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ABOUT ONE BOUNDARY VALUE PROBLEM FOR POLYHARMONIC OPERATORS

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Introduction

A boundary value problem in certain domain $D \subset \mathbb{E}_m$ for the polyharmonic operator Δ^n was studied in [1] by N. Meiman. He reduced the solution of this boundary value problem to a system of integral equations with a weak singularity .

The author finds difficulties in giving a complete list of all papers on the boundary value problem for the polyharmonic operator. More details about this investigations can be found in [2,p. 403]. In [3] and [4] the analogous problem for a sphere was studied in relation with the bigarmonic operator and the polyharmonic operator, respectively .

The boundary value problem for the bigarmonic operator is solved by means of integrals, while the solution of the polyharmonic operator is determined by series .

In this paper the boundary value problem will be reduced to a system of linear algebraical equations having a triangle matrix . This system of equations may be easily solved .

1. In this section we should prove comparatively a simple equality, which is necessary in the following

Lemma 1.1 . If next equality takes place

$$\sum_{k=0}^{n} \alpha_k (1-t)^k = \sum_{k=0}^{n} a_k t^k , \tag{1.1}$$

then the α_k may be defined by the formulas

$$\alpha_k = (-1)^k \sum_{p=k}^{n} \binom{p}{k} a_p , \quad k=0,1,2,....,n \tag{1.2}$$

Proof : Expanding the left-hand side of the equality (1.1) by means of the Newton binomial formula, and changing the order of summation according to

$$\sum_{k=0}^{n} \varphi_k \sum_{p=0}^{k} \lambda_{kp} f_p = \sum_{k=0}^{n} f_k \sum_{p=k}^{n} \lambda_{pk} \varphi_p$$

we obtain

$$\sum_{k=0}^{n} (-1)^k \left[\sum_{p=k}^{n} \binom{p}{k} \alpha_p \right] t^k = \sum_{k=0}^{n} a_k t^k$$

Hence, we have for α_k the following system of linear algebraical equations

$$\sum_{p=k}^{n} \binom{p}{k} \alpha_p = (-1)^k a_k , \quad k=0,1,2,...,n ,$$

whose solutions can be found by the formulas (1.2).

Corollary From the equality

$$\sum_{k=0}^{n} \frac{\alpha_k}{(1-t)^k} = \sum_{k=0}^{n} a_k \left(\frac{t}{t-1}\right)^k$$

it follows (1.2) .

2. Let introduce the following notation: Ω denotes the unit m-dimensional sphere with a centre in the origin of coordinates; S is the surface of this sphere; $\tau = |x-\xi|$ and $\rho = |x|$, where $x \in \Omega$ and $\xi \in S$. Finally, γ stands for the angle between x and ξ .

Thus, we can write

$$A_\beta = \frac{1}{|S|} \int_S \frac{d_\xi S}{\tau^{m+2\beta}} = {}_2F_1(m/2 + \beta, \beta+1; m/2; \rho^2) \tag{2.1}$$

$$B_\beta = \frac{1}{|S|}\int_S \frac{\cos\gamma}{\tau^{m+2\beta}}\, d_\xi S = \frac{m+2}{m}\, {}_2F_1(m/2+\beta+1, \beta+1; m/2+1; \rho^2) \qquad (2.2)$$

The integrals (2.1) and (2.2) are easily calculated if the coordinate system is rotated in such a way that the first coordinate axis go along the vector x. By (2.1) and the results of the first section one has for all $0 \le k \le 2n-1$:

$$A_{k-n,k} = (1-\rho^2)^k A_{k-n} = \begin{cases} (1-\rho^2)^k \sum_{i=0}^{n-k-1} b_i (1-\rho^2)^i & , 0 \le k < n-1 \\ (1-\rho^2)^{n-1} & , k=n-1 \\ (1-\rho^2)^n & , k=n \\ (1-\rho^2)^{n-k-1} \sum_{i=0}^{k-n} b_i (1-\rho^2)^{-i} & , n < k \le 2n-1 \end{cases} \qquad (2.3)$$

