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ANTI-HOLOMORPHIC REFLECTIONS

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ABSTRACT

We treat anti-holomorphic and Ω-reversing reflections with respect to submanifolds in an almost Hermitian manifold (M, g, J) and investigate the relation with isometric reflections when (M, g, J) is a Kähler or a locally Hermitian symmetric space.

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1. INTRODUCTION

Local geodesic symmetries (that is, reflections with respect to a point) and local reflections with respect to a submanifold in a Riemannian or pseudo-Riemannian manifold have been studied intensively and they play an important role at several places. For manifolds or submanifolds of some particular kind, these local diffeomorphisms have some special properties and these properties may in turn be used to characterize some special types of ambient spaces or submanifolds. Isometric reflections are the most well-known examples but also volume-preserving (up to sign) and harmonic reflections have been considered. Moreover, when the ambient space is an almost Hermitian manifold, a Kähler manifold or a Hermitian symmetric space, one has considered holomorphic and symplectic reflections and their relation with isometric ones. We refer to [3], [4], [5], [9], [10], [11], [12], [14], [16], [17], [18], [19] for a collection of results and for further references. Next, we refer to [2], [6], [13], [15], and the included reference lists, where reflections with respect to curves have been used to define φ-symmetric spaces, Killing-transversally symmetric spaces and transversally symmetric

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liations. Finally, we mention [1], [7] for examples where isometric reflections are used to nstruct interesting new examples of a particular class of Riemannian manifolds (namely, eakly symmetric spaces).

In this note we continue our research about reflections in the framework of (almost) comex geometry and concentrate on anti-holomorphic and Ω-reversing reflections with respect submanifolds. We derive some general results and investigate their mutual relation and e connection with isometric reflections when the ambient space is Kählerian or locally isoetric to a Hermitian symmetric space. As we will see and as is already known, totally real bmanifolds of maximal dimension play a crucial role in this context.

The method to derive the results uses Jacobi vector fields and power series expansions. e collect the needed material in Section 2. The main results are derived in Section 3.

2. PRELIMINARIES

We start by recalling some basic facts and refer to [8], [19] for more details and references.

Let (M, g) be a smooth, n-dimensional Riemannian manifold and let P be a connected, relatively compact, topologically embedded submanifold of dimension q. All data are supposed to be analytic where this is needed. Denote by $T^{\perp}P$ the normal bundle of P and by $\exp_P$ the exponential map of this bundle, that is, $\exp_P(m, v) = \exp_m v$ for all $m \in P$ and all $v \in T^{\perp}_m P$. The set $\mathcal{T}_P(s)$ defined by

$$\mathcal{T}_P(s) = \left\{ \exp_P(m, v) \mid v \in T^{\perp}P, \ , \ \|v\| < s \ , \ m \in P \right\}$$

where s is supposed to be smaller than the distance from P to its nearest focal point, is said to be the tubular neighborhood of radius s around P. Now, the mapping φ_P on $\mathcal{T}_P(s)$, defined by

$$\varphi_P : p = \exp_P(m, v) \mapsto \varphi_P(p) = \exp_P(m, -v)$$

for all $m \in P$ and all $v \in T^{\perp}_m P$ such that $\|v\| < s$, is an involutive local diffeomorphism of M. P belongs to its fixed point set. This φ_P is called a (*local*) *reflection* with respect to P.

To describe this map analytically we shall use Fermi coordinates which we introduce now. Let $\{E_1, \ldots, E_n\}$ be a local orthonormal frame field of (M, g) defined along P in a neighborhood of $m \in P$ such that $E_1, \ldots, E_q$ are tangent to P and $E_{q+1}, \ldots, E_n$ are normal to P. Next, let $(y^1, \ldots, y^q)$ be a coordinate system in a neighborhood of m in P for which

$$\frac{\partial}{\partial y^i}(m) = E_i(m)\ , \quad i = 1, \ldots, q.$$

Since every point p in $\mathcal{T}_P(s)$ can be expressed in a unique way as

$$p = \exp_b\Big(\sum_{\alpha=q+1}^{n} t_\alpha E_\alpha\Big)$$

for some $b \in P$, we put

$$\begin{aligned} x^i\Big(\exp_b\Big(\sum_{\alpha=q+1}^{n} t_\alpha E_\alpha\Big)\Big) &= y^i(b), \quad i = 1, \ldots, q, \\ x^a\Big(\exp_b\Big(\sum_{\alpha=q+1}^{n} t_\alpha E_\alpha\Big)\Big) &= t_a, \quad a = q+1, \ldots, n. \end{aligned}$$

