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INTEGRAL REPRESENTATION FOR THE PRODUCT OF TWO EXTENDED JACOBI POLYNOMIALS

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Abstract :

In this note we have obtained an integral representation for the product of two extended Jacobi polynomials. Some particular cases of interest are also pointed out..

I. Introduction

The extended Jacobi polynimials defined by Patil and Thakare [1] are

(1.1) $$F_n(\alpha, \beta; x) = \frac{(-1)^n}{n!}(x-a)^{-\alpha}(b-x)^{-\beta}\left(\frac{\lambda}{b-a}\right)^n$$

$$\times D^n\left[(x-a)^{n+\alpha}(b-x)^{n+\beta}\right], \quad D = \frac{d}{dx}.$$

The aim at presenting the note is to derive an integral representation for the product of two extended Jacobi polynomials as defined in (1.1). In this connection we would like to point it out that the integral representation for the product of two Jacobi polynomials derived by S. K. Chatterjea [2] is incorrect and the correct version of the same has been obtained as a special case of our result. In fact, it may be mentioned here that in [2], the relation (2.9) is wrong and the results (2.13) and (2.14) are incorrect - the correct versions of which are given in (2.9) and (2.10) of this note.

2. DERIVATION

In deriving our result we shall make use of the following formulae [3] :

(2.1) $$\Im_n(\alpha, \gamma, x) = {}_2F_1(-n, \alpha + n; \gamma; x)$$

(2.2) $$F_n(\alpha, \beta; x) = \frac{(1+\beta)_n}{n!} \lambda^n {}_2F_1(-n, 1 + \alpha + \beta + n; 1 + \beta; \frac{x-b}{a-b})$$

(2.3) $$\Phi_n(\alpha, \gamma; x) = \frac{(\alpha)_n}{n!} {}_2F_1(-n, \alpha + n; \gamma; x)$$
$$= \frac{(\alpha)_n}{n!} \Im_n(\alpha, \gamma; x).$$

The definitions (2.1), (2.3) are used in [3] and [2]. The relation (2.2) is proved by the present authors [4].

From (2.3) it follows that

(2.4) $$\Phi_n(\alpha, \gamma; x) = \sum_{r=0}^{n} \frac{(-1)^r (\alpha)_{n+r}}{(n-r)!\,(\gamma)_r\, r!} x^r .$$

Now replacing α by $(1 + \alpha + \beta)$, γ by $(1 + \beta)$ and x by $(\frac{x-b}{a-b})$ in (2.3), we have

(2.5) $$F_n(\alpha, \beta; x) = \frac{(1+\beta)_n}{(1+\alpha+\beta)_n} \lambda^n \Phi_n(1 + \alpha + \beta, 1 + \beta; \frac{x-b}{a-b}).$$

Therefore, with the help of (2.4), we obtain

(2.6) $$F_m(\alpha,\beta;x)\,F_n(\alpha',\beta';y) = \lambda^{m+n}\,\frac{(1+\beta)_m(1+\beta')_n}{(1+\alpha+\beta)_m(1+\alpha'+\beta')_n}$$

$$\times\sum_{r=0}^{m}\sum_{s=0}^{n}\frac{(-1)^{r+s}(1+\alpha+\beta)_{m+r}(1+\alpha'+\beta')_{n+s}}{(m-r)!\,(n-s)!\,(1+\beta)_r(1+\beta')_s}$$

$$\times\frac{1}{r!\,s!}\left(\frac{x-b}{a-b}\right)^r\left(\frac{y-b}{a-b}\right)^s.$$

Now using the results [5] :

$$\frac{\Gamma(\mu+\nu+1)}{\Gamma(\mu+1)\,\Gamma(\nu+1)} = \frac{2^{\mu+\nu}}{\pi}\int_{-\pi/2}^{\pi/2} e^{(\mu-\nu)\theta i}\cos^{(\mu+\nu)}\theta\,d\theta,\ (\mu+\nu>-1)$$

and

$$\frac{\Gamma(\mu)\,\Gamma(\nu)}{\Gamma(\mu+\nu)} = \int_0^1 t^{(\mu-1)}(1-t)^{(\nu-1)}\,dt,\ (\mu>0,\nu>0)$$

we obtain from (2.6)