where the coefficients b_i are determined by the relation

$$b_i = \begin{cases} (-1)^i \sum_{j=0}^{n-k-1-i} \binom{i+j}{j} \frac{(m/2+k-n)_{i+j}\,(k-n+1)_{i+j}}{(m/2)_{i+j}\,(i+j)!} & , 0 \le k < n-1 \\ (-1)^i \sum_{j=0}^{k-n} \binom{i+j}{j} \frac{(n-k)_{i+j}\,(k-n+1)_{i+j}}{(m/2)_{i+j}\,(i+j)!} & , n < k \le 2n-1 \end{cases}$$

where $(a)_i = a(a+1)(a+2)...(a+i-1)$ is the Pochhammer's symbol.

3. **Definition 3.1** We will call the zeros of the polyharmonic operator to the functions having the form $(1-\rho^2)^\alpha\, \tau^{-(m+2\beta)}$ and satisfying the equation

$$\Delta_x^n \frac{(1-\rho^2)^\alpha}{\tau^{m+2\beta}} = 0$$

Lemma 3.1 . The functions

$$P_{n,k}(x,\xi) = \frac{(1-\rho^2)^k}{\tau^{m-2(n-k)}}, \quad k=0,1,2,...,2n-1 \qquad (3.1)$$

are zeros of the polyharmonic operator.

The proof of this Lemma is analogous to the one of Lemma 1 in [1,p.275]. We should remark that a common number of the zeros of the operator is equal to n^2+n .

Lemma 3.2 . For any integer k+0,1,2,....., the equality

$$\Delta_x^{k+1} \frac{(1-\rho^2)^{2k+1}}{\tau^{m+2k}} = 0 \tag{3.2}$$

is valid.

Proof. By virtue of Lemma 3.1, we have

$$\Delta_x^n \frac{(1-\rho^2)^k}{\tau^{m-2(n-k)}} = 0\ , \quad k=0,1,2,....,2n-1 \tag{3.3}$$

If we put k=2n-1 in (3.3), one obtains

$$\Delta_x^n \frac{(1-\rho^2)^{2n-1}}{\tau^{m+2(n-1)}} = 0 \tag{3.4}$$

and the equality holds for any n=1,2,.... . Now, by setting n=k+1, our assertion follows from (3.4).

From the Lemma XII.1 ([5], p. 530), we can establish "

Lemma 3.3 . If for some integer numbers N, n≥1, k≥0 the equality

$$\Delta_x^n \frac{(1-\rho^2)^k}{\tau^N} = 0$$

is valid, then for any integer ν≥0 one has

$$\Delta_x^{n+\nu} \frac{(1-\rho^2)^{k+\nu}}{\tau^N} = 0$$

Lemma 3.4 . Let the domain $D \subset E_m$ be bounded by the sufficiently smooth surface σ .Let the function a(x,ξ), x∈D, ξ∈σ satisfy the following conditions: (i) a(x,ξ) is positive, limited and continuous everywhere except for x=ξ, where it verifies

$$\lim_{D \ni x \to \xi \in \sigma} \frac{1}{|\sigma|} \int_\sigma a(x,\xi)\, d_\xi \sigma = 1$$

(ii) The equality

$$\lim_{D \ni x \to \xi_o \in \sigma} a(x,\xi) = 0$$

fulfils for all $\xi \in S(\xi_o)$, $S(\xi_o)$ being a sufficiently small neighbourhood of the point ξ_o.

Then, we have :

$$\lim_{D \ni x \to \xi_o \in \sigma} \frac{1}{|\sigma|} \int_\sigma a(x,\xi)\ \varphi(\xi)\ d_\xi \sigma = \varphi(\xi_o)\ ,$$

for any function $\varphi(\xi)$ continuous on the surface σ .

