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# THE $D_1A_1$ SYMMETRY ON THE CLASSICAL ELECTROMAGNETIC GAUGE. APPLICATION TO SPHERICAL AND STATIC BLACK HOLES

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**Abstract:** A gauge to electromagnetic field is considered. This gauge is most restrictive as the Lorentz gauge which is verified by the proposed. It is based on several symmetric expressions named $D_mSA_n$ forms. It is included a application to a spherical and static geometry as is generated by a big gravitational object, as a black hole, with central symmetry and electrically charged. The solution is the classical on far space but has strong discrepancy with the classical result near of the event horizon of the black hole.

**Key words:** General Relativity, Field Theory, Gravitation, Black Hole.

## 1. INTRODUCTION

On classical electromagnetic field theory in 4-dimensional riemannian spaces, [4] [6], the potential vector **A** is defined from the electromagnetic field **F** as:

$$F_{ij}=A_{j;i}-A_{i;j}$$

From this definition can be concluded that the potential is not well defined. Several potential transformations with field invariance are possible. A transformation of this type, called $\lambda$-invariance, is valid:

$$\boldsymbol{A}'=\boldsymbol{A}+\nabla\lambda$$

These invariances are due to the presence of several freedom degrees in the potential definition. To reduce some freedom degrees some gauge conditions are imposed. The most common is know as Lorentz gauge, which is useful but do not fix the potential. In this paper all functions are assumed analytic, based in this, a identification of these freedom degrees is show. Let a local frame with cartesian coordinates. This frame is locally plane and the Chistoffel symbols are nulls. Let the electromagnetic field **F** and potential **A** as:

$$F_{ij}(x) = \hat{a}_{ij} + \hat{b}_{ijk}x^k + O(x^2)$$
$$A_i(x) = \alpha_i + \beta_{ij}x^j + \frac{1}{2}\gamma_{ijk}x^j x^k + O(x^3)$$

Based on the antisymmetric property of the electromagnetic field, its coefficients must verifie:

$$\hat{a}_{ij} + \hat{a}_{ji} = 0 \qquad \hat{b}_{ijk} + \hat{b}_{jik} = 0$$

Then, must be:

$$\beta_{ij} - \beta_{ji} = \hat{a}_{ji} \qquad \gamma_{ijk} - \gamma_{jik} = \hat{b}_{jik}$$

Let the following symmetric-antisymmetric decomposition:

$$\beta_{ij} = \bar{\beta}_{ij} + \hat{\beta}_{ij} \qquad \bar{\beta}_{ij} = \bar{\beta}_{ji} \qquad \hat{\beta}_{ij} + \hat{\beta}_{ji} = 0$$

$$\gamma_{ijk} = \bar{\gamma}_{ijk} + \hat{\gamma}_{ijk} \qquad \bar{\gamma}_{ijk} = \bar{\gamma}_{jik} \qquad \hat{\gamma}_{ijk} + \hat{\gamma}_{jik} = 0$$

Then must be:

$$\hat{\beta}_{ij} = \frac{1}{2}\hat{a}_{ji} \qquad \hat{\gamma}_{ijk} = \frac{1}{2}\hat{b}_{jik}$$

And the potential is expressed as:

$$A_i = \alpha_i + \{ \frac{1}{2}\hat{a}_{ji}x^j + \frac{1}{4}\hat{b}_{jik}x^j x^k + O(x^3) \} + \{ \bar{\beta}_{ij}x^j + \bar{\gamma}_{ijk}x^j x^k + O(x^3) \}$$

The potential in this local frame has three terms. The first is an arbitrary constant, the second is a term well correlated with the field coefficients, and the third, which has symmetric coefficients, can be arbitrarily defined. From this is clear that the antisymmetric expression $A_{i;j} - A_{j;i}$ is well defined, but the symmetric expression $A_{i;j} + A_{j;i}$ which do not includes the field coefficients, can be arbitrarily fixed:

$$A_{i;j} + A_{j;i} = \bar{\beta}_{ij} + \bar{\gamma}_{ijk}x^k + O(x^2)$$

This symmetric expression has 10 components which are called in this paper the freedom degrees of the potential definition. The Lorentz gauge partially defines these components. This gauge fix the addition of the components in this way:

$$\frac{1}{2}g^{ij}(A_{i;j} + A_{j;i}) = A^i{}_{;i} = 0$$

This work is related to a gauge based on the previous symmetric expression.

