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ANATOLII PLATONOVICH PRUDNIKOV: LIFE IN SCIENCE

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Abstracts

The biographical data of the Russian mathematician Professor Anatolii Platonovich Prudnikov is presented and his scientific, educational, publishing and public activity is enlightened. A brief survey of Prudnikov's results in operational calculus, integral transforms, special functions and partial differential equations is given together with the list of selected publications in the chronological order.

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Life and activity of A.P.Prudnikov

Anatolii Platonovich Prudnikov, the well-known authority in operational calculus, integral transforms and special functions was born in the city of Ul'janovsk on January 14, 1927. In 1944 he entered the Kuibyshev Aviation Institute. In 1947, after three years studies at this Institute, he continued his education as a forth-year student of the Department of Mathematics and Mechanics in the Kuibyshev Pedagogical Institute. Prudnikov graduated from this Institute in 1949. His professional career as a professor started from a position of Assistant Professor at the Kuibyshev Industrial Institute. In 1952-1954 he took doctor courses in this Institute.

In 1954 Prudnikov continued his doctor courses in Moscow at the Institute of Precise Mechanics and Computing Technique of the Academy of Sciences of the USSR under the supervisorship of Professor Vitalii Arsen'evich Ditkin. He got a position of a Junior Researcher at this Institute. In 1955 he began to work at the Computer Center of Academy of Sciences of the USSR which was organized in this year. He worked there for more than 43 years, starting from Junior Researcher to the head of a department. Prudnikov defended his Ph.D. dissertation "Analytic investigations of processes in heat and mass exchanges" in 1957, and his Dr.Sci. dissertation "On a class of integral transforms of Volterra type and some generalizations of operational calculus" in 1968 (which corresponds to two levels of the doctor degree in the Soviet Union). In 1972 he was awarded Professor's title, which was always considered as a title of a very high level in the USSR.

The first publications of Prudnukov were in partial differential equations, but he became well known due to his papers and books in operational calculus which became the main area of his research interests for many years. This area, closely connected with applied mathematics, was formed under the influence of Professor Ditkin. His interests were based on both the development of theoretical methods and applications to solution of some special problems of importance in science and practice. The result of the close cooperation between Prudnikov and Ditkin was their first book [8] published in 1958 and devoted to operational calculus for two variables. Later investigations were summarized in the books [29] and [30] and in the handbook [16], which were highly estimated by specialists. In particular, Professor Prudnikov together with Professor Ditkin and Professor Maslov (at present the academician of Russian Academy of Sciences) were awarded by the State Prize of the USSR in 1978.

Another area of Prudnikov's mathematical research was connected with integral transforms and special functions. He began to work in the theory of integral transforms together with V.A.Ditkin and continued this with his student Yu.A.Brychkov. Their results in the theory of integral transforms of generalized functions were presented in the book [34]. This was the first monograph in which a survey of properties of various integral transforms in spaces of test and generalized functions was given. Such a survey of the results in the theory of multi-dimensional integral transforms was presented in the book [57] written together with Brychkov, Glaeske and Vu Kim Tuan. This was also the first monograph devoted to the theory of multi-dimensional integral transforms. Later he was an initiator of creating five books [39], [42], [47], [55] and [56] written together with Brychkov and Marichev. These handbooks, devoted to the evaluation of integrals and series and direct and inverse Laplace transforms of elementary and special functions, considerably cover all known results. Unlike other similar handbooks, they contain many new formulas and integrals of general form which can be applied to evaluate wide classes of new integrals. Moreover, they contain properties of many special functions which were not mentioned earlier in monographic literature. These handbooks are very useful not only to mathematicians, but also to specialists in physics, mechanics, chemistry, engineering, etc who apply methods of mathematical analysis. We also mention the book [11] of Prudnikov and Berlyand and Gavrilova where the tables of the repeated integrals of the error functions and of the Hermite polynomials were presented.

The next area of Prudnikov's mathematical research was concentrated in informatics. His interests included a wide spectrum of problems in mathematical modelling, algorithmic approach and numerical analysis in connection with applications in computing technique. In 1992-1996 he developed methods of solving some problems of mathematical modelling and numerical investigations connected with the control of aerodynamic characteristics of flying apparatus. Last years Prudnikov was a supervisor of investigations in medical informatics in the State Research Experimental Institute of Aviation and Cosmic Medicine. Under his guidance the complex of mathematical models, algorithms, programs and nomograms was developed which was aimed to medical investigations on estimation of a professional health of pilots and a level of radiation and electromagnetic pollution of environments.