(2.7) $$F_m(\alpha,\beta;x)\,F_n(\alpha',\beta';y)$$

$$= \lambda^{m+n}\,\frac{(1+\beta)_m(1+\beta')_n}{(1+\alpha+\beta)_m(1+\alpha'+\beta')_n}\;\frac{\Gamma(1+\beta)\,\Gamma(1+\beta')}{\Gamma(1+\alpha+\beta)\,\Gamma(1+\alpha'+\beta')}$$

$$\times\frac{2^{\beta+\beta'+m+n}}{\pi^2}\int_0^1\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2} t^{\alpha+\beta+m}(1-t)^{\alpha'+\beta'+n}$$

$$\times\, e^{(\beta-\beta')\Phi i+(m-n)\theta i}\cos^{(m+n)}\theta\,\cos^{(\beta+\beta')}\Phi$$

$$\times\sum_{k=0}^{(m+n)}\frac{(-1)^k}{k!}\left(\frac{\cos\Phi}{\cos\theta}\right)^k\frac{\Gamma(2+\alpha+\beta+\alpha'+\beta'+m+n+k)}{\Gamma(\beta+\beta'+k+1)\,(m+n-k)!}$$

$$\times\sum_{r+s=k}\binom{k}{r}\left(\frac{t(x-b)}{a-b}\right)^r\left(\frac{(1-t)(y-b)}{a-b}\right)^s e^{(r-s)(\Phi-\theta)i}\,d\Phi\,d\theta\,dt$$

$$= \lambda^{m+n}\,\frac{(1+\beta)_m(1+\beta')_n}{(1+\alpha+\beta)_m(1+\alpha'+\beta')_n}\;\frac{\Gamma(1+\beta)\,\Gamma(1+\beta')}{\Gamma(1+\alpha+\beta)\,\Gamma(1+\alpha'+\beta')}$$

$$\times\frac{\Gamma(2+\alpha+\beta+\alpha'+\beta')}{\Gamma(\beta+\beta'+1)}\;\frac{2^{\beta+\beta'+m+n}}{\pi^2}\int_0^1\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2} t^{\alpha+\beta+m}$$

$$\times (1-t)^{\alpha'+\beta'+n} e^{(\beta-\beta')\Phi i + (m-n)\theta i} \cos^{(m+n)}\theta \cos^{(\beta+\beta')}\Phi$$

$$\times \sum_{k=0}^{(m+n)} \frac{(-1)^k}{k!} \frac{(2+\alpha+\beta+\alpha'+\beta')_{m+n+k}}{(\beta+\beta'+1)_k (m+n-k)!}$$

$$\times \left\{ \frac{\frac{t(x-b)}{(a-b)} e^{(\Phi-\theta)i} + (1-t)\left(\frac{y-b}{a-b}\right) e^{(\theta-\Phi)i}}{\cos\theta} \cos\Phi \right\}^k d\Phi\, d\theta\, dt$$

$$= \lambda^{(m+n)} \frac{(1+\beta)_m (1+\beta')_n \Gamma(1+\beta)\Gamma(1+\beta')\Gamma(2+\alpha+\beta+\alpha'+\beta')}{(1+\alpha+\beta)_m (1+\alpha'+\beta')_n \Gamma(1+\alpha+\beta)\Gamma(1+\alpha'+\beta')\Gamma(\beta+\beta'+1)}$$

$$\times \frac{2^{(\beta+\beta'+m+n)}}{\pi^2} \int_0^1 \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} t^{\alpha+\beta+m} (1-t)^{\alpha'+\beta'+n}$$

$$\times e^{(\beta-\beta')\Phi i + (m-n)\theta i} \cos^{(m+n)}\theta \cos^{(\beta+\beta')}\Phi \times \Phi_{m+n}(2+\alpha+\beta+\alpha'+\beta',$$

$$\beta+\beta'+1, \frac{\{\frac{t(x-b)}{(a-b)} e^{(\Phi-\theta)i} + (1-t)\left(\frac{y-b}{a-b}\right) e^{(\theta-\Phi)i}\} \cos\Phi}{\cos\theta}) \, d\Phi\, d\theta\, dt$$

obtained by making use of (2.4).