## 2. THE $SA_n$ Y $D_mSA_n$ SYMMETRIES

The $SA_n$ tensor symmetries are a generalization of symmetric and antisymmetric tensors. Let all tensors on $C^\infty$.

**Definition 2.1:** A Tensor, $X_{i1\ i2\ \dots\ in}$ , has the $S_n$ symmetry if all its components in a index cyclic rotation are equal:

$$X_{i1\ i2...in}=X_{i2...in\ i1}=...=X_{in\ i1...in-1}$$

**Definition 2.2:** A Tensor, $X_{i1\ i2\ \dots\ in}$ , has the $A_n$ symmetry if the addition of all its components in a index cyclic rotation is null:

$$X_{i1\ i2...in}+X_{i2...in\ i1}+...+X_{in\ i1...in-1}=0$$

From these definitions is founded that the second degree symmetric and antisymmetric tensors have the $S_2$ and $A_2$ symmetries respectively.

**Definition 2.3:** The expression of an arbitrary tensor **X** as the sum of $S_n$ and $A_n$ tensors is called its $SA_n$ Form:

$$X_{i1\ i2...in}=S_{i1\ i2...in}+A_{i1\ i2...in}$$

Where **S** and **A** are $S_n$ and $A_n$ tensors, and are the $S_n$ y $A_n$ parts of **X** respectively.

**Theorem 2.1:** The parts of a $SA_n$ form are univócaly determined.

**Proof:** Let **X** an arbitrary tensor where **S** and **A** are its $S_n$ and $A_n$ parts, as done on Definition 2.3. Adding a cyclic index rotation:

$$X_{i1\ i2...in}+X_{i2...in\ i1}+...+X_{in\ i1...in-1}=(S_{i1\ i2...in}+S_{i2...in\ i1}+...+S_{in\ i1...in-1})+(A_{i1\ i2...in}+A_{i2...in\ i1}+...+A_{in\ i1...in-1})=nS_{i1\ i2...in}$$

Can be obtained that:

$$S_{i1\ i2...in}=\frac{X_{i1\ i2...in}+X_{i2...in\ i1}+...+X_{in\ i1...in-1}}{n}$$

$$A_{i1\ i2...in}=X_{i1\ i2...in}-\frac{X_{i1\ i2...in}+X_{i2...in\ i1}+...+X_{in\ i1...in-1}}{n}$$

The parts are well defined from **X**. ■

**Definition 2.4:** An arbitrary tensor has the $D_mS_n$ symmetry, if its m-th derivate tensor has the $S_{m+n}$ symmetry.

**Definition 2.5:** An arbitrary tensor has the $D_mA_n$ symmetry, if its m-th derivate tensor has the $A_{m+n}$ symmetry.

**Definition 2.6:** The $D_mSA_n$ Form of an arbitrary tensor is its expression as sum of $D_mS_n$ y $D_mA_n$ tensors.

**Theorem 2.2:** All tensors with the $D_1S_1$ symmetry can be expressed as a scalar gradient. That is, if is verified:

$$X_{i;j}=X_{j;i}$$

Then:

$$X_i=\partial_i S(x^k)$$

**Proof:** The $D_1S_1$ definition is exactly the integrability condition for the next expression, [5], a particular case of Poincare Lemma, [2] [3] :

$$dS(x^k)=X_i dx^i \quad \blacksquare$$

**Theorem 2.3:** All tensors with the $D_1A_1$ symmetry have null divergence. That is, if is verified:

$$X_{i;j}+X_{j;i}=0$$

Then:

$$X^i_{\ ;i}=0$$

**Proof:** Reducing the i and j index in the $D_1A_1$ definition:

$$g^{ij}(X_{i;j}+X_{j;i})=2X^i_{\ ;i}=0 \quad \blacksquare$$

**Lemma 2.1:** If a tensor $X_i$ and its $D_1SA_1$ parts are analytics, and is known one of the its $D_1A_1$ or $D_1S_1$ parts in a point, then the $D_1SA_1$ parts are well defined.