The results of Prudnikov's scientific activity have been published in more than 100 articles including 10 monographs translated in England, France, Germany, Japan, Poland and USA. His books became a table handbooks for researches and engineers.

From the very beginning of his start at Kuibyshev Industrial Institute, Anatolii Platonovich paid a lot of attention to mathematical education, devoted much strength and energy to the supervision of postgraduates and trainees, conducting seminars in the Computer Center and giving simultaneously lectures in the All-Union Institute of Food Industry. 15 of his students defended PhD thesis under his supervisorship and two of them defended the second-level dissertation of Doctor of Science.

Prudnikov was widely known in Russia and other countries of the former USSR as well as abroad. He was invited to give talks at many International Conferences, Seminars and Workshops in the former USSR, in Bulgaria, Germany, Great Britain, Japan, Poland, Spain and Yugoslavia. He was Visiting Professor in Fukuoka (Japan), Jena (Germany) and Valencia (Spain), and many times was invited by his colleagues in Finland, Germany, Great Britain, Hungary, Poland and Yugoslavia. He actively took part in the organization of many USSR and international conferences as a member of Organizing Committee. In particular, he was a Vice-director of International Conference on Generalized Functions and their Applications in Mathematical Physics held in Moscow in 1980.

Prudnikov gave a lot of his energy to publishing activity. For a long time he was a member of the Editorial Board of the All-Union journal *Inzhenerno-Fizicheskii Zhurnal* translated in English, a member of the Publishing Council in the Ministry of Education of the USSR, Chief of Publishing Council of Computer Center and Editor-in-Chief of all their Proceedings. He was well known as a good organizer and a person producing a positive impression on any partner in negotiations. Last years Prudnikov was very busy with Editor's work under the "Gordon & Breach Science Publishers". He was the organizer of the international journal *Integral Transforms and Special Functions* which started in 1993, and was an active Editor-of-Chief of this journal, devoting much time to his duties in this journal till the last days. Professor Prudnikov paid a lot of attention to the monograph series "Analytical Methods and Special Functions" published by "Gordon & Breach Science Publishers", of which he was also the Editor-in-Chief. He was also engaged into the activity connected with the Information Bulletin *Integral Transforms and Special Functions (in Russian)* prepared by the research group of the journal *Integral Transforms and Special Functions* and published by the Computer Center of the Russian Academy of Sciences.

Prudnikov took a very active part in public and scientific life in Russia (in the USSR before) and abroad. He was a Scientific Secretary of Computer Center and a Chief of the post-graduate courses there, a member of Directory Board of Scientific and Methodological Council in Mathematics, Physics and Astronomy of Russian society "Znanie", a member and Vice-Chief of Expert Council in the Supreme Attestation Commission of the USSR and later Russia in mathematics and mechanics. Last years he was a Chief of such an Expert Council in the Supreme Attestation Commission of Russia and a member of Dissertation Councils of Experts in Computer Center and in Moscow State University. He was a member of London and American Mathematical Societies.

Prudnikov had a bright and original personality, was a joyful and witty person, and was very communicable. Unexpected death of Professor Prudnikov on January 10, 1999 was really a great loss for all people who collaborated with him.

The interested reader can find some additional information about Prudnikov in the Obituary

"Anatolii Platonovich Prudnikov", Integral Transform and Special Functions **8** (1999), *no. 1-2*, to appear,

and in the paper

E.I.Moiseev, K.Skornik and W.Kierat "Life and work of Professor Anatolii Platonovich Prudnikov (1927-1999)", Fractional Calculus and Applied Analysis **2** (1999), *no. 1*, 97-106.

