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AN EXTENSION OF BILINEAR GENERATING FUNCTION OF CERTAIN SPECIAL FUNCTION - III

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Abstract: In this paper we derive some more general generating functions of ${}_2F_1(-n,\beta;v;x)$ from the existence of quasi bilinear generating function involving the special functions under consideration from the Lie group view point.

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1. Introduction : In [1], Chatterjea and Chakraborty defined the quasi-bilateral generating function as follows :

(1.1) $$G(x,u,w)=\sum_{n=0}^{\infty} a_n p_n^{(a)}(x) q_m^{(n)}(u) w^n$$

where $P_n^{(\alpha)}(x), q_m^{(n)}(u)$ are two special functions of order n, m and of parameters α, n respectively.

Here we have obtained some more general generating functions of ${}_2F_1(-n,\beta;v;x)$ from the existence of a quasi bilinear generating function by group-theoretic method [4]. Some particular eases of interest are also pointed out.

Theorem: If there exists a generating relation of the form

(1.2) $$G(x,\xi,w)=\sum_{n=0}^{\infty} a_n \, {}_2F_1(-n,\beta;v;x)\, {}_2F_1(-m,n;v;\xi) w^n$$

then

(1.3) $$\left(1+\frac{wy}{zu}(1-x)\right)^{v-1}\left(1-\frac{wxy}{zu}\right)^{\beta-v}$$

$$\times G\left(x(1+\frac{wy}{zu}(1-x),\frac{\xi}{1-w\eta},\frac{wy\eta v}{(1-w\eta)(1-\frac{wxy}{zu})}\right)$$

$$=\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_n \frac{w^{n+p+q}}{p!q!}(yv)^n (n)_p (-1)^q (-v+1)_q \left(\frac{y}{zu}\right)^q$$
$$\times {}_2F_1(-(n+q),\beta-q;v-q;x)\, {}_2F_1(-m,n+p;v;\xi).$$

The importance of our theorem lies in the fact that one can get a large number of bilinear generating relations from (1.3) by attributing different suitable values to a_n in (1.2).

2. Proof of the theorem: We now consider the following two linear partial differential operators [2,3]

(2.1) $$R_1 = x(1-x)yz^{-1}u^{-1}\frac{\partial}{\partial x}+xy^2z^{-1}u^{-1}\frac{\partial}{\partial y}-xyu^{-1}\frac{\partial}{\partial z}+yz^{-1}\frac{\partial}{\partial u}-(1-x)yz^{-1}u^{-1}$$

and

(2.2) $$R_2=\xi\eta\frac{\partial}{\partial\xi}+\eta^2\frac{\partial}{\partial\eta}$$

such that

(2.3) $$R_1\left(y^n z^{\beta} u^{v}\, {}_2F_1(-n,\beta;v,x)\right)$$
$$=(v-1)y^{n+1}z^{\beta-1}u^{v-1}\, {}_2F_1(-(n+1),\beta-1;v-1;x)$$

and

(2.4) $$R_2\left(\eta^n\, {}_2F_1(-m,n;v,\xi)\right)=n\eta^{n+1}\, {}_2F_1(-m,n+1;v,\xi)$$

and also

(2.5) $$e^{wR_1} f(x,y,z,u)=\frac{zu}{zu+wy(1-x)}$$

$$\times f\left(\frac{(zu+wy(1-x))x}{zu},\frac{y\eta u}{zu-wxy},\frac{zu-wxy}{zu},\frac{u(zu+wy(1-x))}{zu-wxy}-\right)$$

and

(2.6) $$e^{wR_2} f(\xi,\eta)=f\left(\frac{\xi}{1-\omega\eta},\frac{\eta}{1-\omega\eta}\right).$$

Now we consider the following quasi bilinear generating relation.

