1 \in R^+$, $(\alpha + \beta)x = \alpha x + \beta x$, $\alpha(x + y) = \alpha x + \alpha y$, $1x = x$, $0x = \theta$.

Definition 2.2. A semi algebra X over $R^+U\{0\}$ is a semilinear space over $R^+U\{0\}$ where there is an additional composition o : XxX $\rightarrow$ X, called multiplication such that

(i) $x\circ(y\circ z)=(x\circ y)\circ z$

(ii) $x\circ(y+z)=x\circ y+x\circ z$, $(x+y)\circ z=x\circ z+y\circ z$

(iii) $\theta\, x=\theta=x\,\theta$

(iv) $\alpha(x\circ y)=(\alpha\, x)\circ y=x\circ(\alpha\, y)$, $\forall\, x\, ,\, y\, ,\, z\, \in\, X\, ,\, \alpha\, \in\, R^+U\{0\}$.

Definition 2.3. Let X be an unital semialgebra. The multiplicative identity I of X is called a S-identity, if for each $r \in X$, there exists a unique $r^0 \in X$ such that $r + r^0 = I$.

Definition 2.4. A paranorm on a linear space X over a field F is a function

$p : X \rightarrow R^+U\{0\}$ such that

(i) $p(x) = o$ iff $x = \theta$

(ii) $p(x+y) \leq p(x) + p(y)$

(iii) $p(-x) = p(x)$

(iv) $|\lambda_n - \lambda| \rightarrow 0,\ p(x_n - x) \rightarrow 0 \Rightarrow p(\lambda_n x_n - \lambda x) \rightarrow 0$ as $n \rightarrow \infty$

where $\{x_n\} \subset X$, x, $y \in X$, $\{\lambda_n\} \subset F$, $\lambda \in F$,. θ is the additive identity of X.

(X, p) is called a paranormed space over F.

Remark 2.1. Every linear metric space/metric linear space (X,ρ) is a paranormed space (X,p) where

$p(x)=\rho(x\ ,\ \theta)$, $x\in X$ [θ being the additive identity of X].

Examples of paranormed spaces

Example 2.1. $l_\infty(r) = \left[x=\{x_k\},\ x_k \in R : \operatorname*{Sup}_k |x_k|^{r_k} < \infty\right]$

where $\{r_k\}$ is a bounded sequence of reals $0 < r_k \leq H$, $H = \operatorname*{Sup}_k r_k$, $\inf r_k > 0$. $l_\infty(r)$ is a paranormed space over R under componentwise operations, where paranorm p is given by

$p(x) = \sup_k |x_k|^{r_k/M}$, M = max (1, H).

Example 2.2. $l(r) = \left[x=\{x_k\},\ x_k \in R;\ \sum_1^{\infty} |x_k|^{r_k} \text{ exists} \right]$, where $\{r_k\}$ is defined as in example 2.1.

l(r) is paranormed space over R under componentwise operations with paranorm

$$p(x) = \left(\sum_1^{\infty} |x_k|^{r_k} \right)^{1/M}.$$

Example 2.3. $X = \left[x=\{x_k\},\ x_k \in R,\ \sup_k |x_k| < \infty \right]$.

X is a paranormed space over R under componentwise operations, with paranorm

$$p(x) = \sup_k \frac{|x_k|}{1+|x_k|}.$$

Remark 2.2. Every normed linear space is a paranormed space, but the converse is not true.

Remark 2.3. A linear space endowed with an invariant metric may not induce a paranorm.

Examples 2.1. 2.2. and 2.3. are all linear spaces with translation invariant metrics

$$\rho(x,y) = \sup_k |x_k - y_k|^{r_k/M}, \quad \rho(x,y) = \left(\sum_1^{\infty} |x_k - y_k| \right)^{1/M} \text{ and}$$

$\rho(x,y) = \sup_k \frac{|x_k - y_k|}{1+|x_k - y_k|}$ respectively, where all the metrics ρ induce p given by $p(x) = \rho(x,\theta)$. But if in Example 2.1. $\inf r_k > 0$ then corresponding metric ρ, although translation invariant, can not induce the paranorm p.