Brief survey of the scientific results of A.P.Prudnikov

The first investigations of Prudnikov were connected with mixed boundary value problems for partial differential equations. In [1] he studied the system of two parabolic equations

$$\frac{\partial u}{\partial t} = a_1\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + b_1\frac{\partial v}{\partial t},$$

$$\frac{\partial v}{\partial t} = a_2\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + b_2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \qquad (1)$$

a_1, b_1, a_2 and b_2 being constants such that

$$(a_1 + a_2 + b_1 b_2)^2 \neq 4a_1 a_2, \tag{2}$$

with the initial and boundary conditions $u(x, y, 0) = f_1(x, y)$, $v(x, y, 0) = f_2(x, y)$, $u'_x(0, y, t) = u'_y(x, 0, t) = v'_x(0, y, t) = v'_y(x, 0, t) = 0$ $u'_x(c, y, t) = g_1(y, t)$, $u'_y(x, d, t) = h_1(x, t)$ $(t \geq 0)$, $v'_x(c, y, t) = g_2(y, t)$, $v'_y(x, d, t) = h_2(x, t)$ $(t \geq 0)$ where $f_k(x, y)$ $(k = 1, 2)$ and $h_i(x, t)$ $(i = 1, 2)$ are given functions. Applying two-dimensional Fourier transform, he reduced this problem to a system of ordinary differential equations and obtained the solution $u(x, y, t)$ and $v(x, y, t)$ in closed form in terms of theta-functions.

In [3] Prudnikov studied two-dimensional system of the form (1)

$$\frac{\partial u}{\partial t} = a_1 \frac{\partial^2 u}{\partial x^2} + b_1 \frac{\partial v}{\partial t}, \ \frac{\partial v}{\partial t} = a_2 \frac{\partial^2 v}{\partial x^2} + b_2 \frac{\partial^2 u}{\partial x^2}, \tag{3}$$

where the constants a_1, b_1, a_2 and b_2 satisfy (2), with the conditions

$$u(x, 0) = f_1(x), \ v(x, 0) = f_2(x) \ (0 < x < l) \tag{4}$$

$$u(0, t) = g_1(t), \ v(0, t) = g_2(t), \ u(l, t) = g_3(t), \ v(l, t) = g_4(t), \tag{5}$$

$$a_{k1} u(0, t) + a_{k2} v(0, t) + a_{k3} u_x(0, t) + a_{k4} v_x(0, t) = h_k(t) \ (k = 1, 2, 3, 4), \tag{6}$$

$$u(0, t) = h_1(t), \ v(0, t) = h_2(t), \ u(l, t) = h_3(t), \ v(l, t) = h_4(t), \tag{7}$$

where a_{kj} $(1 \leq k, j \leq 4)$ are constants and $f_k(x)$ $(k = 1, 2)$, $g_k(t)$ $(1 \leq k \leq 4)$ and $h_k(t)$ $(1 \leq k \leq 4)$ are given functions for $x \in (0, l)$ and $t \in (0, \infty)$. Applying to (3)-(6) the double mixed Fourier and Laplace transform, he represented the solutions $u(x, t)$ and $v(x, t)$ as sums of integrals containing unknown functions which are determined form the boundary condition (7) through a system of Volterra integral equations.

In [3] Prudnikov showed some application of the problem (4)-(7) in the thermo-diffusion theory. His papers [2], [4]-[7] and [9]-[10] were devoted to applications of partial differential equations to investigation of other applied problems. In [2] he dealt with a solution of some problems in the theory of molecular transfer. Some study of certain processes of heat and mass transfer was given in [4]-[7]. Some problems in the theory of filtration of fluid were considered in [9] and a problem in the theory of heat conduction was discussed in [10].

The above applied results led Prudnikov to the main area of his research in the field of operational calculus, in which he had a close cooperation with Ditkin. Ditkin and Prudnikov developed the operational calculus by Heavisade and by Mikusinski. In [8] they extended the investigations by Voelker and Doetsch [*Die zweidimensionale Laplace-Transformation*, Verlag Brkhauser, Basel, 1950, MR **12**, 699], concerning the properties of two-dimensional Laplace transform, and constructed the operational calculus in two variables on the basis of the two-dimensional Laplace convolution. These investigations were generalized in [30] where they developed the operational calculus based on the Volterra-type two-dimensional transform

$$F \bigotimes G = \frac{\partial^2}{\partial x \partial y} \int_0^x \int_0^y F(x-t, y-s) G(t,s) dt ds. \tag{8}$$

In [29] using the idea of the generalized Laplace transform, Ditkin and Prudnikov extended Mikusinski's operational calculus to the complex domain. They showed that the set of all Laplace transformable operators forms a field $\mathcal{M}_0 \subset \mathcal{M}$, where $\mathcal{M}$ is the Mikusinski's quotient field. This field is isomorphic to the field $\mathcal{M}_0$ of all functions of a complex variable represented in the form $F^*(z)/G^*(z)$ where $F^*(z)$ and $G^*(z)$ are the Laplace integrals of f and g, respectively, in some half-plane $\mathrm{Re}(z) > c$, c is real constant. Ditkin and Prudnikov studied the representation of the whole field $\bar{\mathcal{M}}$ of Mikusinski's operators by functions of a complex variable and proved that $\mathcal{M}$ is isomorphic to the field $\bar{\mathcal{M}}$ which consists of elements closely related to functions of the form $F^*(z)/G^*(z)$.

