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ON PARTIAL QUASIBILATERAL GENERATING FUNCTION INVOLVING BESSEL POLYNOMIAL

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Abstract :

In [1], Mandal introduced the term 'Partial quasibilateral generating function' as follows

$$G(x,w,z)=\sum_{n=0}^{\infty}a_n w^n p_{m+n}^{(\alpha)}(x) q_r^{m+n}(z)$$

where $p_{m+n}^{(\alpha)}(x)$, $q_r^{(m+n)}(z)$ are two special functions of orders m+n, r and of parameters α, m+n and the coefficients are quite arbitrary. If $q_r^{(m+n)}(z)=p_r^{(m+n)}(z)$ then it is called partial quasi-bilinear. In the present paper we shall show that the existence of a partial quasi-bilinear generating function involving Bessel polynomial implies the existence of a more general generating function by means of one parameter group of continuous transformations.

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1. Introduction

In [2], the author has proved the following theorem on quasi bilinear generating function involving Bessel polynomial.

Theorem -1.

If

(1.1) $$G(x,u,w)=\sum_{n=0}^{\infty}a_n w^n Y_n^{(\alpha)}(x) Y_p^{(n)}(u)$$

then

(1.2) $$(1-wt)^{1-p}\left(1-wxy^{-1}z\right)^{1-\alpha}\exp\left(\beta wy^{-1}z\right)G\left(\frac{x}{1-wxy^{-1}z},\frac{u}{1-wt},\frac{wvzt}{(1-wt)\left(1-wxy^{-1}z\right)}\right)$$

$$=\sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{s=0}^{\infty}a_n\frac{w^{n+s}}{\ell!s!}\beta^{\ell}(p+n-1)_s\left(wy^{-1}z\right)^{\ell}(zv)^n t^{n+s}Y_{n+\ell}^{(\alpha-\ell)}(x)Y_p^{n+s}(u) \quad .$$

In the present paper we have extended the above mentioned theorem by using the definition of partial quasi bilinear generating function of Bessel polynomial.

Theorem -1.

If

(1.1) $$G(x,u,w)=\sum_{n=0}^{\infty}a_n w^n Y_n^{(\alpha)}(x)Y_p^{(n)}(u)$$

then

(1.2) $$(1-wt)^{1-p}\left(1-wxy^{-1}z\right)^{1-\alpha}\exp\left(\beta wy^{-1}z\right)G\left(\frac{x}{1-wxy^{-1}z},\frac{u}{1-wt},\frac{wvzt}{(1-wt)\left(1-wxy^{-1}z\right)}\right)$$

$$=\sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{s=0}^{\infty}a_n\frac{w^{n+s}}{\ell!s!}\beta^{\ell}(p+n-1)_s\left(wy^{-1}z\right)^{\ell}(zv)^n t^{n+s}Y_{n+\ell}^{(\alpha-\ell)}(x)Y_p^{n+s}(u)\ .$$

In the present paper we have extended the above mentioned theorem by using the definition of partial quasi bilinear generating function of Bessel polynomial.

Theorem -2

If there exists following partial quasi-bilinear generating function

(1.3) $$G(x,u,w)=\sum_{n=0}^{\infty}a_n w^n Y_{n+m}^{(\alpha)}(x)Y_p^{(m+n)}(u)$$

then the following more general generating function can be obtained

(1.4)

$$(1-wt)^{1-(p+m)}\left(1-wxy^{-1}z\right)^{1-(\alpha+m)}\exp\left(\beta xy^{-1}z\right)\times G\left(\frac{x}{1-wxy^{-1}z},\frac{u}{1-wt},\frac{wvzt}{(1-wt)\left(1-wxy^{-1}z\right)}\right)$$

$$=\sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{s=0}^{\infty}a_n\frac{w^{n+s}\beta^{\ell}}{\ell!s!}(p+m+n-1)_s\left(wy^{-1}z\right)^{\ell}(zv)^n t^{n+s}Y_{n+m+\ell}^{(\alpha-\ell)}(x)Y_p^{n+m+s}(u)$$