Definition 2.5. Let X be a linear space over F. A function $p : X \to R^+ \cup \{0\}$ is called a l-paranorm on X, if for x, y $\in$ X, $\alpha \in F$

(i) p(x) = 0 iff $x = \theta$ (ii) $p(x+y) \le p(x)+p(y)$ (iii) p(-x) = p(x)

(iv) $p(\alpha x) \le |\alpha|\, p(x)$, $|\alpha| \ge 1$

$|\alpha|\, p(x) < p(\alpha x) < p(x)$, $|\alpha| < 1$.

EXAMPLES OF I-PARANORMED SPACES.

The spaces $l_\infty(r)$ (example 2.1.) and X (example 2.3.) are l-paranormed spaces.

Remark 2.4. A normed linear space is not a l-paranormed space and a l-paranormed space is not a normed linear space.

Remark 2.5. A paranormed space may not be I-paranormed space. In fact, I(r) (example 2.2) is not a I-paranormed space.

Remark 2.6. Every I-paranormed space (X , p) is a paranormed space.

Let $\{x_n\} \subset X$ and $x \in X$ such that $p(x_n - x) \to 0$ as $n \to \infty$;

let $\{\alpha_n\} \subset F$ and $\alpha \in F$ be such that $|\alpha_n - \alpha| \to 0$ as $n \to \infty$

Now $p(\alpha_n x_n - \alpha x) \leq p[\alpha_n(x_n - x)] + p[(\alpha_n - \alpha)x]$

If $|\alpha_n| \geq 1$, then $p[\alpha_n(x_n - x)] \leq |\alpha_n| p(x_n - x)$; also as $|\alpha_n - \alpha| < 1$, so

$p[(\alpha_n - \alpha)x] < p(x)$, hence $p[(\alpha_n - \alpha)x] \leq |\alpha_n - \alpha|.\beta.p(x)$ for some $\beta > 1$.

Hence $p(\alpha_n x_n - \alpha x) \to 0$ as $n \to \infty$.

If $|\alpha_n| < 1$, then $p[\alpha_n(x_n - x)] < p(x_n - x)$ and

$p[(\alpha_n - \alpha)x] \leq |\alpha_n - \alpha| \beta p(x)$, $\beta > 1$.

So, $p(\alpha_n x_n - \alpha x) \to 0$ as $n \to \infty$. Hence X is a paranormed space.

Remark 2.7. Normed linear spaces and I-paranormed spaces form two distinct subsets of the set of all paranormed spaces.

Definition 2.6. A semiparanorm on a semilinear space X over $R^+U\{0\}$ is a function $p : X \to R^+U\{0\}$ such that for all x, y $\in$ X

(i) $p(x) = 0$ iff $x = \theta$ (ii) $p(x+y) \leq p(x) + p(y)$.

(iii) For every $\{x_n\} \subset X$ and $x \in X$ for which $|p(x_n) - p(x)| \to 0$ as $n \to \infty$

and for every $\{\lambda_n\} \subset R^+U\{0\}$, $\lambda \in R^+U\{0\}$ for which $|\lambda_n - \lambda| \to 0$ as $n \to \infty$,

it is implied that $\left| p(\lambda_n x_n) - p(\lambda x) \right| \to 0$ as $n \to \infty$.

(X,p) is called a semiparanormed space over $R^+U\{0\}$.

Remark 2.8. A semiparanorm on a semilinear space X can never induce an invariant metric on X, as for x, y $\in$ X, x-y is undefined. However, a semilinear space X endowed with an invariant metric ρ may or may not induce a semiparanorm p on X.

A semilinear space X equipped with an invariant metric ρ is called a metric semilinear space if ρ induces a semiparanorm p on X. In this case, it is also called a semiparanormed space and is denoted by (X, p).

EXAMPLES OF SEMIPARANORMED SPACES.

