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γ-CONVERGENCE OF NETS AND FILTERS

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Abstract : In this paper we discuss the notion of γ- convergence of nets and filters. We also introduce the concept of γ- compact spaces and characterize such spaces.

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1 Introduction

In [5] Min first introduced the notion of γ- open sets (originally called γ- sets). In the same paper Min has also introduced and discussed the notion of γ- continuous functions and γ- irresolute functions. Later in [1] Ganguly and Sen have introduced the concept of γ- convergence of nets.

The aim of this paper is to introduce γ- convergence of filters and discuss γ- convergence of nets and filters.

Throughout this paper, (X, τ) (simply X) and (Y, τ^*) always mean topological spaces. Let S be a subset of X. The closure (resp. interior) of S will be denoted by $cl(S)$ (resp. $int(S)$).

A subset S of X is called a semi-open set [4] if $S \subseteq cl(int(S))$. The complement of a semi-open set is called a semi-closed set. The family of all semi-open sets in a topological space

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(X,τ) will be denoted by $SO(X)$. A subset $M(x)$ of a space X is called a semi-neighborhood of a point $x \in X$ if there exists a semi-open set S such that $x \in S \subseteq M(x)$. In [2] Latif introduced the notion of semi-convergence of filters. Let $S(x) = \{A \in SO(X) : x \in A\}$ and let $S_x = \{A \subseteq X :$ there exists $\mu \subseteq S(x)$ such that μ is finite and $\cap\mu \subseteq A\}$. Then S_x is called the semi-neighborhood filter at x. For any filter Γ on X we say that Γ semi-converges to x if and only if Γ is finer than the semi-neighborhood filter at x.

Definition 1.1 [3] A subset U of X is called a γ open set if whenever a filter Γ semi-converges to x and $x \in U$, $U \in \Gamma$. The complement of a γ- open set is called a γ- closed set.

The intersection of all γ- closed sets containing A is called the γ- closure of A, denoted by $cl_\gamma(A)$. A subset A is γ- closed iff $A = cl_\gamma(A)$. We denote the family of all γ- open sets of (X,τ) by τ^γ. It is shown in [5] that τ^γ is a topology on X. In a topological space (X,τ), it is always true that $\tau \subseteq S(X) \subseteq \tau^\gamma$.

Definition 1.2 [5] A function $f : X \to Y$ is said to be γ- continuous if the inverse image of every open set of Y is γ- open in X.

The set of all γ- continuous functions from X into Y is denoted by $\gamma C(X,Y)$.

Definition 1.3 [5] A function $f : X \to Y$ is said to be γ- irresolute if the inverse image of every γ- open set of Y is γ- open in X.

Definition 1.4 A net $\{x_\lambda : \lambda \in \Lambda\}$ in X is said to converge to a limit $x \in X$ (in symbol, $x_\lambda \to x$) if for every neighborhood V of x, $\exists$ a $\lambda_0 \in \Lambda$ such that $\lambda \geq \lambda_0$ implies $x_\lambda \in V$.

Definition 1.5 [1] A net $\{x_\lambda : \lambda \in \Lambda\}$ in X is said to γ- converge to a limit $x \in X$ (in symbol, $x_\lambda \to^\gamma x$) if for every γ- open set V containing x, $\exists$ a $\lambda_0 \in \Lambda$ such that $\lambda \geq \lambda_0$ implies $x_\lambda \in V$.

Definition 1.6 Let Γ be a filter on a topological space X. Then Γ is said to converge to x if each neighborhood of x is a member of Γ, i.e., $\mathcal{N}_x \subseteq \Gamma$.

2 γ- convergence of nets and filters

In this article we first discuss the γ- convergence of nets. Then we discuss the γ- convergence of filters.

Definition 2.1 Let X be a topological space. A subset $A \subseteq X$ is said to be a γ- neighborhood of a point $x \in X$ if and only if there exists a γ- open set U such that $x \in U \subseteq A$.

The set of all γ- neighborhoods of a point $x \in X$ is denoted by $\gamma\mathcal{N}_x$.