**Proof:** Let a local frame centered at point p where a part is known. Then:

$$\begin{gathered} X_i=\bar{X}_i+\hat{X}_i \\ \partial_j\bar{X}_i=\partial_i\bar{X}_j \\ \partial_j\hat{X}_i+\partial_i\hat{X}_j=0 \\ \hat{X}_i(p)=Z_i \end{gathered}$$

In this case is known the $D_1A_1$ part. Let:

$$\begin{gathered} X_i(x)=a_i+b_{ij}x^j+\frac{1}{2}c_{ijk}x^jx^k+O(x^3) \\ \bar{X}_i(x)=\bar{a}_i+\bar{b}_{ij}x^j+\frac{1}{2}\bar{c}_{ijk}x^jx^k+O(x^3) \\ \hat{X}_i(x)=Z_i+\hat{b}_{ij}x^j+\frac{1}{2}\hat{c}_{ijk}x^jx^k+O(x^3) \end{gathered}$$

Then, must be:

$$\begin{gathered} \bar{b}_{ij}=\bar{b}_{ji} \qquad \bar{c}_{ijk}=\bar{c}_{jik} \\ \hat{b}_{ij}+\hat{b}_{ji}=0 \qquad \hat{c}_{ijk}+\hat{c}_{jik}=0 \\ a_i=\bar{a}_i+Z_i \qquad b_{ij}=\bar{b}_{ij}+\hat{b}_{ij} \qquad c_{ijk}=\bar{c}_{ijk}+\hat{c}_{ijk} \end{gathered}$$

And the solution is:

$$\begin{gathered} \bar{X}_i(x)=(a_i-Z_i)+\frac{1}{2}(b_{ij}+b_{ji})x^j+\frac{1}{4}(c_{ijk}+c_{jik})x^jx^k+O(x^3) \\ \hat{X}_i(x)=Z_i+\frac{1}{2}(b_{ij}-b_{ji})x^j+\frac{1}{4}(c_{ijk}-c_{jik})x^jx^k+O(x^3) \end{gathered}$$

All the parts are well defined. $\blacksquare$

## 3. THE $D_mSA_n$ PARTS OF ELECTROMAGNETIC FIELD. KERNEL POTENTIAL

The electromagnetic field **F** has the $A_2$ and $D_1A_2$ symmetries. It is verified that:

$$F_{ij}+F_{ji}=0$$
$$F_{ij;k}+F_{jk;i}+F_{ki;j}=0$$

The electromagnetic potential **A** has not these symmetries. Let the $D_1SA_1$ form of **A** as:

$$\boldsymbol{A}=\nabla S+\hat{\boldsymbol{A}}$$

Where S is a scalar and **Â** is a $D_1A_1$ tensor called in this paper the *kernel potential*. If in a physical problem the potential has solution, then, based on the Lemma 2.1 and some contour conditions the kernel potential has also solution. The primitive relation between **F** and **A** is transformed on a relation between **F** and **Â** as follow:

$$F_{ij}=A_{j;i}-A_{i;j}=\hat{A}_{j;i}-\hat{A}_{i;j} \qquad [3.1]$$
$$\hat{A}_{i;j}+\hat{A}_{j;i}=\partial_j\hat{A}_i+\partial_i\hat{A}_j-2\Gamma^k{}_{ij}\hat{A}_k=0 \qquad [3.2]$$