Example 2.4. $l_\infty^+(r) = \left[x=\{x_k\},\ x_k > 0 : \sup_k (x_k)^{r_k} < \infty\right]$

is a semiparanormed space over $R^+ U\{0\}$ under componentwise operations with semiparanorm $p(x) = \sup_k (x_k)^{r_{k/M}}$, $\inf r_k > 0$, where $\{r_k\}$ is defined as in $l_\infty(r)$.

It is not a semiparanormed space if $\inf r_k > 0$.

Example 2.5. $l^+(r) = \left[x= \{x_k\},\ x_k > 0 : \sum_1^\infty (x_k)^{r_k} < \infty\right]$

is a paranormed space over $R^+ U\{0\}$ under componentwise operations with semiparanorm

$$p(x) = \left(\sum_1^\infty (x_k)^{r_k}\right)^{1/M}$$

Definition 2.7. Let X be an algebra over F with invariant metric ρ. It is called a metric paraalgebra over F if it is a paranormed space with paranorm $p(x) = \rho(x, \theta)$, where for all x, y, z, $\in$ X,

$$\rho(xy,\ xz) \le \rho(x,\ \theta)\ \rho(y, z)\ ;\ \rho(xy,\ zy) \le \rho(x, z)\ \rho(y, \theta).$$

Definition 2.8. let X be an algebra over F. It is called a paranormed algebra over F if it is a paranormed space with paranorm p where

$$p(xy) \le p(x)\ p(y),\ \forall\ x,\ y \in X.$$

Remark 2.9. Every metric paraalgebra is a paranormed algebra.

EXAMPLES OF METRIC PARAALGEBRAS

Example 2.1. and example 2.2. are metric paraalgebras under an additional composition of multiplication defined componentwise.

Definition 2.9. Let X be an algebra over F with invariant metric ρ. It is called a parametric paraalgebra or a paranormed paraalgebra over F if it is a paranormed space with paranorm $p(x) = \rho(x, \theta)$ where the operation of miltiplication is continuous with respect to the paranorm p, i.e.

(i) $\{x_n\} \subset X$, $x \in X$, $p(x_n - x) \to 0 \Rightarrow p(x_n y - xy) \to 0$ as $n \to \infty$, for each $y \in X$.

(ii) $\{y_n\} \subset X$, $y \in X$, $p(y_n - y) \to 0 \Rightarrow p(xy_n - xy) \to 0$ as $n \to \infty$ for each $x \in X$.

Remark 2.10. A paranormed algebra is always a paranormed paraalgebra but the converse is not true. Example 2.3. under componentwise operation of multiplication is a paranormed paraalgrbra which is not a paranormed algebra.

Indeed, let $\{f_i\} = \{x_{in}\} \subset X$ and $f = \{x_n\}$ belong to X such that $p(f_i - f) \to 0$ as $i \to \infty$.

Then $\sup_n \frac{|x_{in} - x_n|}{1 + |x_{in} - x_n|} \to 0$ as $i \to \infty$.

Thus $|x_{in} - x_n| \to 0$ i.e. $|x_{in} y_n - x_n y_n| \to 0$ as $i \to \infty$, for each bounded $\{y_n\} \subset X$.

Hence if $g = \{y_n\}$, then $p(f_i g - fg) \to 0$ as $i \to \infty$, for each $g \in X$.

It may be similarly shown that if $g_i = \{g_{in}\} \subset X$, $g = \{g_n\} \subset X$ be such that $p(g_i - g) \to 0$ as $i \to \infty$, then $p(fg_i - fg) \to 0$ as $i \to \infty$ for each $f \in X$. So X is a paranormed para-algebra.

Now we show that $p(fg) \leq p(f)\, p(g)$ does not hold for all f, $g \in X$.

In fact, for

$$f = \{x_n\},\ p(ff) = \sup_n \left[x_n^2 / (1 + x_n^2)\right]$$

$$\geq \sup_n \left[\, |x_n| / (1 + |x_n|)\, \right] \sup_n \left[\, |x_n| / (1 + |x_n|)\, \right]$$

$$= p(f)\, p(f)$$

Hence X is not a paranormed algebra.