Theorem 2.2 [1] A function $f : X \to Y$ (where X and Y are topological spaces) is γ-irresolute at a point $x \in X$ iff for any net $\{x_\lambda : \lambda \in \Lambda\}$ in X γ- converging to x, the net $\{f(x_\lambda) : \lambda \in \Lambda\}$ γ- converges to $f(x)$ in Y.

Theorem 2.3 A function $f : X \to Y$ is γ- continuous at a point $x \in X$ iff for any net $\{x_\lambda : \lambda \in \Lambda\}$ in X γ- converging to x, the net $\{f(x_\lambda) : \lambda \in \Lambda\}$ converges to $f(x)$ in Y.

Proof : First assume that f is γ- continuous at $x \in X$. Let $\{x_\lambda : \lambda \in \Lambda\}$ be a net in X γ-converging to x. Let V be an open set in Y containing $f(x)$. Now there exists a γ- open set U containing x in X such that $f(U) \subset V$. Now $\{x_\lambda : \lambda \in \Lambda\}$ γ- converges to x implies that there exists a $\lambda_0 \in \Lambda$ such that $x_\lambda \in U$ for all $\lambda \geq \lambda_0$. Hence, for all $\lambda \geq \lambda_0$, $f(x_\lambda) \in V$. This shows that $\{f(x_\lambda) : \lambda \in \Lambda\}$ lies eventually in V and hence it converges to $f(x)$.

Conversely, let f be not γ- continuous at x. Then there exists an open set W containing $f(x)$ in Y such that from every γ- open set U containing $x \in X$, there exists an element x_U with $f(x_U) \notin W$. Let $\gamma\mathcal{N}_x$ be the γ- neighborhood system at x. So, $\{x_U : U \in \gamma\mathcal{N}_x\}$ is a net in X γ- converging to x, but the net $\{f(x_U) : U \in \gamma\mathcal{N}_x\}$ in Y does not lie eventually in W and consequently it cannot converge to $f(x)$.

Next we form the γ- neighborhood filter at a point $x \in X$ and discuss the notion of γ-convergence of filters.

Definition 2.4 Let (X, τ) be a topological space. For $x \in X$, let $\gamma(x) = \{A \subseteq \tau^\gamma : x \in A\}$. Then $\gamma(x)$ has the finite intersection property. Thus $\gamma(x)$ is a filter subbasis on X. Let γ_x be the filter generated by $\gamma(x)$, i.e., $\gamma_x = \{A \subseteq X :$ there exists $\mu \subseteq \tau^\gamma$ such that μ is finite and $\cap\mu \subseteq A\}$. γ_x will be called the γ- neighborhood filter at x.

Definition 2.5 Let Γ be a filter on (X, τ). Γ is said to γ- converge to $x \in X$ iff Γ is finer than the γ- neighborhood filter at x.

Definition 2.6 Let Γ be a filter on (X, τ). Γ is said to γ- cluster to $x \in X$ iff every $F \in \Gamma$ intersects each $A \in \gamma(x)$.

Example 2.7 Consider $X = \{1,2,3,4\}$, $\tau = \{\Phi, X, \{1\}, \{1,2\}, \{1,2,3\}\}$. Then $\tau^\gamma = \{\Phi, X, \{1\}, \{1,2\}, \{1,3\}, \{1,4\}, \{1,2,3\}, \{1,2,4\}\}$. Then the γ- neighborhood system at 4 consists of $\{1,4\}$, $\{1,2,4\}$ and X. Now, $\gamma(4) = \{\{1,4\}, \{1,2,4\}, X\}$ and $\gamma_4 = \{\{1,4\}, \{1,2,4\}, X\}$. Then γ_4 is the γ- neighborhood filter at 4.

Consider the filter $\Gamma = \{\{1,4\}, \{1,2,4\}, \{1,3,4\}, X\}$. Then Γ is finer than γ_4 and hence Γ γ- converges to 4. Also every $F \in \Gamma$ meets each $A \in \gamma(4)$. Hence Γ γ- clusters at 4.