In the previously considered local frame, the kernel potential and scalar S are:

$$\hat{A}_i(x)=\hat{\alpha}_i+\{\ \frac{1}{2}\hat{a}_{ji}x^j+\frac{1}{4}\hat{b}_{jik}x^jx^k+O(x^3)\ \} \qquad S(x)=\delta+\bar{\alpha}_ix^i+\{\ \bar{\beta}_{ij}x^ix^j+\bar{\gamma}_{ijk}x^jx^kx^k+O(x^4)\ \} \qquad \alpha_i=\bar{\alpha}_i+\hat{\alpha}_i$$

**Definition 3.1:** In this context, the freedom degrees of a tensor $X_i$ is the number of $\partial_iX_j$ components which can be arbitrarily defined.

**Theorem 3.1:** The maximum freedom degrees of a tensor with $D_1A_1$ symmetry is 6.

**Proof:** The number of $\partial_iX_j$ is 16, but the $D_1A_1$ symmetry definition provides 10 linear equations about $\partial_iX_j$. Then is concluded that only 6 components can be arbitrarily defined. ■

**Theorem 3.2:** If the electromagnetic field **F** is known, then the kernel potential has not freedom degrees, that is, all the $\partial_j\hat{A}_i$ are fixed.

**Proof:** The linear equations system [3.1] and [3.2] has 6 equations from the field definition and 10 equations from the $D_1A_1$ definition. Then, there are 16 linearly independent equations about the 16 $\partial_j\hat{A}_i$. ■

**Theorem 3.3:** If in a Space the Cristoffel symbols are independents from a coordinate, $x^A$, then the freedom degrees of tensor **X** with $D_1A_1$ symmetry is reduced to 2.

**Proof:** If the Cristoffel symbols are independents of a coordinate, $x^A$, then the lineal equations system [3.2] has this solution:

$$X_i(x^j)=\exp(\gamma x^A)Y_i(\gamma,x^k) \qquad k\neq A$$

It is possible to define 4 additional equations related to the $\partial_jX_i$ as:

$$\partial_A X_i = \gamma X_i$$

The more general solution is:

$$X_i(x^j) = \sum_u \exp(\gamma_u x^A) Y_i(\gamma_u, x^k) \qquad k \neq A$$

Where $\gamma_u$ are the eigenvalues. Based on Theorem 3.1 an joining these 4 additional equations, then, only is possible to arbitrarily define 2 components $\partial_j X_i$ . ■

**Theorem 3.4:** If in a Space the Cristoffel symbols are independents from a coordinate, $x^A$, then only 2 components of the electromagnetic field can be arbitrarily defined.

**Proof:** In this case the freedom degrees of the kernel potential is reduced to 2, due to Theorem 3.3, and only can be defined 2 equations including the electromagnetic field, because any more equations can be generate an incompatible system.■

**Theorem 3.5:** If in a Space the Cristoffel symbols are independents from two coordinates, $x^A$ and $x^B$, then can not be defined any arbitrary component of the electromagnetic field.

**Proof:** In this case, there are 10 equations about the $D_1A_1$ symmetry definition and 8 equations about the independent coordinates. If this system with 18 equations is incompatible then a null solution is provided to the kernel potential and electromagnetic field. If the system is compatible, then a solution is obtained for the electromagnetic field, and this solution is the only solution. Any more compatible equations are lineal combinations of the previous, and the solutions are the same. In this case the electromagnetic field is an intrinsic property of the space. ■

Based on these considerations is proposed this Lemma:

**Lemma 3.1:** *If an electromagnetic field F is physically possible in a space, then there is solution for its kernel potential $\hat{\mathbf{A}}$. That is, there is solution to the equations system [3.1][3.2].*

This Lemma is called the $D_1A_1$ gauge in this paper.

## 4. ELECTROMAGNETIC FIELD BASED ON $D_1A_1$ GAUGE

A physical case with two independent coordinates exist in a spherical, static and charged black hole. The solution to a $D_1A_1$ tensor for this case is obtained in the Appendix.