The proof of this lemma is analogous to the one of the assertion [6,p.228].

<u>**Corollary**</u> . If the first condition in Lemma 3.4 is replaced by

$$\lim_{D \ni x \to \xi \in \sigma} \frac{1}{|\sigma|} \int_\sigma a(x,\xi)\ d_\xi \sigma = \alpha$$

then, the following equality

$$\lim_{D \ni x \to \xi_o \in \sigma} \frac{1}{|\sigma|} \int_\sigma a(x,\xi)\ \varphi(\xi)\ d_\xi \sigma = \alpha\ \varphi(\xi_o)$$

is valid.

4. Consider the following boundary value problem : to find the function $u(x) \in C^{2n}(\Omega) \cap C^{n-1}(\overline{\Omega})$ satisfying the equation

$$\Delta^n\ u = 0 \tag{4.1}$$

and the boundary value problem

$$\frac{\partial^\mu\ u}{\partial \nu^\mu}\Big|_S = \varphi_\mu(\xi)\ ,\ \mu=0,1,2,...,n-1 \tag{4.2}$$

where ν is the external normal to S and the functions $\varphi_\mu(\xi) \in C(S)$.

Let us construct the function u(x) by the formula

$$u(x) = \sum_{k=n}^{2n-1} (A_{n,k}\ \Psi_{k-n})(x) \tag{4.3}$$

where

$$(A_{n,k}\ f)(x) = \frac{1}{|S|} \int_S P_{n,k}(x,\xi)\ f(\xi)\ d_\xi S\ , \tag{4.4}$$

the functions $P_{n,k}(x,\xi)$ being defined by (3.1). From the Lemma 3.4 follows

that the zeros of $P_{n,k}$ when k<n, can not be used in constructing solutions of the above boundary value problem since either these solutions or their derivatives will not satisfy the conditions (i) or (ii) of Lemma 3.4 .

Let us choose the functions $\Psi_k(x)$,k=0,1,...,n-1, so that the conditions

$$\sum_{k=n}^{2n-1} \frac{\partial^{\mu}}{\partial \nu^{\mu}} (A_{n,k} \Psi_{k-n})\big|_S = \varphi_{\mu}(\xi) \ , \ \mu=0,1,2,\dots,n-1 \qquad (4.5)$$

are satisfied.

The boundary conditions can be written in the form

$$\lim_{D\ni x\to\xi\in S} \sum_{k=n}^{2n-1} \frac{\partial^{\mu}}{\partial \nu^{\mu}} (A_{n,k} \Psi_{k-n}) = \varphi_{\mu}(\xi) \ , \ \mu=0,1,2,\dots,n-1$$

By virtue of Lemma 3.4 and its Corollary ,we obtain

$$\sum_{k=n}^{2n-1} \Psi_{k-n}(\xi) \lim_{D\ni x\to\xi\in S} \frac{\partial^{\mu}}{\partial \nu^{\mu}} B_k(x) = \varphi_{\mu}(\xi) \ , \ \mu=0,1,2,\dots,n-1$$

where

$$B_k(x) = (A_{n,k+n} 1)(x) = \frac{1}{|S|} \int_S \frac{(1-\rho^2)^{k+n}}{\tau^{m+2k}} d_{\xi}S \quad , k=0,1,\dots,n-1$$

Now, using the results of Section 2, it follows

$$B_k(x) = \begin{cases} \displaystyle\sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i} \rho^{2i} \ , \ k=0 \\ \displaystyle\sum_{i=0}^{k} b_i \sum_{j=0}^{n-1-i} (-1)^j \binom{n-1-i}{j} \rho^{2j} \ , \ k=1,2,\dots,n-1 \end{cases} \qquad (4.6)$$

where

$$b_i = \frac{1}{(m/2)_i \ i!} \sum_{j=0}^{k-i} (-1)^j \frac{(k+i+j)!}{j!(m/2+i)_j \ (k-i-j)!} \ , \ i=1,..,k; \ k=1,..,n-1, \qquad (4.7)$$