Proposition 2.8 Let Γ be a filter on (X, τ) γ- converging to x in X. Then Γ also converges to x.

Proof : Obvious.

Proposition 2.9 Let Γ be a filter on (X, τ) which γ- clusters to x in X. Then $x \in \cap\{cl_\gamma(F) : F \in \Gamma\}$.

Proof : Obvious.

Definition 2.10 Let (X, τ) be a topological space. Let Γ be a filter on X. Then a point $x \in X$ is said to be a strong γ- cluster point of Γ iff every $F \in \Gamma$ intersects each $A \in \gamma_x$.

Proposition 2.11 If Γ is a filter on (X, τ) strongly γ- clustering to $x \in X$, then Γ also γ- clusters at x.

Proof : Obvious.

That the converse may not be true is proved by the following example.

Example 2.12 Consider $X = \{1,2,3,4\}$, $\tau = \{\Phi, X, \{2,3\}, \{4\}, \{2,3,4\}\}$. Then the γ- open sets are Φ, X, $\{2,3\}$, $\{4\}$, $\{2,3,4\}$, $\{1,4\}$, $\{1,2,3\}$. Then $\gamma(1) = \{\{1,4\}, \{1,2,3\}, X\}$ and the γ- neighborhood filter at 1 is $\gamma_1 = \{\{1,4\}, \{1,2,3\}, \{1\}, X\}$. Consider the filter $\Gamma = \{\{2,4\}, \{1,2,4\}, \{2,3,4\}, X\}$. Clearly Γ γ- clusters to 1 since every $F \in \Gamma$ meets each $A \in \gamma(1)$. But Γ does not strongly γ- cluster to 1 since $\{2,4\} \in \Gamma$, $\{1\} \in \gamma_1$, but $\{2,4\} \cap \{1\} = \phi$.

Definition 2.13 Let $f : X \to Y$ be a function and Γ be a filter on X. Let $f(\Gamma) = \{f(F) : F \in \Gamma\}$. Then $f(\Gamma)$ may not be a filter on Y, but it is a base for some filter on Y and this filter is denoted by the symbol $f^\sharp(\Gamma)$ and is called the image of the filter Γ under the map f.

Theorem 2.14 Let $f : X \to Y$ be a function where X and Y are topological spaces. Then f is γ- continuous at $x \in X$ iff for any filter Γ on X γ- converging to x, the filter $f^{\sharp}(\Gamma)$ on Y converges to $f(x)$.

Proof : First assume that f is γ- continuous at x. Let Γ be a filter on X γ- converging to x. Let W be a neighborhood of $f(x)$ in Y. Then there exists a γ- open set V containing x in X with $f(V) \subset W$. Now since Γ γ- converges to x, so $V \in \Gamma$ and hence $f(V) \in f(\Gamma) \Rightarrow W \in f^{\sharp}(\Gamma)$. So, $f^{\sharp}(\Gamma)$ converges to $f(x)$.

To prove the converse, assume that the given condition holds. Let W be a neighborhood of $f(x)$ in Y and Γ be the filter of all γ- neighborhoods of x in X. Then clearly Γ γ- converges to x and hence by the assumed condition, the filter $f^{\sharp}(\Gamma)$ on Y converges to $f(x)$ and so $W \in f^{\sharp}(\Gamma)$. Now $W \in f^{\sharp}(\Gamma) \Rightarrow W \supset f(V)$ for some γ- neighborhood V of x, i.e., $W \supset f(V)$ for some γ- open set V containing x. This shows that f is γ- continuous at x.

Theorem 2.15 Let $f : X \to Y$ be a function where X and Y are topological spaces. Then f is γ- irresolute at $x \in X$ iff for any filter Γ on X γ- converging to x, the filter $f^{\sharp}(\Gamma)$ on Y γ- converges to $f(x)$.

Proof : Let f be γ- irresolute at x and let Γ be a filter on X γ- converging to x. Let V be a γ- open set containing $f(x)$. Since f is γ- irresolute at x, there exists a γ- open set U containing x such that $f(U) \subset V$. Now since Γ γ- converges to x, $U \in \Gamma$ and hence $f(U) \in f(\Gamma) \Rightarrow V(\supset f(U)) \in f^{\sharp}(\Gamma)$. Thus the filter $f^{\sharp}(\Gamma)$ on Y γ- converges to $f(x)$.