**Theorem 4.1:** In a Spherical and Static Geometry asymptotically plane, the asymptotically null solution to electromagnetic field based on the $D_1A_1$ gauge is:

$$F_{10}=\frac{d}{dr}[K_0e^{2\Phi(r)}]$$

The metric is [6]:

$$ds^2=-e^{2\Phi(r)}dt^2+e^{2\Lambda(r)}dr^2+r^2(d\theta^2+sen^2\theta d\varphi^2)$$

**Proof:** Based on the Appendix results, if the metric is asymptotically plane must be:

$$\lim_{r\to\infty}\Phi(r)=\lim_{r\to\infty}\Lambda(r)=0$$

Then:

$$\lim_{r\to\infty}\frac{d\Phi(r)}{dr}=\lim_{r\to\infty}\frac{d\Lambda(r)}{dr}=0$$

If the next condition is imposed:

$$\lim_{r\to\infty}\boldsymbol{F}=0$$

Then must be $K_2=K_3=0$, and the solution obtained is:

$$A_0(r)=K_0e^{2\Phi(r)} \qquad A_\alpha=0 \qquad \alpha=1,2,3$$

From which is obtained the Theorem result. ■

The results provides by the classical electromagnetic field theory to the metric and field for a spherical and static geometry is known as the Reissner-Nordstrøm solution [7][8] which is:

$$e^{2\Phi(r)}=e^{-2\Lambda(r)}=1-\frac{r_m}{r}+\frac{r_e^2}{r^2}$$

$$F_{10}=\frac{r_e}{r^2}$$

This result are obtained from the Einstein equation $G_{ij}=kT_{ij}$ , which determines both the metric an field. The work here shown only provides the field if the metric is defined. The obtained solution based on Lemma 3.1 using the metric of classical solution with $K_0=r_e/r_m$ is:

$$F_{10}=K_0\left(\frac{r_m}{r^2}-2\frac{r_e^2}{r^3}\right)=\frac{r_e}{r^2}-\left(2\frac{r_e^3}{r_m}\right)\frac{1}{r^3}$$

The complete solution with the classical metric and the field based on Lemma 3.1 is:

$$g_{00}(r) = \frac{(r-r_+)(r-r_-)}{r^2}$$

$$F_{01}(r) = \frac{r_e}{r^2}\left(1 - \frac{r_s}{r}\right)$$

$$r_\pm = \frac{r_m \pm \sqrt{r_m^2 - 4r_e^2}}{2}$$

$$r_s = \frac{2r_e^2}{r_m} = \frac{2r_- r_+}{r_- + r_+} \qquad r_- \le r_s \le r_+$$

This result has discrepancy with the exact solution provided by the classical theory. The discrepancy is weak, about $O(1/r^3)$, on the far space. Near the horizon of a spherical static and charged black hole the field discrepancy is strong as is shown the Figure 1., where $0 \le 2|r_e| \le r_m$. The solution is similar to a space polarization in a dielectric with a relative permittivitie done as:

$$\epsilon_r(r) = \frac{r}{r - r_s}$$

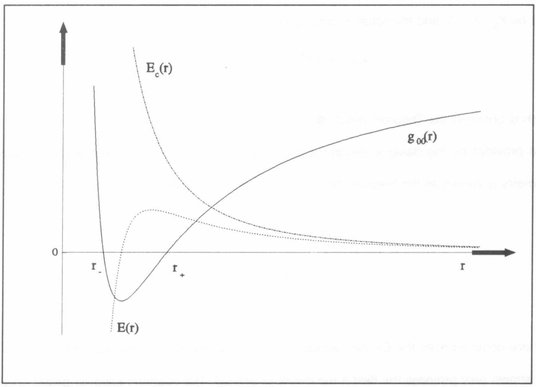


**Figure. 1.** The metric, $g_{00}(r)$, and the field, $E_c(r)$, solutions to Reissner-Nordstrøm geometry. $E(r)$ is the solution based on Lemma 3.1.