2. **Proof of the Theorem - 2**

We now define the following operators

(2.1) $$R_1=x^2y^{-1}z\frac{\partial}{\partial x}+xz\frac{\partial}{\partial y}+xy^{-1}z^2\frac{\partial}{\partial z}+(\beta+mx-x)y^{-1}z$$

and

(2.2) $$R_2=ut\frac{\partial}{\partial u}+t^2\frac{\partial}{\partial t}+(p+m-1)t$$

such that

(2.3) $$R_1\left(Y_{n+m}^{(\alpha)}(x)y^{\alpha}z^n\right)=\beta Y_{n+m+1}^{(\alpha-1)}(x)y^{\alpha-1}z^{n+1}$$

(2.4) $$R_2\left(Y_p^{(n+m)}(u)t^n\right)=(p+n+m-1)Y_p^{m+n+1}(u)t^{n+1}$$

and

(2.5) $$\exp(wR_1)\,f(x,y,z)=\left(1-wxy^{-1}z\right)\exp\left(\beta wy^{-1}z\right)\times f\left(\frac{x}{1-wxy^{-1}z},\frac{y}{1-wxy^{-1}z},\frac{z}{1-wxy^{-1}z}\right)$$

(2.6) $$\exp(wR_2)\,f(u,t)=(1-wt)^{1-(p+m)}f\left(\frac{u}{1-wt},\frac{t}{1-wt}\right)$$

Firstly, we consider the following generating relation

$$G(x,u,w)=\sum_{n=0}^{\infty} a_n w^n Y_{n+m}^{(\alpha)}(x) Y_p^{(n+m)}(u) \tag{2.7}$$

Replacing w by wvtz and then multiplying both sides of (2.7) by y^α, we get

$$y^\alpha G(x,u,wvzt)=\sum_{n=0}^{\infty} a_n (wv)^n \left(Y_{n+m}^{(\alpha)}(x) y^\alpha z^n\right)\left(Y_p^{(m+n)}(u) t^n\right) \tag{2.8}$$

Operating $e^{wR_1}e^{wR_2}$ on both sides of (2.8), we obtain

$$e^{wR}\, e^{wR_2}\left(y^\alpha G(x,u,wvzt)\right) \tag{2.9}$$

$$= e^{wR_1}e^{wR_2}\left[\sum_{n=0}^{\infty} a_n (wv)^n \left(Y_{n+m}^{(\alpha)}(x) y^\alpha z^n\right)\left(Y_p^{m+n}(u) t^n\right)\right]$$

The left member of (2.9), with the help of (2.5) and (2.6), becomes

$$e^{wR_1}e^{wR_2}\left[y^\alpha G(x,u,wvzt)\right] \tag{2.10}$$

$$= e^{wR_1}\left[y^\alpha (1-wt)^{1-(p+m)} G\left(x, \frac{u}{1-wt}, \frac{wvzt}{1-wt}\right)\right]$$

$$= y^\alpha (1-wt)^{1-(p+m)} \left(1-wxy^{-1}z\right)^{1-(\alpha+m)} \exp\left(\beta w y^{-1} z\right)$$

$$\times G\left(\frac{x}{1-wxy^{-1}z}, \frac{u}{1-wt}, \frac{wzvt}{(1-wt)\left(1-wxy^{-1}z\right)}\right).$$

On the other hand the right member of (2.9), with the help of (2.3) and (2.4), becomes

$$e^{wR_1}e^{wR_2}\left[\sum_{n=0}^{\infty} a_n (wv)^n \left(Y_{n+m}^{(\alpha)}(x) y^\alpha z^n\right)\left(Y_p^{(m+n)}(u) t^n\right)\right] \tag{2.11}$$

$$= \sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{s=0}^{\infty} a_n (wv)^n \frac{w^\ell}{\ell!}\cdot\frac{w^s}{s!}\beta^\ell \cdot Y_{m+n+\ell}^{(\alpha-\ell)}(x) y^{\alpha-\ell} z^{n+\ell} t^{n+s} \times (p+m+n-1)_s\, Y_p^{(m+n+s)}(u).$$