Now from (2.7), by using (2.5), we obtain

(2.8) $$F_m(\alpha,\beta;x)\, F_n(\alpha',\beta';y)$$

$$= \frac{2^{(\beta+\beta'+m+n)}}{\pi^2} \frac{\Gamma(\alpha+\alpha'+\beta+\beta'+m+n+2)}{\Gamma(\alpha+\beta+m+1)\,\Gamma(\alpha'+\beta'+n+1)}$$

$$\times \frac{\Gamma(\beta+m+1)\,\Gamma(\beta'+n+1)}{\Gamma(\beta+\beta'+m+n+1)} \int_0^1 \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} t^{\alpha+\beta+m} (1-t)^{\alpha'+\beta'+n}$$

$$\times e^{(\beta-\beta')\Phi i + (m-n)\theta i} \cos^{(m+n)}\theta \cos^{(\beta+\beta')}\Phi$$

$$\times F_{m+n}(\alpha+\alpha'+1, \beta+\beta';$$

$$b + \frac{\{t(x-b) e^{(\Phi-\theta)i} + (1-t)(y-b) e^{(\theta-\Phi)i}\} \cos\Phi}{\cos\theta}) \, d\Phi\, d\theta\, dt$$

which is our desired result.

Some special cases :

Case - I. Now putting $b = -a = 1$ and $\lambda = 1$ (in 2.8) and finally interchanging α, β and α', β', we have

$$(2.9) \qquad P_m^{(\alpha,\beta)}(x)\, P_n^{(\alpha',\beta')}(y) = \frac{2^{(\alpha+\alpha'+m+n)}}{\pi^2} \frac{\Gamma(\alpha+\alpha'+\beta+\beta'+m+n+2)}{\Gamma(\alpha+\beta+m+1)\,\Gamma(\alpha'+\beta'+n+1)}$$

$$\times \frac{\Gamma(\alpha+m+1)\,\Gamma(\alpha'+n+1)}{\Gamma(\alpha+\alpha'+m+n+1)} \int_0^1 \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} t^{\alpha+\beta+m} (1-t)^{\alpha'+\beta'+n}$$

$$\times\, e^{(\alpha-\alpha')\Phi i + (m-n)\theta i} \cos^{(m+n)}\theta \cos^{(\alpha+\alpha')}\Phi$$

$$\times\, P_{m+n}^{(\alpha+\alpha',\, \beta+\beta'+1)}\left(1 + \frac{\{t(x-1)e^{(\Phi-\theta)i} + (1-t)(y-1)e^{(\theta-\Phi)i}\}\cos\Phi}{\cos\theta}\right) d\Phi\, d\theta\, dt,$$

which is the correct version of the integral representation of the product of two jacobi polynomials derived by S. K. Chatterjea in (2.13) of [2].

Case-II Now putting $\alpha = \beta = \alpha' = \beta' = 0$ in (2.9), we have

$$(2.10) \qquad P_m(x)\, P_n(y) = \frac{2^{m+n}}{\pi^2}(m+n+1) \int_0^1 \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} t^m$$

$$\times\, (1-t)^n e^{(m-n)\theta i} \cos^{(m+n)}\theta$$

$$\times\, P_{m+n}^{(0,1)}\left(1 + \frac{\{t(x-1)e^{(\Phi-\theta)i} + (1-t)(y-1)e^{(\theta-\Phi)i}\}\cos\Phi}{\cos\theta}\right) d\Phi\, d\theta\, dt$$

which is correct version of (2.14) of [2].

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