Conversely, let the given condition holds. Let V be a γ- open set in Y containing $f(x)$ and Γ be the filter of all γ- neighborhoods of x in X. Then clearly Γ γ- converges to x and by the given condition, the filter $f^{\sharp}(\Gamma)$ on Y γ- converges to $f(x)$ and hence $V \in f^{\sharp}(\Gamma)$. This implies that $V \supset f(U)$ for some γ- open set U containing x. Hence the function $f : X \to Y$ is γ- irresolute at x.

3 Characterizations of γ- compact spaces

In this article we first introduce the notion of γ- compact spaces and then try to characterize such spaces.

Definition 3.1 A topological space (X, τ) is called γ- compact if every γ- open cover of X, i.e., a cover of X by γ- open sets in X has a finite subcover.

Result 3.2 It has been shown in [3] that γ- compactness implies compactness and Latif has given an example to show that the converse may not be true.

Definition 3.3 Let $\{x_\lambda : \lambda \in \Lambda\}$ be a net in X. Then the family of tails $\{T_\lambda : \lambda \in \Lambda\}$ where, $T_\lambda = \{x_\mu : \mu \geq \lambda\}$ is a base for some filter on X. This filter is called the filter generated by the net $\{x_\lambda : \lambda \in \Lambda\}$.

Definition 3.4 Let Γ be a filter on X. Let $D_\Gamma = \{(x, F) : x \in F \in \Gamma\}$. If we set for any two (x, F), $(y, G) \in D_\Gamma$, $(x, F) \geq (y, G)$ iff $F \subseteq G$, then $(D_\Gamma, \geq)$ becomes a directed set. Then the map $P_\Gamma : D_\Gamma \to X : (x, F) \to x$ is a net in X which is called the net defined by the filter Γ.

Theorem 3.5 Let (X, τ) be a topological space. A net $\{x_\lambda : \lambda \in \Lambda\}$ in X γ- converges to x in X iff the filter generated by the net γ- converges to x.

Proof : Let the net $\{x_\lambda : \lambda \in \Lambda\}$ in X γ- converges to x. We have to show that the filter Γ generated by the net also γ- converges to x. Choose any γ- neighborhood U of x. Since $\{x_\lambda : \lambda \in \Lambda\}$ γ- converges to x, there exists a $\lambda_0 \in \Lambda$ such that $x_\lambda \in U$, for all $\lambda \geq \lambda_0$, i.e., $T_{\lambda_0} \in U$. Since T_{λ_0} is the base for the filter Γ, so $T_{\lambda_0} \in \Gamma$. Hence $U \in \Gamma$. Thus the filter Γ generated by the net $\{x_\lambda : \lambda \in \Lambda\}$ γ- converges to x.

Conversely, let the filter Γ generated by the net $\{x_\lambda : \lambda \in \Lambda\}$ γ- converges to x. Let U be a γ- neighborhood of x. Since Γ γ- converges to x, $U \in \Gamma$. Now since $\{T_\lambda : \lambda \in \Lambda\}$ is a base for Γ, there exists a $\lambda_0 \in \Lambda$ such that $T_{\lambda_0} \subseteq U$, i.e., $\{x_\lambda : \lambda \geq \lambda_0\} \subseteq U$. Thus $x_\lambda \in U$, for all $\lambda \geq \lambda_0$. Hence the net $\{x_\lambda : \lambda \in \Lambda\}$ γ- converges to x.

Theorem 3.6 Let (X, τ) be a topological space. A filter Γ on X γ- converges to x iff the net $P_\Gamma : D_\Gamma \to X$ defined by the filter γ- converges to x.

Proof : Let Γ γ- converges to x. Choose a γ- neighborhood U of x. Then $U \in \Gamma$ and so $(x, U) \in D_\Gamma$. Let $(y, V) \in D_\Gamma$ be such that $(y, V) \geq (x, U)$. Then $y \in V \subseteq U$, i.e., $P_\Gamma(y, V) \in U$. This shows that the net P_Γ lies eventually in U and hence the net P_Γ γ- converges to x.