Equating (2.10) and (2.11), we obtain

$$(1-wt)^{1-(p+m)}\left(1-wxy^{-1}z\right)^{1-(\alpha+m)} \exp\left(\beta w y^{-1} z\right) \times G\left(\frac{x}{1-wxy^{-1}z}, \frac{u}{1-wt}, \frac{wvzt}{(1-wt)\left(1-wxy^{-1}z\right)}\right) \tag{2.12}$$

$$= \sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{s=0}^{\infty} a_n \frac{(zv)^n w^{n+s}}{\ell!\, s!} \beta^\ell \left(wy^{-1}z\right)^\ell (p+m+n-1)_s\, t^{n+s} Y_{m+n+\ell}^{(\alpha-\ell)}(x) Y_p^{(m+n+s)}(u).$$

which is our desired result.

Special Cases :

a) Putting m=0 in the above mentioned result (2.12), we obtain

Result - 1

If

$$G(x,u,w)=\sum_{n=0}^{\infty} a_n w^n Y_n^{(\alpha)}(x) Y_p^{(n)}(u)$$

then

$$(1-wt)^{1-p}\left(1-wxy^{-1}z\right)^{1-\alpha}\exp\left(\beta wy^{-1}z\right)G\left(\frac{x}{1-wxy^{-1}z},\frac{u}{1-wt},\frac{wvzt}{(1-wt)\left(1-wxy^{-1}z\right)}\right)$$

$$=\sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{s=0}^{\infty}a_n\frac{w^{n+s}}{\ell!s!}\beta^{\ell}(p+n-1)_s\left(wy^{-1}z\right)^{\ell}(zv)^n t^{n+s}Y_{n+\ell}^{(\alpha-\ell)}(x)Y_p^{(n+s)}(u).$$

which is theorem - 1

b) Putting p=0, y=z=t=1 and setting $Y_0^{n+s}(u)=1$ in the above mentioned result (2.12), we get the following result :

Result -2.

$$(1-w)^{1-p}(1-wx)^{1-\alpha}\exp(\beta x)G\left(\frac{x}{1-wx},\frac{wv}{(1-w)(1-wx)}\right)$$

$$=\sum_{n=0}^{\infty}\sum_{s=0}^{\infty}\sum_{\ell=0}^{\infty}a_n(wv)^n\frac{w^s}{s!}(n-1)_s\frac{(w\beta)^{\ell}}{\ell!}Y_{n+\ell}^{(\alpha-\ell)}(x)$$

$$=\sum_{\ell=0}^{\infty}\frac{(w\beta)^{\ell}}{\ell!}\sum_{n=0}^{\infty}\sum_{s=0}^{\infty}a_n\frac{w^s}{s!}(wv)^n(n-1)_s Y_{n+\ell}^{(\alpha-\ell)}(x)$$

$$=\exp(\beta w)\sum_{n=0}^{\infty}\sum_{s=0}^{\infty}a_n\frac{w^s}{s!}(wv)^n(n-1)_s Y_{n+\ell}^{(\alpha-\ell)}(x)$$

$$\therefore\ (1-w)^{1-p}(1-wx)^{1-\alpha}\exp\{\beta(x-w)\}G\left(\frac{x}{1-wx},\frac{wv}{(1-w)(1-wx)}\right)$$

$$=\sum_{n=0}^{\infty}\sigma_n(x,v)w^n$$

where

$$\sigma_n(x,v)=\sum_{s=0}^{n}a_n\frac{w^s}{s!}(n-1)_s Y_{n+\ell}^{(\alpha-\ell)}(x)v^n$$

which is found derived in [3].

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