Prudnikov paid a lot of attention to operational calculus of Bessel type operators. Ditkin first constructed operational calculus for the Bessel operator $B_2 = D_x x D_x$, $D_x = d/dx$, by defining ring M_{B_2} with the convolution

$$F \bigotimes G = B_2 \int_0^x dt \int_0^1 F(ty) G[(x-t)(1-y)] dy, \tag{9}$$

as a multiplication of the elements F and G of M_{B_2}. Developing the results of Ditkin and Meller who studied an operational calculus for the operators closely connected with B_2, in [12] Prudnikov investigated the operational calculus for the operator $B_3 = D_x x D_x x D_x$. He defined a ring M_{B_3} with the convolution product

$$F \bigotimes G = B_3 \int_0^x \int_0^1 dy \int_0^1 F(tys) G[(x-t)(1-y)(1-s)] ds \tag{10}$$

and showed that in such an operational calculus the integral transform

$$2 \int_0^\infty M(xt) \omega(t) dt, \qquad M(u) = 2 \int_0^\infty \exp\left(-4u/t^2\right) K_0(t) \frac{dt}{t}, \tag{11}$$

with McDonald function $K_0(z)$ plays the same role as the Laplace transform in Mikusinski's operational calculus. The results obtained were extended by Ditkin and Prudnikov to the Bessel operators $B_k = (1/x)(xD)^k$ $(k = 2, 3, \cdots)$, see [13], where application to the differential equation $B_3 y + \lambda y = 0$ was also considered. In [14] they gave a Mikusinskı-type treatment of an operational calculus based on the operator B_2 and indicated that such a calculus is related to the Meijer integral transform

$$(\mathbf{K}_0\omega)(x) = 2\int_0^\infty K_0\left(2(xt)^{1/2}\right)\omega(t)dt \tag{12}$$

in the same way as the classical operational calculus with the Laplace transform.

Ditkin and Prudnikov also considered discrete analogies of the constructions above and developed the operational calculus of functions of one and two integer arguments in [15] and [18], respectively, with indication of applications in discrete analysis. In [19] they suggested the method of inclusion of non-integrable functions into operational calculus based on Hadamard's ideas on finite parts of divergent integrals. In [26]-[27] they developed the operational calculus based on the generalization of the relations (9) by

$$F\bigotimes G = U^{-1}\frac{1}{\omega}\int_0^x F(x-t)G(t)\omega(x-t)\omega(t)dt \tag{13}$$

and

$$F\bigotimes G = U^{-1}\frac{1}{\omega}\int_0^x dt\int_0^1 F(ty)G[(x-t)(1-y)]\omega[(x-t)(1-y)]\omega(ty)dy, \tag{14}$$

respectively, where U^{-1} is the operator inverse to a certain linear operator U and $\omega(x)$ is a function for which there exists the Laplace transform in the former case and the Meijer transform (12) in the latter. We also note Prudnikov's papers [24] and [25] where some problems in the theory of operational calculus were discussed.

In [28] Ditkin and Prudnikov suggested a general method for developing an operational calculus on the basis of a generalization of (13) and (14) in the form

$$F\bigotimes G = U^{-1}(FG). \tag{15}$$

Here U^{-1} is the operator inverse to the linear operator U defined by $Uf = l_0 f$ for a function f in a commutative ring L with a multiplication $fg = w^{-1}(\omega f * \omega g)$, $\omega f * \omega g$ being the Laplace convolution, ω is a linear operator defined on L with values in a set of functions f defined on $(0,\infty)$ and Lebesgue integrable on any interval $(0, A)$, and F and G are elements of the ideal of the ring L generated by a fixed element $l_0 \in L$.