Remark 2.11. Definitions and examples of l-paranormed paraalgebra, l-paranormed algebra may be similarly given. Moreover, all these definitions and examples may be given in the setting of semilinear space as well.

3. REGULAR AND SINGULAR ELEMENTS OF AN UNITAL METRIC PARA-ALGEBRA.

Theorem 3.1. Let (X, P) be a complete metric paraalgebra with identity I. Then each $r \in X$ for which $\rho(r, I) < 1$ is regular and its inverse s is given by $s = I + \sum_{n=1}^{\infty}(I-r)^n$

Theorem 3.2. Let G and S denote respectively the set of regular and singular elements of a complete metric paraalgebra (X, P). Then G is an open set and S is a closed set.

Theorem 3.3. The mapping $r \to r^{-1}$ of G into G is continuous. As an application of theorem 3.2. the following theorem follows :

Theorem 3.4. Let R be the radical of a complete metric paraalgebra (X, P) where R is taken as the intersection of all maximal left ideals of X. Then X/R is a complete semisimple metric paraalgebra.

We simply state the theorems 3.1. - 3.4. , as the proofs are parallel in case of a Banach algebra.

We now define a topological divisor of zero in a metric l-paraalgebra and obtain one of its important properties.

Definition 3.1. An element z of a metric l-paraalgebra (X, P) called a topological divisor of zero, if there exists a sequence $\{z_n\} \subset X$ such that $p(z_n) \to 0$ as $n \to \infty$ but either zz_n or $z_n z$ tends to zero as $n \to \infty$.

Theorem 3.5. Let Z denote the set of all topological divisors of zero of a metric l-paraalgebra (X, P), then $Z \subseteq S$; also some boundary points of S form a subset of Z.

Proof. $Z \subseteq S$ is clear, because if $z \in Z$ but $z \notin S$ then z^{-1} exists. As $z \in Z$, so there exists $\{z_n\} \subset X$ such that $p(z_n) \to 0$ as $n \to \infty$ but $zz_n \to 0$ or $z_n z \to 0$ as $n \to \infty$.

So $p(z_n) = p((z^{-1}z)z_n) = p(z^{-1}(zz_n)) \leq p(z^{-1})\, p(zz_n) \to 0$ as $n \to \infty$.

This is a contradiction. So $z \in S$. Now to prove the other part, we take $z \in \text{bd } S$. As S is closed, so there exists $\{r_n\} \subset G$ such that $P(r_n - z) \to 0$ as $n \to \infty$. Also the sequence $\{p(r_n^{-1})\}$ is unbounded, for otherwise, if $p(r_n^{-1}) < \infty$ then using $p(r_n - z) \to 0$ as $n \to \infty$. the inequality $p(r_n^{-1}z - I) = p(r_n^{-1}(z - r_n)) \leq p(r_n^{-1})\, p(z - r_n)$, it may be shown that $p(r_n^{-1}z - I) < 1$. So $r_n^{-1}z \in G$.

Hence $z = r_n(r_n^{-1}z) \in G$. This is a contradiction. Therefore $p(r_n^{-1}) \to \infty$ as $n \to \infty$.

Let $z_n = \left(\dfrac{r_n^{-1}}{p(r_n^{-1})}\right)$.

So $p(z_n) = p\left(\dfrac{r_n^{-1}}{p(r_n^{-1})}\right)$. As $p(r_n^{-1}) \to \infty$ as $n \to \infty$,

so $\exists\ N(>0)$ such that $p(r_n^{-1}) > M$, M being large at pleasure, $\forall\ n \geq N$.

So, $\dfrac{1}{p(r_n^{-1})} < \dfrac{1}{M} < 1$. Using $|\alpha|\ p(x) < p(\alpha x) < p(x)$, if $|\alpha| < 1$ and taking $x = r_n^{-1}$,

We get $\dfrac{1}{p(r_n^{-1})}\ p(r_n^{-1}) < p(z_n) < p(r_n^{-1}), \forall\ n \geq N$.

i.e., $1 < p(z_n) < p(r_n^{-1})$, $\forall\ n \geq N$.