## 5. CONCLUSION

A study about the freedom degrees in the computation of the electromagnetic potential in riemannian spaces is presented. A proposal to reduce these freedom degrees is presented, based on several tensor symmetries which define a new gauge type. For several space metrics types, this gauge determines the field from the intrinsic space properties. A study case is resolved, showing similarities and differences with the classsically accepted solution. The study case concerning with a electrically charged black hole can not be successfully resolved based only on the paper thesis, because the presented framework determines, for this case, the field from the space metric, and in order to obtain the resolution of the problem is need to determine also the metric from the field, that is performed by the classical theory.

## 6. APPENDIX: THE $D_1A_1$ TENSORS IN SPHERICAL AND STATIC GEOMETRIES

A solution to the next equations system is provided in a spherical and static 4-dimensional riemannian geometry:

$$X_{i;j}+X_{j;i}=\partial_j X_i+\partial_i X_j-2\Gamma^k{}_{ij}X_k=0 \qquad [6.1]$$

The metric with the $(t,r,\theta,\phi)$ Schwarzschild coordinates is:

$$ds^2=-e^{2\Phi(r)}dt^2+e^{2\Lambda(r)}dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2)$$

Using this (- + + +) metric, the non null Cristoffel symbols are [4][1]:

$$\begin{matrix} \Gamma^1{}_{00}=e^{-2\Phi+2\Lambda}\dfrac{d\Phi}{dr} & \Gamma^0{}_{01}=\dfrac{d\Phi}{dr} & \Gamma^1{}_{11}=\dfrac{d\Lambda}{dr} & \Gamma^2{}_{12}=\Gamma^3{}_{13}=\dfrac{1}{r} \\ \Gamma^1{}_{22}=-re^{-2\Lambda} & \Gamma^3{}_{23}=\cot\theta & \Gamma^1{}_{33}=-e^{-2\Lambda}r\sin^2\theta & \Gamma^2{}_{33}=-\sin\theta\cos\theta \end{matrix} \qquad [6.2]$$

The [6.1] equations system has 10 homogeneous differential equations and the Chistoffel symbols are independents from the $(t,\phi)$ coordinates, then the solutions have this form:

$$X_i(t,r,\theta,\phi)=e^{\alpha t+\beta\phi}Y_i(r,\theta) \qquad [6.3]$$

From [6.1], [6.2] and [6.3] the next equations system is obtained:

$$\alpha Y_0=\Gamma^1{}_{00}(r)Y_1 \qquad [6.4]$$

$$\partial_r Y_0+\alpha Y_1=2\Gamma^0{}_{01}(r)Y_0 \qquad [6.5]$$

$$\partial_\theta Y_0+\alpha Y_2=0 \qquad [6.6]$$

$$\beta Y_0+\alpha Y_3=0 \qquad [6.7]$$

$$\partial_r Y_1=\Gamma^1{}_{11}(r)Y_1 \qquad [6.8]$$

$$\partial_\theta Y_1+\partial_r Y_2=2\Gamma^2{}_{12}(r)Y_2 \qquad [6.9]$$

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[1] Landau-Lifshitz use a metric (+ - - -).

$$\beta Y_1+\partial_r Y_3=2\Gamma^3{}_{13}(r)Y_3 \qquad [6.10]$$

$$\partial_\theta Y_2=\Gamma^1{}_{22}(r)Y_1 \qquad [6.11]$$

$$\beta Y_2+\partial_\theta Y_3=2\cot\theta Y_3 \qquad [6.12]$$

$$\beta Y_3=\Gamma^1{}_{22}(r)\sin^2\theta Y_1-\sin\theta\cos\theta Y_2 \qquad [6.13]$$

From [6.8] is obtained:

$$Y_1(r,\theta)=e^{\Lambda(r)}Z_1(\theta) \qquad [6.14]$$

From [6.11] and [6.14] is deduced:

$$Y_2(r,\theta)=-re^{-\Lambda(r)}Z_2(\theta)+U_2(r) \qquad Z_2'(\theta)=Z_1(\theta) \qquad [6.15]$$