Then $(x^1, \ldots, x^n)$ is a coordinate system on $\mathcal{T}_P(s)$, called a *Fermi coordinate system* (relative to m, $(y^1, \ldots, y^q)$ and $(E_{q+1}, \ldots, E_n)$). With respect to such a Fermi coordinate system, the reflection φ_P takes the following (local) form:

$$\varphi_P : (x^1, \ldots, x^q, x^{q+1}, \ldots, x^n) \mapsto (x^1, \ldots, x^q, -x^{q+1}, \ldots, -x^n).$$

Further, there exists a strong relation between the basic vector fields $\frac{\partial}{\partial x^\alpha}$ of the Fermi coordinate system and some special Jacobi vector fields along geodesics through m in M. To describe this relation, we choose a fixed unit normal vector u at m, $u \in T_m^\perp P \subset T_m M$, and consider the geodesic $\gamma(t) = \exp_m(tu)$. Further, we adapt the frame field $(E_1, \ldots, E_n)$ such that $E_n(m) = u = \gamma'(0)$. Finally, we denote by Y_α the Jacobi vector fields along γ satisfying the following initial conditions:

$$\begin{aligned} Y_i(0) &= E_i(m)\ , & Y_i'(0) &= \nabla_u \frac{\partial}{\partial x^i}, \\ Y_a(0) &= 0\ , & Y_a'(0) &= E_a(m) \end{aligned}$$

for all $i = 1, \ldots, q$ and $a = q+1, \ldots, n-1$. Here, ∇ denotes the Levi Civita connection of (M, g). Then we have

$$Y_i(t) = \frac{\partial}{\partial x^i}(\gamma(t)), \quad Y_a(t) = t\frac{\partial}{\partial x^a}(\gamma(t)).$$

Next, let $(F_1, \ldots, F_n)$ be the frame field along γ obtained by parallel translation of $\{E_1, \ldots, E_n\}$ and define the endomorphism-valued function $t \mapsto D_u(t)$ by

$$Y_\alpha(t) = D_u(t) F_\alpha, \quad \alpha = 1, \ldots, n-1.$$

Then this function satisfies the Jacobi equation

$$D_u'' + R \circ D_u = 0$$

where $t \mapsto R(t)$ is the endomorphism-valued function on $\{\gamma'(t)\}^{\perp} \subset T_{\gamma(t)}M$ defined by

$$R(t)x = R_{\gamma'(t)x}\gamma'(t), \quad x \in \{\gamma'(t)\}^{\perp}.$$

Here, R is the Riemannian curvature tensor taken with the sign convention

$$R_{UV} = \nabla_{[U,V]} - [\nabla_U, \nabla_V]$$

for all smooth vector fields U, V. Further we put $R_{XYZW} = R(X,Y,Z,W) = g(R_{XY}Z, W)$.

The initial conditions for D are given by

$$D(0) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad D'(0) = \begin{pmatrix} T(u) & 0 \\ -{}^t\bot(u) & I \end{pmatrix}$$

where T and $\bot$ are defined, via the Levi Civita connection $\tilde{\nabla}$ of P, by

$$\begin{aligned} \nabla_X Y &= \tilde{\nabla}_X Y + T_X Y, \\ \nabla_X N &= T(N)X + \bot_X N \end{aligned}$$

for all smooth X, Y tangent to P and all smooth N normal to P, and

$$\begin{aligned} T(u)_{ij} &= g(T(u)E_i, E_j)(m), \\ \bot(u)_{ia} &= g(\bot_{E_i} E_a, E_n)(m). \end{aligned}$$

$T_X Y = T(X,Y)$ is the second fundamental form operator of P and $T(N)$ is the shape operator of P with respect to N. They are related by $g(T(N)X,Y) = -g(T(X,Y),N)$. Further, $\bot_X N = \nabla_X^{\bot} N$ where $\nabla^{\bot}$ is the normal connection along P.