So, $p(z_n) \to 0$ as $n \to \infty$. Moreover, it is concluded that $\{z_n\}$ may not always tend to infinity as $n \to \infty$.

In case, $p(z_n) \to \infty$ as $n \to \infty$, we show that $p(zz_n) \to 0$ as $n \to \infty$.

In fact, $p(zz_n) = p(z - r_n + r_n, z_n) \leq p\{(z - r_n)z_n\} + p(r_n z_n) \leq p(z - r_n)\ p(z_n) + p(r_n z_n)$.

Now, $p(r_n z_n) = p(r_n \cdot \dfrac{r_n^{-1}}{p(r_n^{-1})}) = p(\alpha_n I)$ where $\alpha_n = \dfrac{1}{p(r_n^{-1})} \to 0$ as $n \to \infty$.

As p is paranorm so $p(\alpha_n I) \to p(0I) = 0$

So, $p(zz_n) \to 0$ as $n \to \infty$ but $p(z_n) \to 0$ as $n \to \infty$, so z becomes a topological divisor of zero. As z_n depends on r_n^{-1} and hence on r_n and as r_n, in its term, is chosen corresponding to the boundary point z, so it may be concluded that there are some points on the boundary of the set of singular elements such that they are topological divisors of zero.

4. REGULAR AND SINGULAR ELEMENTS OF A COMPLETE METRIC I- PARA-SEMIALGEBRA

4.1 Properties of regular and singular elements.

Theorem 4.1.1. Let (X,ρ) be a complete metric para-semialgebra with I as its S-identity. Then each $r \in X$, for which $\rho(I, r) < 1$, is regular. Further, the inverse of r is given by $r' = I + \sum_{1}^{\infty} t^n$ where $r + t = I$.

Proof. By given conditions $p(t) = \rho(t, \theta) = \rho(r + t, r) = \rho(I, r) < 1$. As $p(t^n) \le (p(t))^n < 1$, so partial sums of $\sum_{1}^{\infty} t^n$ from a Cauchy sequence in X. As X is complete, so $\sum_{1}^{\infty} t^n$ converges to some element of X. We denote it by $\sum_{1}^{\infty} t^n$. Now

$$\rho(rr', I) = \rho\left(r + \sum_{1}^{\infty} r\, t^n,\ r + t\right) = \rho\left(\sum_{1}^{\infty} r\, t^n,\ t\right)$$

$$= \rho\left(\sum_{1}^{\infty} r\, t^n + \sum_{2}^{\infty} t^n,\ t + \sum_{1}^{\infty} t^n\right)$$

$$= \rho\left(\sum_{1}^{\infty} r\, t^n + \sum_{2}^{\infty} t^n,\ \sum_{1}^{\infty} t^n\right)$$

Again $I = r + t \Rightarrow t^n = r\, t^n + t^{n+1}$. So $\sum_{1}^{\infty} t^n = \sum_{1}^{\infty} r\, t^n + \sum_{1}^{\infty} t^{n+1} = \sum_{1}^{\infty} r\, t^n + \sum_{2}^{\infty} t^n$

Hence $\rho(r\, r', I) = \rho\left(\sum_{1}^{\infty} t^n, \sum_{1}^{\infty} t^n\right) = 0$. So $r\, r' = I$. It may be similarly shown that $r'\, r = I$. Thus r' is the inverse of r. This proves the theorem.

Theorem 4.1.2. Let (X,ρ) be defined as in theorem 4.1.1. Then the set of regular elements G is open and the set of singular elements S is closed.

Proof : We only prove that G is open. Let $r \in G$ be arbitrary and let r' be its inverse. We show that the open ball with centre r and radius $1/p(r')$ lies in G. Let $y \in X$ such that $\rho(y, r) < 1/p(r')$.

We prove that $y \in G$.

Now $\rho(yr', I) = \rho(yr', r\, r') \le \rho(y, r)\ \rho(r', \theta) \le (1/p(r'))\ p(r') = 1.$

Hence $yr' \in G$. As composed of two regular elements is a regular element,

so $(yr')r = y(rr') = y \in G$. Thus G is open and hence S is closed.