From [6.9] and [6.15] is obtained:

$$e^{\Lambda(r)}Z_2''(\theta)+e^{-\Lambda(r)}(1+r\frac{d\Lambda}{dr})Z_2(\theta)+U_2'(r)=\frac{2}{r}U_2(r)$$

There are two possible solutions based on factoring variables in the previous equation. First solution is obtained when factoring the r variable. It is verified that:

$$\omega^2e^{\Lambda(r)}=e^{-\Lambda(r)}(1+r\frac{d\Lambda}{dr})$$

Where $\omega$ is a constant. This solution is possible only when the metric is restricted to a particular case. The second solution, the only here considered, is obtained when factoring with the $\theta$ variable. It is verified:

$$Z_2''(\theta)=\omega^2Z_2(\theta)=cte$$

The solution for this case must be:

$$Z_2(\theta)=0 \qquad Z_1(\theta)=0 \qquad U_2(r)=K_2r^2$$

Then the partial solution is:

$$Y_1(r,\theta)=0 \qquad Y_2(r,\theta)=K_2r^2 \qquad [6.16]$$

From [6.4], [6.5] and [6.6] is obtained:

Case 1: $\alpha=0$

$$Y_0(r,\theta)=K_0e^{2\Phi(r)}$$

Case 2: $\alpha\neq0$, From [6.4], [6.6], [6.7] and [6.16] is obtained:

$$Y_0(r,\theta)=Y_1(r,\theta)=Y_2(r,\theta)=Y_3(r,\theta)=0$$

There is a solution only if $\alpha=0$. In this case from [6.10] is obtained:

$$Y_3(r,\theta)=r^2Z_3(\theta)+U_3(\theta)$$

From [6.13] is obtained:

$$\beta(r^2 Z_3(\theta)+U_3(\theta)) = -\sin\theta\cos\theta K_2 r^2$$

From the previous equation two cases are considered :

Case 3: $\beta=0$, in this case $K_2=0$. From [6.10], [6.12] and [6.16] is obtained:

$$Y_2(r,\theta)=0 \qquad Y_3(r,\theta)=K_3 r^2\sin^2\theta$$

Case 4: $\beta\neq 0$, must be:

$$U_3(\theta)=0 \qquad \beta Z_3(\theta)=-\sin\theta\cos\theta K_2 \tag{6.17}$$

For this case, from [6.7], [6.12], [6.16] and [6.17] is deduced that:

$$Y_0(r,\theta)=0 \qquad \beta^2=-1$$

The final solutions are:

Solution 1: $\alpha\neq 0$:

$$X_i(t,r,\theta,\phi)=0$$

Solution 2: $\alpha=0$, $\beta\neq 0$, $\beta\neq\pm i$

$$X_i(t,r,\theta,\phi)=0$$

Solution 3: $\alpha=0$, $\beta=\pm i$

$$X_0(t,r,\theta,\phi)=X_1(t,r,\theta,\phi)=0 \qquad X_2(t,r,\theta,\phi)=K_2 r^2 e^{i\phi} \qquad X_3(t,r,\theta,\phi)=K_2 r^2\sin\theta\cos\theta e^{i(\phi+\frac{\pi}{2})}$$

Solution 4: $\alpha=0$, $\beta=0$

$$X_0(t,r,\theta,\phi)=K_0 e^{2\Phi(r)} \qquad X_1(t,r,\theta,\phi)=X_2(t,r,\theta,\phi)=0 \qquad X_3(t,r,\theta,\phi)=K_3 r^2\sin^2\theta$$

The most general solution is:

$$X_0(t,r,\theta,\phi)=K_0 e^{2\Phi(r)}$$

$$X_1(t,r,\theta,\phi)=0$$

$$X_2(t,r,\theta,\phi)=K_2 r^2 e^{i\phi}$$

$$X_3(t,r,\theta,\phi)=K_2 r^2\sin\theta\cos\theta e^{i(\phi+\frac{\pi}{2})}+K_3 r^2\sin^2\theta$$


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