The last interest of Prudnikov in operational calculus was connected with perfect operators, and his investigations with Ryabtsev in this field were presented in two monographs [53] and [54]. The first one deals with a rigorous exposition of the basis and principal results of the theory of operational calculus of perfect operators. The whole construction is based on distribution theory, and the perfect operators are introduced as operators in a commutative algebra without divisors of zero. Among the discussed problems there one can find an algebraization of the initial system with respect to a linear operator, the fundamental formula of the operational calculus with applications to differential and integro-differential equations, theory and applications of systems that are left compact supported, the convergence of sequences and series of perfect operators and the continuity properties of the corresponding algebraic operations, the infinitesimal properties of the ring of perfect abstract operators. The second book [2] is devoted to differentiation and integration of operator functions and to the method of superposition. The definitions of differentiation and integration with respect to one and two real parameters are given, theorems concerning ordinary and partial linear differential equations are proved with applications to solving, in a purely algebraic way, the wave, heat and telegraph equations of two variables and equations with delay.

In [61] Prudnikov presented a comprehensive survey of latest results in the theory of operational calculus. This paper can be considered as a continuation of the survey articles [17] and [22] written together with Ditkin and the survey paper [35] with Brychkov and Shilov. Beginning from a short historical comment on the Heaviside classical operational calculus, he gave Plesner's rigorous description of the operational calculus based on the theory of operators developed by Mikusinski, characterized Mikusinski operational calculus and its development by Ditkin and Prudnikov, reviewed the operational calculus based on the Schwartz theory of distribution and on Antosik's sequential approach to distributions, paid the special attention to the convolution calculus studied by Dimovski and presented his and Ryabtsev operational calculus of perfect operators. Prudnikov indicated general applications of operational calculus to nonlinear systems of automatic control, linear escape problem, and linear problem of tracking motion. He also formulated a new problem for four nonlinear partial differential equations, and suggested that these equations need solutions based on the new convolution in the construction of an operational calculus as a branch of nonlinear functional analysis. This Prudnikov's article is very informative and, surely, it will provide new stimulating interest in the advanced study and research in operational calculus.

The above Prudnikov's investigations in operational calculus were always connected with problems in applications. In the eighties he paid a special attention to two-dimensional boundary value problems for the Laplace and the Poisson equations. In [40] Prudnikov together with his student Vlasov considered the boundary value problems for the Laplace equation in a class of two-dimensional plane domains with complicated boundary, often arising in physics and mechanics, with a non-homogeneous Dirichlet condition on a part of boundary and the homogeneous Dirichlet or Neumann condition on the rest of boundary. They developed the method for solution of this problem based on the representation of the solution in the form of a series in powers of a conformal mapping of an extension of the original domain. They considered a generalization of such a method in [43], and in [49] applied this method to solve some boundary value problems for the Poisson equation in the above domains. Vlasov, Volkov and Prudnikov in [48] considered the Dirichlet problem for the Laplace equation in a disk with a round angular notch and gave the solution in closed form that is infinitely differentiable in the domain and on the contour of the rounded notch. In [44] and [45] they investigated the Dirichlet problem for L-type domain and for a circle with two notches, respectively. In [46] Vlasov, Volkov, Prudnikov and Yakovleva obtained the so called fundamental frequencies and the eigen-functions for the torsional oscillation, when the cross-section of the cylindrical shaft is a disc with a notch that is bounded by a rounded corner.

Numerical examples for the solution of the Dirichlet problem for the Laplace equation in considered domain were also presented in [48]. These investigations were continued by Vlasov, Volkov, Prudnikov, Yakovleva and Vladimirova in [51]. They gave a description of a complex of algorithms and programs for the solution of the Dirichlet problem for the Poisson equation in domains whose boundary is a polygon of an arbitrary form, one of the angles of which is smoothly rounded-off by an arc of a circle. The developed algorithms allowed to obtain both the solution of the boundary value problem itself and its derivatives in the domain and its boundary with high degree of accuracy. We also note that earlier Prudnikov together with Zurina and Popova applied the Pade approximate methods to numerical solution of the integral equations of transfer theory and quantum mechanics in [31] and [32], respectively, see also [33] in this connection.