Using the initial conditions for $D_u(t)$, one obtains the following useful power series expansions:

$$(1) \qquad \begin{cases} D_u(t)F_i = E_i(m) + t(TE_i - {}^t\bot E_i)(m) - \frac{t^2}{2}(RE_i)(m) + O(t^3), \\ D_u(t)F_a = tE_a(m) - \frac{t^3}{6}(RE_a)(m) + O(t^4), \end{cases}$$

for $i = 1, \ldots, q$ and $a = q+1, \ldots, n-1$.

We finish this section with a criterion for isometric reflections with respect to P.

Theorem 2.1. [3] *Let (M, g) be a Riemannian manifold and P a submanifold. Then the reflection φ_P is an isometry if and only if*

(i) *P is totally geodesic;*

(ii) *$(\nabla^{2k}_{u\ldots u}R)_{uv}u$ is normal to P,*

$(\nabla^{2k+1}_{u\ldots u}R)_{uv}u$ is tangent to P and

$(\nabla^{2k+1}_{u\ldots u}R)_{ux}u$ is normal to P

for all normal vectors u, v of P, any tangent vector x of P and all $k \in \mathbb{N}$.

Then we get at once:

Corollary 2.1. [3] *Let (M, g) be a locally symmetric Riemannian manifold and P a submanifold. Then the reflection φ_P is an isometry if and only if*

(i) *P is totally geodesic;*

(ii) *$R_{uv}u$ is normal to P for all $u, v \in T^{\perp}P$.*

3. ANTI-HOLOMORPHIC AND Ω-REVERSING REFLECTIONS

Now, we turn to the main contents of this note. So, let (M, g, J) be an almost Hermitian manifold and P a submanifold. Then the reflection φ_P is said to be *anti-holomorphic* (or *J-reversing*) if

(2)
$$\varphi_{P*} \circ J = -J \circ \varphi_{P*}$$

and *Ω-reversing* if

(3)
$$\varphi_P^* \Omega = -\Omega$$

where Ω denotes the Kähler form on (M, g, J) defined by $\Omega(X, Y) = g(X, JY)$ for all vectors X, Y tangent to M. Further, P is called a *totally real* or *anti-invariant* submanifold of (M, g, J) if $JT_mP \subset T_m^{\perp}P$ for all $m \in P$ [20].

First, we prove

Theorem 3.1. *Let (M, g, J) be an almost Hermitian manifold, P a submanifold and suppose that the reflection φ_P is anti-holomorphic or Ω-reversing. Then P is totally geodesic and totally real with $2\dim P = \dim M$.*

Proof. First, let φ_P be Ω-reversing. Then, for arbitrary tangent X, Y on M along P we have from (3)

$$g(\varphi_{P*}X, J\varphi_{P*}Y) = -g(X, JY).$$

Further, if X, Y are both tangent or both normal to P, we also have

$$g(\varphi_{P*}X, J\varphi_{P*}Y) = g(X, JY)$$

and hence, $g(X, JY) = 0$. This implies that P is totally real and $2\dim P = \dim M$.

Next, put

$$\Omega_{\alpha\beta} = \Omega\Big(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}\Big), \quad \alpha, \beta = 1, \ldots, n.$$

Then (3) yields

$$\Omega_{ia}(\varphi_P(p)) = \Omega_{ia}(p)\ , \quad \Omega_{in}(\varphi_P(p)) = \Omega_{in}(p) \tag{4}$$

for all $i = 1, \ldots, q$, $a = q+1, \ldots, n-1$ and $p = \exp_m(tu)$, $u \in T_m^\perp P$, $\|u\| = 1$. Using the formulas from Section 2 we further have

$$\begin{aligned}\Omega_{ia}(p) &= \frac{1}{t}g(D_u(t)F_i, JD_u(t)F_a),\\ \Omega_{in}(p) &= g(D_u(t)F_i, Ju).\end{aligned}$$