Inasmuch $\partial/\partial\nu=\partial/\rho$ on the spherical surface, then from (4.6) we have

$$\frac{\partial^{\mu}}{\partial^{\ \mu}} B_k(x) = \begin{cases} \displaystyle\sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i} \frac{(2i)!}{(2i-\mu)!} \rho^{2i-\mu} \ , \ k=0 \\ \displaystyle\sum_{i=0}^{k} b_i \sum_{j=0}^{n-1-i} (-1)^j \binom{n-1-i}{j} \frac{(2j)!}{(2j-\mu)!} \rho^{2j-\mu} \ , \ k=1,2,\dots,n-1 \end{cases} \qquad (4.8)$$

On the other hand, the point x tends to the point ξ of the spherical surface from within the sphere and therefore the radius of the sphere will be extended to 1 and then

$$\lim_{D\ni x\to\xi\in S} \frac{\partial^{\mu}}{\partial \nu^{\mu}} B_k(x) = \lim_{\rho\to 1} \frac{\partial^{\mu}}{\partial \rho^{\mu}} B_k(x) = \frac{\partial^{\mu}}{\partial \nu^{\mu}} B_k(x)$$

Hence, taking into account the formula 4.2.5.61 ([7], p. 620), we deduce from (4.8) ,for any $\mu=0,1,2,\dots,n-1$:

$$\frac{\partial^{\mu}}{\partial v^{\mu}} B_{k}\Big|_{S} = \mu! \begin{cases} (-1)^{n-1} \, 2^{2(n-1)-\mu} \binom{n-1}{\mu-n+1} & ,k=0 \\ \sum_{i=0}^{k} (-1)^{n-1-i} \, 2^{2(n-1-i)-\mu} \binom{n-1-i}{\mu-n+1+i} b_{i} & ,k=1,2,...,n-1 \end{cases} \tag{4.9}$$

where b_i are defined by formulas (4.7). By virtue of the binomial coefficients and (4.9), system (4.5) will take the form

$$\begin{cases} B^{(o)}_{n-1} \Psi_{n-1} = \alpha_{o} \varphi_{o} \\ B^{(1)}_{n-2} \Psi_{n-2} + B^{(1)}_{n-1} \Psi_{n-1} = \alpha_{1} \varphi_{1} \\ \dots\dots\dots \\ B^{(k-1)}_{n-k} \Psi_{n-k} + B^{(k-1)}_{n-k+1} \Psi_{n-k+1} + \dots + B^{(k-1)}_{n-1} \Psi_{n-1} = \alpha_{k-1} \varphi_{k-1} \\ \dots\dots\dots \\ B^{(n-1)}_{o} \Psi_{o} + B^{(n-1)}_{1} \Psi_{1} + \dots + B^{(n-1)}_{n-1} \Psi_{n-1} = \alpha_{n-1} \varphi_{n-1} \end{cases} \tag{4.10}$$

where

$$\begin{aligned} \alpha_{\mu} &= \frac{(-1)^{n-1}}{2^{2(n-1)-\mu} \, \mu!} , \quad \mu=0,1,...,n-1 \\ B^{(\mu)}_{o} &= \binom{n-1}{\mu-n+1} , \quad \mu=0,1,...,n-1 \\ B^{(\mu)}_{k} &= \sum_{i=0}^{k} (-1)^{i} \, 2^{-2i} \binom{n-1-i}{\mu-n+1+i} b_{i} \quad ,k=1,...,n-1;\mu=0,1,..,n-1 \end{aligned} \tag{4.11}$$

From (4.11), on account of the formula (4.7), it follows that the diagonal elements of system (4.10) are not equal to zero.

Solving system (4.10), what it is easily, we shall deduce $\Psi_k(\xi)$. By substituting these values of Ψ_k in (4.3) , we obtain the solution of the boundary value problem (4.1)-(4.2).

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