Conversely, let the net $P_\Gamma : D_\Gamma \to X$ generated by the filter Γ γ- converges to x. Let U be a γ- neighborhood of x. Then the net P_Γ lies eventually in U, i.e., there exists $(y, V) \in D_\Gamma$ such that for all $(z, G) \geq (y, V)$, $P_\Gamma(z, G) = z \in U$. In particular, for any point $k \in V$,

$(k, V) \geq (y, V)$, so that $k \in U$. This shows that $V \subseteq U$. But $(y, V) \in D_\Gamma \Rightarrow V \in \Gamma \Rightarrow U \in \Gamma$. Hence the filter Γ γ- converges to x.

Definition 3.7 A net $\{x_\lambda : \lambda \in \Lambda\}$ in (X, τ) is said to γ- cluster at a point $x \in X$ if given a γ- neighborhood U of x and $\mu \in \Lambda$, there exists a $\lambda \in \Lambda$ such that $x_\lambda \in U$, for all $\lambda \geq \mu$. We say that the net lies frequently in U.

Theorem 3.8 Let (X, τ) be a topological space. A net $\{x_\lambda : \lambda \in \Lambda\}$ in X γ- clusters at a point $x \in X$ iff the filter generated by the net γ- clusters at x.

Proof : Suppose $\{x_\lambda : \lambda \in \Lambda\}$ γ- clusters at x. Let U be a γ- neighborhood of x. Let Γ be the filter generated by the net $\{x_\lambda : \lambda \in \Lambda\}$ and $F \in \Gamma$. We have to show that $U \cap F \neq \phi$. If possible, let $U \cap F = \phi$. Since $F \in \Gamma$, there exists a $\lambda_0 \in \Lambda$ such that $T_{\lambda_0} \subseteq F$, i.e., $\{x_\lambda : \lambda \geq \lambda_0\} \subseteq F$. Therefore, $\{x_\lambda : \lambda \geq \lambda_0\} \cap U = \phi$. Hence for $\lambda_0 \in \Lambda$, there does not exist $\lambda \in \Lambda$ such that $x_\lambda \in U$, for $\lambda \geq \lambda_0$. This implies that $\{x_\lambda : \lambda \in \Lambda\}$ does not γ- cluster at x, a contradiction. Hence $U \cap F \neq \phi$ and thus the filter Γ γ- clusters at x.

Conversely, let the filter Γ generated by the net $\{x_\lambda : \lambda \in \Lambda\}$ γ- clusters at x. Let U be a γ- neighborhood of x and $\lambda_0 \in \Lambda$. Since Γ γ- clusters at x, $U \cap F \neq \phi$, for all $F \in \Gamma$. Now $T_{\lambda_0} \in \Gamma \Rightarrow T_{\lambda_0} \cap U \neq \phi \Rightarrow$ there exists $\lambda \in \Lambda$ such that $x_\lambda \in U$ for $\lambda \geq \lambda_0 \Rightarrow \{x_\lambda : \lambda \in \Lambda\}$ γ- clusters at x.

Theorem 3.9 Let (X, τ) be a topological space. A filter Γ on X γ- clusters at x iff the net $P_\Gamma : D_\Gamma \to X$ defined by the filter γ- clusters to x.

Proof : Let Γ γ- clusters at x. Let U be a γ- neighborhood of x and $(y, F) \in D_\Gamma$. Then $y \in F \in \Gamma$. Since Γ γ- clusters at x, $F \cap U \neq \phi$ and so we can choose a point $z \in F \cap U$. Then $(z, F) \in D_\Gamma$ and $(z, F) \geq (y, F)$ implies that $P_\Gamma(z, F) = z \in U$. This shows that the net P_Γ is frequently in U and hence P_Γ γ- clusters at x.