Prudnikov's interest to the theory of integral transforms was caused by an operational calculus, see (11) and (12), and applied problems. His results in this field were presented in the book [29] and the survey articles [20] and [23] written together with Ditkin. We also note the book [34] and the review paper [41] written with Brychkov devoted to the theory of integral transforms of generalized functions, and the monograph [57] with Brychkov, Glaeske and Vu Kim Tuan dealt with multidimensional integral transforms. The special interest of Prudnikov was connected with the Watson transform. Such a transform was introduced by Watson as the generalization of the Fourier cosine transform $\mathcal{F}_c f$ given by

$$(\mathcal{F}_c f)(x) = \left(\frac{2}{\pi}\right)^{1/2} \frac{d}{dx} \int_0^\infty \frac{\sin(xt)}{t} f(t)dt \tag{16}$$

by replacing the kernel $(2/\pi)^{1/2} \sin x/x$ by the function $\psi(x)$. The Watson transform can be deined by

$$(Wf)(x) = \frac{d}{dx}\left(x \int_0^\infty \psi(xt) f(t)dt\right) \tag{17}$$

where the function $\psi(x)$ satisfies the condition

$$\int_{-\infty}^{\infty} \psi(xt)\psi(yt)dt = \begin{cases} 0, & \text{if } xy < 0, \\ 1/[\max(|x|,|y|)] & \text{if } xy > 0. \end{cases} \tag{18}$$

Prudnikov investigated the properties of such an operator in $L_2(-\infty,\infty)$ in [52] and together with Skornik in [64]. In particular, it was proved that the Watson operator W can be represented in a unique way as a series of the operators of the form $(TS)^nT$ and $S(TS)^n$ $(n = 0, 1, 2, \cdots)$, where

$$(Tf)(x) = \frac{d}{dx}\left(x\int_0^{1/x} f(t)dt\right), \qquad (Sf)(x) = \frac{1}{x}f\left(\frac{1}{x}\right). \tag{19}$$

Another generalizations of cosine- and sine-transforms given by the integral transform with confluent hypergeometric functions as kernels, were investigated by Prudnikov together with Moiseev and Skornik in [63].

Prudnikov came to the Watson transform while studying in [36]-[37] certain problems of the theory of heat conduction. He reduced these problems to the heat equation

$$\frac{\partial u(,t)}{\partial t} = \lambda^2\frac{\partial^2 u(x,t)}{\partial x^2} \tag{20}$$

with the boundary conditions $u_x + au|_{x=l_1+bt} = h_1(t)$, $u|_{x=l_2+bt} = h_2(t)$, λ and a being real numbers and $h_i(t)$ $(i = 1, 2)$ given functions, and with the initial condition $u|_{t=-\infty} = 0$, and obtained the explicit solution $u(x,t)$ of this problem in terms of the Watson transform (17) with some special $\psi(x)$. Prudnikov and Bartoshevich [38] solved some problems of mathematical physics by using the method of decomposition of certain integral operators into orthogonal Watson operators.

The main interests of Prudnikov in the theory of special functions were connected with calculation of integrals of special functions. In this connection the reader may be referred to the books [39], [42], [47], [54], [55] and a survey article [50] written together with Brychkov and Marichev. In the nineties he especially interested in problems concerning orthogonal systems of polynomials and special functions. His first results in this direction were obtained in [21] where he gave the explicit expressions for the series

$$\sum_{n=0}^{\infty}\lambda^n L_n(t)P_n(x), \qquad \sum_{n=0}^{\infty}\lambda^n P_n(t)P_n(x), \; \sum_{n=0}^{\infty}\frac{L_n(t)L_n(x)}{n+1/2} \tag{21}$$

with Laguerre $L_n(t)$ and Legendre $P_n(x)$ polynomials, in terms of the Bessel function of the first kind $J_0(z)$, the modified Bessel function $I_0(z)$ and McDonald function $K_0(z)$.

Prudnikov carried out investigations in this area together with Moiseev. In [59] they introduced a new system of orthonormal functions defined in terms of the Gauss hypergeometric function ${}_2F_1(a,b;;z)$ by

$$\rho_n^\alpha(x,a) = A_n^\alpha \; {}_2F_1\left[-n, 1+\frac{\alpha}{2}; \alpha+1; \left(\frac{1}{2}+\frac{ix}{2a}\right)\right]\left(\frac{1}{2}+\frac{ix}{2a}\right)^{-1-\alpha/2}, \tag{22}$$

where

$$A_n^\alpha = \frac{\Gamma(1+\alpha/2)}{2\Gamma(1+\alpha)}\left(\frac{\Gamma(\alpha+n+1)}{n!\pi a}\right), \; -\infty < x < \infty, \; n = \pm 1, \pm 2, \cdots. \tag{23}$$

When $\alpha = 0$ and $a = 1$, the system (22) is reduced to the the orthogonal system

$$\rho(x) \equiv \rho_n^0(x,1) = \pi^{-1/2}\frac{(ix-1)^n}{(ix+1)^{n+1}}, \; -\infty < x < \infty, \; n = \pm 1, \pm 2, \cdots, \tag{24}$$

introduced by Wiener.