Using (1) and taking into account that P is totally real, we obtain

$$\begin{aligned}\Omega_{ia}(p) &= g(E_i, JE_a)(m) + t\Big\{g(TE_i, JE_a) + g(E_i, J'E_a)\Big\}(m) + O(t^2),\\ \Omega_{in}(p) &= g(E_i, Ju)(m) + t\Big\{g(TE_i, Ju) + g(E_i, J'u)\Big\}(m) + O(t^2)\end{aligned}$$

where $T = T(u)$. Then (4) yields

$$\begin{aligned}g(TE_i, JE_a) + g(E_i, J'E_a) &= 0,\\ g(TE_i, Ju) + g(E_i, J'u) &= 0.\end{aligned}$$

So, we have $g(TX, JN) = -g(X, J'N)$ for all vectors X tangent to P and all normal vectors N. Now, put $Y = JN$. Then we have

$$g(T(X,Y), u) = -g(X, J'JY)$$

and since the right-hand side is skew-symmetric in X, Y, it follows that $T = 0$ and hence, P is totally geodesic.

Now, we consider the case where φ_P is anti-holomorphic. Then (2) implies

$$\varphi_{P*}JX = -J\varphi_{P*}X = -JX$$

for all X tangent to P. Hence, JX is normal. Similarly, we obtain $JT^{\perp}P \subset TP$. So, P is totally real and $2\dim P = \dim M$.

To prove that P is totally geodesic we first note that the components J_α^β of J with respect to the Fermi coordinates satisfy

$$J_\alpha^\beta = -\Omega_{\alpha\gamma}g^{\gamma\beta} \quad , \quad \alpha, \beta, \gamma = 1, \ldots, n.$$

So, since

$$g^{\alpha n}(p) = 0 \quad , \quad g^{nn}(p) = 1 \quad , \quad \alpha = 1, \ldots, n-1,$$

we get

$$J_i^n = \Omega_{ni} \quad , \quad J_i^a = \Omega_{ki}g^{ka} + \Omega_{bi}g^{ba}.$$

Next, (2) implies

$$\begin{aligned} J_i^n(\varphi_P(p)) &= J_i^n(p), \\ J_i^a(\varphi_P(p)) &= J_i^a(p) \end{aligned}$$

and using the power series expansions for $g^{\alpha\beta}$ and $\Omega_{\alpha\beta}$, we get

$$J_i^a(p) = -g(E_i, JE_a)(m) - t\Big\{g(TE_i, JE_a) + g(E_i, J'E_a)\Big\}(m) + O(t^2)$$

for $a = q+1, \ldots, n$, which yields as before that P is totally geodesic. ∎

Now, we suppose that (M, g, J) is a Kähler manifold (that is, $\nabla J = 0$) and prove

Theorem 3.2. *Let P be a totally real submanifold of a Kähler manifold with $2\dim P = \dim M$. If the reflection φ_P is an isometry, then it is anti-holomorphic or equivalently, Ω-reversing.*

Proof. Since φ_P is an isometry and P belongs to the fixed point set of it, it is a totally geodesic submanifold. In this case, and using the differential equation for D and its initial values, we have

$$D_u^{l+2}(0) = -\sum_{k=0}^{l} C_l^k R^{(l-k)}(0) D_u^k(0).$$

Further, this and the conditions (ii) in Theorem 2.1 yield (see [3])

$$D_u^{(2l)}(0)v \text{ is tangent}, \qquad D_u^{(2l)}(0)x \text{ is tangent},$$
$$D_u^{(2l+1)}(0)v \text{ is normal}, \qquad D_u^{(2l+1)}(0)x \text{ is normal}$$

for $v \in T_m^\perp P$, $x \in T_m P$. Hence, we get

$$\begin{aligned} D_u(t)F_i &= \alpha_i(t) + \beta_i(t), \\ \frac{1}{t} D_u(t)F_a &= \alpha_a(t) + \beta_a(t) \end{aligned}$$

where α_i, α_a are tangent and β_i, β_a normal along P. (Here, we used the identification of the spaces $\{\gamma'(t)\}^\perp$ via parallel translation.) Moreover, α_i, β_a are even functions of t and β_i, α_a are odd functions of t. Since (M, g, J) is Kählerian and P totally real, we have