Conversely, let the net $P_\Gamma : D_\Gamma \to X$ γ- clusters at x. Given any γ- neighborhood U of x, let G be an arbitrary member of Γ. We have to show that $U \cap G \neq \phi$. Choose any $y \in G$ [since $G \in \Gamma$, $G \neq \phi$]. Since P_Γ γ- clusters at x and $(y, G) \in D_\Gamma$, there exists $(z, H) \in D_\Gamma$ with $(z, H) \geq (y, G)$. Therefore, $z \in H \subseteq G$ such that $P_\Gamma(z, H) = z \in U$. Hence $G \cap U \neq \phi$. Thus the filter Γ γ- clusters at x.

Theorem 3.10 A topological space (X, τ) is γ- compact iff for every collection of γ- closed sets $\{F_\alpha : \alpha \in \Lambda\}$ in X possessing finite intersection property (in short, f.i.p.), the intersection $\bigcap_{\alpha\in\Lambda} F_\alpha$ of the entire collection is nonempty.

Proof : Let (X, τ) be γ- compact and let $\{F_\alpha : \alpha \in \Lambda\}$ be a collection of γ- closed sets having f.i.p. If possible, let $\bigcap_{\alpha\in\Lambda} F_\alpha = \phi$. Then $X = X \setminus \bigcap_{\alpha\in\Lambda} F_\alpha = \bigcup_{\alpha\in\Lambda}(X \setminus F_\alpha)$. Thus $\{X \setminus F_\alpha : \alpha \in \Lambda\}$ is a γ- open cover of X. Thus there exists a finite subset Λ_0 of Λ such that $X = \bigcup_{\alpha\in\Lambda_0}(X \setminus F_\alpha) = X \setminus \bigcap_{\alpha\in\Lambda_0} F_\alpha$, i.e., $\bigcap_{\alpha\in\Lambda_0} F_\alpha = \phi$, a contradiction. Hence $\bigcap_{\alpha\in\Lambda} F_\alpha \neq \phi$.

Conversely, let $\{G_\alpha : \alpha \in I\}$ be a γ- open cover of X. Then $\phi = X \setminus \bigcup_{\alpha\in I} G_\alpha = \bigcap_{\alpha\in I}(X \setminus G_\alpha)$. Since $\{X \setminus G_\alpha : \alpha \in I\}$ is a collection of γ- closed sets, by hypothesis, it cannot have f.i.p. Thus there exists a finite subset I_0 of I such that $\bigcap_{\alpha\in I_0}(X \setminus G_\alpha) = \phi$, then $X = \bigcup_{\alpha\in I_0} G_\alpha$, proving that X is γ- compact.

Theorem 3.11 A topological space (X, τ) is γ- compact iff every filter in X has a γ- cluster point in X.

Proof : First let X be γ- compact. Let Γ be a filter on X. Then $cl_\gamma\Gamma = \{cl_\gamma F : F \in \Gamma\}$ is a family of γ- closed subsets of X with f.i.p. Since X is γ- compact, $\bigcap_{F\in\Gamma} cl_\gamma F \neq \phi$. Take a point $x \in \bigcap_{F\in\Gamma} cl_\gamma F$. Then x is a γ- cluster point of Γ.

To prove the converse, let the condition holds. Let $\{G_\alpha : \alpha \in I\} = \mathcal{B}$ be a family of γ- closed sets in X with f.i.p. Now there exists a filter Γ on X such that $\mathcal{B} \subseteq \Gamma$. By the assumed condition, Γ γ- clusters at $x \in X$. Then $x \in cl_\gamma F$, for all $F \in \Gamma$. But each $G_\alpha \in \Gamma \Rightarrow x \in cl_\gamma(G_\alpha)$, for all $\alpha \in I$, i.e., $x \in G_\alpha$, for all $\alpha \in I$. So $\bigcap_{\alpha\in I} G_\alpha \neq \phi$. Hence X is γ- compact.

With the help of the above proved theorems, we can now state that :

Theorem 3.12 The following conditions are equivalent for a topological space X :

(a) X is γ- compact.

(b) Every filter in X has a γ- cluster point.

(c) Every net in X has a γ- cluster point.

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