Moiseev and Prudnikov [59] proved that $\rho_n^\alpha(x,a)$ is a complete orthonormal system in $L^2(0,\infty)$ as well as the systems $\sqrt{2}C_n^\alpha(x,a)$ and $\sqrt{2}S_n^\alpha(x,a)$ of real functions $C_n^\alpha(x,a)$ and $S_n^\alpha(x,a)$ defined via (22) by

$$C_n^\alpha(x,a) + iS_n^\alpha(x,a) = \sqrt{2}\rho_n^\alpha(x,a). \tag{25}$$

They also constructed the corresponding differential equation and the generating function for $\rho_n^\alpha(x,a)$. These investigations were continued in [60] where the representation for the Wiener function $\rho(x)$ in (24) via Chebyshev polynomials of the first and second kind $T_{2n+1}\left((1+t^2)^{1/2}\right)$ and $U_{2n+1}\left((1+t^2)^{1/2}\right)$ and the differentiation relation between McDonald function $K_{n+1/2}(x)$ and Laguerre polynomial $L_n(x)$ were applied to construct several complete orthogonal systems of functions in $L^2(0,\infty)$ and $L^2(0,1)$. In [62] Moiseev and Prudnikov investigated the system $\sin[(n-\beta/2)\theta]$ $(n = 1, 2, \cdots)$ of sines and the system 1, $\cos[(n-\beta/2)\theta]$ $(n = 1, 2, \cdots)$ of cosines with real β on the interval $[0,\pi]$. They proved that these systems form a basis in some subspaces of the Sobolev space $W_p^l(0,\pi)$ and the corresponding series are convergent, provided that $p \in (1,\infty)$ and β satisfy additional conditions. They also obtained similar results for the associated system of exponential functions $\exp[i(n-\beta/2)sgn(n)]$ $(n = 0, \pm 1, \pm 2, \cdots)$ in a subspace of $W_p^l(-\pi,\pi)$.

In [58] Prudnikov considered the orthonormal system of polynomials $V_0(x,k) = 1$, $V_1(x,k), \cdots, V_n(x,k)$ with $k > 0$ on the interval $[0,\infty)$ such that

$$\int_0^\infty V_n(x,k)V_m(x,k)\zeta(x,k)dx = \delta_{mn} = \begin{cases} 0, & \text{if } m \neq n, \\ 1 & \text{if } m = n, \end{cases} \tag{26}$$

with respect to the weight function $\zeta(x,k)$ defined by the Mellin-Barnes integral

$$\zeta(x,k) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \Gamma^k(s)x^{-s}ds, \qquad x>0,\ a>0,\ \mathrm{Re}(s)>0. \tag{27}$$

The system $V_n(x,k)$ is a generalization of the system of Laguerre polynomials $(-1)^n L_n(x)$ with respect to the weight $\zeta(x,1) = e^{-x}$ which corresponds to $k=1$. When $k=2$, the relations (26) and (27) give a new system of orthogonal polynomials $V_n(x) = V_{n,2}(x)$ with respect to the weight $\zeta(x,2) = 2K_0(\sqrt{x})$. Prudnikov formulated the problem to find the generating function, an analogue of Rodrigues' formula, the recurrence relation and a differential equation for the orthogonal polynomials $V_n(x)$ and for more general polynomials $V_{n,k}(x)$ with $k>2$. He discussed the recurrent formula for $V_n(x)$, and found such a recurrent relation for the weight function $2|x|K_0(|x|)$ on the whole axis $(-\infty,\infty)$ and for the corresponding orthogonal polynomials $W_n(x)$ symmetric with respect to the origin.

The above problems as well as other problems and ideas by Professor Anatolii Platonovich Prudnikov will stimulate both the further development of methods of operational calculus, integral transforms and special functions and their applications to applied problems in physics, mechanics and other sciences.

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