Theorem 4.1.3. Let (X, ρ) be defined as in theorem 4.1.1., then the mapping $r \to r^{-1}$ of G into G is continuous.

Proof : Let $r, r_0 \in G$ such that $\rho(r, r_0) < (1/2)\, p(r_0^{-1})$; then

$\rho\left(r_0^{-1} r, I\right) = \rho\left(r_0^{-1} r, r_0^{-1} r_0\right) \le p\left(r_0^{-1}\right) \rho\left(r, r_0\right) < 1/2$. So $r_0^{-1} r \in G$.

Now $r^{-1} r_0 = \left(r_0^{-1} r\right)^{-1} = I + \sum_1^{\infty} t^n$ where $r_0^{-1} r + t = I$. Hence

$p(t) = \rho(t, \theta) = \rho\left(r_0^{-1} r + t, r_0^{-1} r\right) = \rho\left(I, r_0^{-1} r\right) < 1/2$.

$$\text{Now } \rho\left(r^{-1}, r_0^{-1}\right) = \rho\left(r^{-1} r_0 r_0^{-1}, I r_0^{-1}\right) \le p\left(r_0^{-1}\right) \rho\left(r^{-1} r_0, I\right)$$

$$= p\left(r_0^{-1}\right) \rho\left(I + \sum_1^{\infty} t^n, I\right)$$

$$= p\left(r_0^{-1}\right) p\left(\sum_1^{\infty}(t^n)\right) \le p\left(r_0^{-1}\right) \left(\sum_1^{\infty} p(t)\right)^n$$

$$= p\left(r_0^{-1}\right) \left[p(t) / (1 - p(t))\right], \text{ [as } p(t) < 1 \text{]}$$

$$< 2\, p\left(r_0^{-1}\right) p(t), \text{ [as } p(t) < 1/2 \text{]}$$

$$\text{But } p(t) = \rho(t, \theta) = \rho\left(t + r_0^{-1} r, r_0^{-1} r\right) = \rho\left(I, r_0^{-1} r\right) = \rho\left(r_0^{-1} r_0, r_0^{-1} r\right)$$

$$\le p\left(r_0^{-1}\right) \rho\left(r_0, r\right).$$

$$\text{So } \rho\left(r^{-1}, r_0^{-1}\right) < 2p\left(r_0^{-1}\right) \rho\left(r_0, r\right) p\left(r_0^{-1}\right) = 2\left(p\left(r_0^{-1}\right)\right)^2 \rho\left(r, r_0\right)$$

$$= k\, \rho\left(r, r_0\right),\ k > 0.$$

Hence $r \to r^{-1}$ of G into G is a continuous map.

4.2 RADICAL OF A SEMIALGEBRA WITH S-IDENTITY

In this article, it is shown that the usual conceps of radical of an algebra fails for a semialgebra. In fact, it is nolonger a two sided ideal and hence the corresponding quotient algebra with respect to this radical can not be obtained. However, left and right radicals for the semialgebra may be defined and their characterizations may be obtained.

Definition 4.2.1. Let, X be a semialgebra over a field F. A left radical R_l of X is defined as the intersection of all maximal left ideals. A right radical R_r is defined as the intersection of all maximal right ideals,

Definition 4.2.2. A semialgebra is called semisimple if each of R_l and R_r is a zero ideal of the semialgebra.

Theorem 4.2.1. Let R_l be the left radical of a semialgebra with I as its S-identity. Then R_l consists of precisely those elements r of X such that each $r^0 \in X$ for which $x r + r^0 = I$, for some $x \in X$ is left regular, without being right regular.

Proof : We first prove the following lemmas :

Lemma 1. If $r \in R_l$ and if $r + r^0 = I$, then r^0 is left regular. If possible let r^0 be not left regular. Then

$[x(r^0) : x \in X]$ is a proper left ideal of X containing r^0. We now imbed $[x(r^0) : x \in X]$ in a maximal left ideal M of X. Obviously r, $r^0 \in M$. So $r+r^0 = I$ is impossible, as M is proper. Hence r^0 is left regular.