$$\begin{aligned} \Omega_{ij}(p) &= g(\alpha_i(t), J\beta_j(t)) + g(\beta_i(t), J\alpha_j(t)), \\ \Omega_{ab}(p) &= g(\alpha_a(t), J\beta_b(t)) + g(\beta_a(t), J\alpha_b(t)), \\ \Omega_{ia}(p) &= g(\alpha_i(t), J\beta_a(t)) + g(\beta_i(t), J\alpha_a(t)), \\ \Omega_{in}(p) &= g(\alpha_i(t), Ju), \\ \Omega_{an}(p) &= g(\alpha_a(t), Ju) \end{aligned}$$

and hence,

$$\begin{aligned} \Omega_{ij}(\varphi_P(p)) &= -\Omega_{ij}(p), & \Omega_{ia}(\varphi_P(p)) &= \Omega_{ia}(p), \\ \Omega_{in}(\varphi_P(p)) &= \Omega_{in}(p), & \Omega_{an}(\varphi_P(p)) &= -\Omega_{an}(p), \\ \Omega_{ab}(\varphi_P(p)) &= -\Omega_{ab}(p). \end{aligned}$$

This expresses that φ_P is Ω-reversing. ■

Remark 3.1. Using the same technique as in the proof of Theorem 3.2 one can also prove a corresponding result for holomorphic submanifolds: *Let P be a holomorphic submanifold in a Kähler manifold such that φ_P is isometric. Then φ_P is holomorphic or equivalently, symplectic.* This result extends the similar one obtained in [3] for locally symmetric Kähler manifolds.

To prove our next result, we consider

Lemma 3.1. *Let P be a submanifold of a Kähler manifold (M, g, J) such that the reflection φ_P is anti-holomorphic or Ω-reversing. Then R_{uvu} is normal to P for all $u, v \in T^{\perp}P$.*

Proof. First, let φ_P be Ω-reversing. Then (3) implies

$$\Omega_{an}(\varphi_P(p)) = -\Omega_{an}(p) \tag{5}$$

where $a = q+1, \ldots, n-1$. Since P is totally real and $2\dim P = \dim M$ we have

$$\Omega_{an}(p) = tg(E_a, J'u)(m) - \frac{t^2}{6}g(RE_a, Ju)(m) + O(t^3).$$

So, this and (5) imply

$$R_{uvuJu} = 0 \tag{6}$$

for all normal vectors u, v. Now, put $u = \alpha w + \beta z$ in (6) for arbitrary $\alpha, \beta \in \mathbb{R}$ and for arbitrary normal vectors w, z. Using the first Bianchi identity and the Kähler identity $R_{xyJzJw} = R_{xyzw}$, we then get by considering the coefficient of $\alpha^2\beta$:

$$3R_{wvwJz} - R_{wzvJw} = 0. \tag{7}$$

Interchanging v and z in (7) yields

$$3R_{wzvJw} - R_{wvwJz} = 0 \tag{8}$$

and so, from (7) and (8), we get $R_{wvwJz} = 0$ or equivalently, R_{wvw} is normal to P along P.

Finally, if φ_P is anti-holomorphic, a same procedure and $J_a^n = \Omega_{na}$, $J_a^n(\varphi_P(p)) = -J_a^n(p)$, yields the required result. ■

From this Lemma 3.1, Corollary 2.1, Theorem 3.1 and Theorem 3.2 we now derive at once

Theorem 3.3. *Let P be a totally real submanifold of a locally Hermitian symmetric space such that $2\dim P = \dim M$. Then the following statements are equivalent:*

(i) *φ_P is an isometry;*

(ii) *φ_P is anti-holomorphic;*

(iii) *φ_P is Ω-reversing.*

Corollary 3.1. *Let P be a submanifold of a locally Hermitian symmetric space (M, g, J). Then the reflection φ_P is anti-holomorphic if and only if it is Ω-reversing.*

Finally, using [3, Corollary 4] we have

Corollary 3.2. *Let (M, g, J) be a Kähler manifold of constant holomorphic sectional curvature $c \neq 0$. Then φ_P is anti-holomorphic if and only if P is a totally geodesic and totally real submanifold with $2 \dim P = \dim M$.*

Remark 3.2. For more information about the existence of fixed point sets of anti-holomorphic involutions (that is, *real forms*) in Hermitian symmetric spaces and for further references, we refer to [1]. There one also finds references concerning the theory of *locally* and *globally reflective* submanifolds P in M, that is, submanifolds P such that the reflection φ_P is a well-defined local or global isometry with P as fixed point set.

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