Lemma 2. If $r \in R_l$ and if $xr + r^0 = I$ for some $x \in X$, then each such $r^0 \in X$ is left regular but r^0 is not right regular.

Proof : Since $r \in R_l \Rightarrow xr \in R_l$, so by Lemma 1, r^0 is left regular. If possible let r^0 be right regular also. Then there exists a unique $s \in X$ such that $r^0 s = I$. Now $r^0 + xr = I \Rightarrow I + (xr)s = s$. As s is unique, so for given r and x, (xr)s is also unique. Hence $I + (xr)s = s$ means that the cancellation law for addition holds in X. But this is impossible for X. So r^0 is not right regular.

Lemma 3. Let $r \in X$. Let $r^0 \in X$ be left regular and let r^0 be not right regular.
If $xr + r^0 = I$ for some $x \in X$ then $r \in R_l$.

Proof : If possible let $r \notin R_l$. Then r does not belong to some maximal left ideal M of X. So $\{m+xr : m \in M, x \in X\}$ is a left ideal containing both M and r. So $\{m + xr\}$ is improper. Hence $xr+m=I$ for some $m \in M$ and some $x \in X$. Therefore by Lemma 2, m is left regular and m is not right regular. But this implies that m can not belong to a proper left ideal. Hence we get a contradiction as $m \in M$. So $r \in R_l$.

Proof of the main theorem follows directly from Lemma 2 and Lemma 3.

Theorem 4.2.2. Let R_r be the right radical of a semialgebra X with I as the S-identity. Then R_r consists of precisely those elements $r \in X$ such that each $r^{00} \in X$ for which $rx + r^{00} = I$ for some $x \in X$ is right regular but r^{00} is not left regular.

Proof follows as in Theorem 4.2.1.

Theorem 4.2.3. $R_l \cap R_r = \{\theta\}$ where R_l and R_r are defined as above.

Proof : Let $r \neq \theta \in R_l$. Then each $r^0 \in X$ such that $xr + r^0 = I$, for some $x \in X$ is left regular (not right regular). So there exists a unique $s \in X$ such that $s r^0 = I$. Now let $r^{00} \in X$ be such that $rx + r^{00} = I$, for the same r and x taken above. We show that r^{00} can not be right regular. If possible let r^{00} be right regular. Then there exists a unique $s' \in X$ such that

$r^{00} s' = I$. Now $rx + r^{00} = I \Rightarrow (rsx)\, rx + (rsx)r^{00} = rsx$

$\Rightarrow r(sxr)x + (rsx)r^{00} = rsx$

$\Rightarrow r(sxr)x + rx + (rsx)r^{00} = rsx + rx$

$\Rightarrow rsx + (rsx)r^{00} = rsx + rx \; [xr + r^0 = I \Rightarrow sxr + I = s]$

$\Rightarrow (rsx)\, s' + rsx = (rsx)\, s' + rx\, s'$.

As cancellation law does not hold in x, so $rsx \neq rx\,s$. Naturally, any s' can act as the right inverse of r^{00} provided $rsx \neq rx\,s'$. Thus s' fails to be unique. This is a contradiction. Hence $r \notin R_r$. So $R_l \cap R_r = \{\theta\}$.

4.3. Radical of a metric para-semialgebra with S-identity

Theorem 4.3.1. The left radical R_l of a complete metric para-semialgebra (X, ρ) with S-identity is a proper closed left ideal of (X, ρ).

Proof : We show that L is closed. If L is not closed, then $L \subset \bar{L}$ (closure of L). As L is proper, so $L \subset S$ (the set of all singular elements). As S is closed, so $\bar{L} \subset S$. Hence $\bar{L}$ is also proper. This contradicts maximality of L, as $L \subset \bar{L}$. So L is closed. Thus R_l is the intersection of closed ideals. Hence R_l is closed.

Theorem 4.3.2. The right ideal R_r of a complete metric para-semialgebra with S-identity is a proper closed right ideal.

Proof is parallel to Theorem 4.3.1.

4.4. Topological divisor of zero in a metric I-para-semialgebra

Theorem 4.4.1. Let (X, ρ) be a complete metric I-parasemialgebra with I as the S indentity. Let Z and S denote respectively the set of all topological divisors of zero and that of all singular elements of X. Then $Z \subseteq S$, also some boundary points of S may form a subset of Z.

Proof : $Z \subseteq S$ is obvious. To prove the second part, we take $z \in$ bd S. We show that

$z \in Z$. Now S being closed, there exists $\{r_n\} \subset G$ such that $\rho(r_n, z) \to 0$ as $n \to \infty$. We can prove that $p(r_n^{-1}) \to \infty$ as $n \to \infty$.

In fact, $\rho\left(r_n^{-1}z\ ,\ I\right)=\rho\left(r_n^{-1}z\ ,\ r_n^{-1}r_n\right)\le\rho\left(r_n^{-1},\ \theta\right)\rho\left(z\ ,\ r_n\right)$

So, $\rho\left(r_n^{-1}z\ ,\ I\right)\le p\left(r_n^{-1}\right)\rho\left(z\ ,\ r_n\right)\to 0$ as $n\to\infty$ $\left(\text{ if } p\left(r_n^{-1}\right)<\infty\right)$

It may be shown that $\rho\left(r_n^{-1}z\ ,\ I\right)<1$. So $r_n^{-1}z \in G$. Hence $z = r_n\left(r_n^{-1}z\right)\in G$. This is a contradiction.

Therefore $p\left(r_n^{-1}\right)\to\infty$ as $n\to\infty$.

We now write $z = r_n + r_n^0$. Then

$p\left(r_n^0\right)=\rho\left(r_n^0,\ \theta\right)=\rho\left(r_n+r_n^0,\ r_n\right)=\rho\left(z\ ,\ r_n\right)\to 0$ as $n\to\infty$.

Next let $z_n=\dfrac{r_n^{-1}}{p\left(r_n^{-1}\right)}$. Then as in theorem 3.5, it may be shown that

$p\left(z_n\right)\to 0$ as $n\to\infty$, $\left[1<p(z_n)<p\left(r_n^{-1}\right),\ \forall n\ge N \text{ ,say}\right]$

Now, $z\, z_n=\dfrac{z\, r_n^{-1}}{p\left(r_n^{-1}\right)}=\dfrac{\left(r_n+r_n^0\right)r_n^{-1}}{p\left(r_n^{-1}\right)}=\dfrac{I}{p\left(r_n^{-1}\right)}+r_n^0 z_n$.

So $p(z\, z_n) \le p\left(\frac{1}{p(r_n^{\ 1})}\right) + p(r_n^{\ 0})\ p(z_n) = p(\alpha_n\ 1) + p(r_n^{\ 0})\ p(z_n)$

where $\alpha_n = \frac{1}{p(r_n^{\ 1})}$

As $\alpha_n \to 0$ where $n \to \infty$, so $p(\alpha_n\ 1) \to p(0\ 1) = 0$ as $n \to \infty$. Also $p(r_n^{\ 0}) \to 0$ as $n \to \infty$, if $p(z_n) \to \infty$ Then $p(z\, z_n) \to 0$ as $n \to \infty$. Hence $z z_n \to 0$ as $n \to \infty$. Thus for those $z \in \text{bd } S$, for which corresponding $z_n \to \infty$,

we see that $z \in Z$. This completes the proof.

CONCLUDING REMARKS :

1. Regular and singular elements (in general) can be studied on metric paraalgebra (paranormed algebra) and on metric para-semialgebra. Even characterization of radicals can be obtained in such cases.

2. Special type of singular elements (topological divisors of zero) can be studied only on a metric l-paraalgebra and on a metric l-para-semialgebra. For a Banach algebra, the definition of topological divisors of zero is more restricted compaired to that given for a metric l-para-algebra and l-para-semialgebra. For a Banach algebra topological divisors of zero are always permanent singular elements, but no such definite conclusion can be drawn for each singular element of a l-paranormed algebra and a l-paranormed semialgebra.

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