(2.7) $$G(x,\xi,w)=\sum_{n=0}^{\infty} a_n\, {}_2F_1(-n,\beta;v;x)_2F_1(-m,n;v;\xi)w^n.$$

Replacing w by $w\,y\,\eta\,v$ and then multiplying both sides of (2.7) by $z^\beta u^v$ we get

(2.8) $$z^\beta u^v G(x,\xi,wy\eta v)=\sum_{n=0}^{\infty} a_n\left(y^n z^\beta u^v\, {}_2F_1(-n,\beta,v,x)\eta^n\, {}_2F_1(-m,n;v;\xi)\right)(wv)^n.$$

Operating $e^{wR_1}e^{wR_2}$ on both sides of (2.8), we obtain

(2.9) $$e^{wR_1}e^{wR_2}\left(z^\beta u^v G(x,\xi,wy\eta v)\right)$$

$$=e^{wR_1}e^{wR_2}\left(\sum_{n=0}^{\infty} a_n\left(y^n z^\beta u^v\, {}_2F_1(-n,\beta,v,x)\right)\left(\eta^n\, {}_2F_1(-m,n;v;\xi)\right)(wv)^n\right).$$

Now the left member of (2.9) becomes

(2.10) $$e^{wR_1}e^{wR_2}\left(z^\beta u^v G(x,\xi,wy\eta v)\right)=e^{wR_1}\left[z^\beta u^v G\left(x,\frac{\xi}{1-w\eta},\frac{wy\eta v}{1-w\eta}\right)\right]$$

$$=z^\beta u^v\left(1+\frac{wy}{zu}(1-x)\right)^{v-1}\left(1-\frac{wxy}{zu}\right)^{\beta-v}$$

$$\times G\left(\left(x\left(1+\frac{wy}{zu}(1-x)\right),\frac{\xi}{1-w\eta},\frac{wy\eta v}{(1-w\eta)\left(1-\frac{wxy}{zu}\right)}\right)\right).$$

On the other hand the right member of (2.9) becomes

(2.11) $$e^{wR_1}e^{wR_2}\left[\sum_{n=0}^{\infty} a_n\left(y^n z^\beta u^v\, {}_2F_1(-n,\beta,v,x)\right)\left(\eta^n\, {}_2F_1(-m,n;v;\xi)\right)(wv)^n\right]$$

$$= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_n (wv)^n \frac{w^{p+q}}{p!q!}(n)_p(-1)^q(-v+1)_q \eta^{n+p} y^{n+q} z^{\beta-q} u^{v-q}$$

$$\times {}_2F_1(-(n+q),\beta-q;v-q;x)\,{}_2F_1(-m,n+p;v;\xi),$$

Equating (2.10) and (2.11) and then simplifying we get

$$(2.12) \qquad \left(1+\frac{wy}{zu}(1-x)\right)^{v-1}\left(1-\frac{wxy}{zu}\right)^{\beta-v}$$

$$\times G\left(x\left(1+\frac{wy}{zu}(1-x)\right),\frac{\xi}{1-w\eta},\frac{wy\eta v}{(1-w\eta)\left(1-\frac{wxy}{zu}\right)}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_n \frac{w^{n+p+q}}{p!q!}(yv)^n (n)_p(-1)^q(-v+1)_q$$

$$\times\left(\frac{y}{zu}\right)^q {}_2F_1(-(n+q),\beta-q;v-q;x)\,{}_2F_1(-m,n+p;v;\xi),$$

which is our desired result.

3. Particular Case: Putting m = 0, y = z = u = η = 1, in the above mentioned result (2.12), we have

$$(3.1) \qquad \{1+w(1-x)\}^{v-1}(1-wx)^{\beta-v} G\left(x(1+w(1-x),\frac{wv}{(1-w)(1-wx)}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_n \frac{w^{n+p+q}}{p!q!}(n)_p(-1)^q(-v+1)_q\,{}_2F_1(-(n+q),\beta-q;v-q;x)v^n$$

$$= \sum_{p=0}^{\infty}\frac{w^p(n)_p}{p!}\sum_{n=0}^{\infty}\sum_{q=0}^{\infty} a_n(-1)^q w^{n+q}\frac{(-v+1)_q}{q!}\,{}_2F_1(-(n+q),\beta-q;v-q;x)v^n$$

$$= (1-w)^{-n}\sum_{n=0}^{\infty} w^n \sum_{q=0}^{\infty} a_{n-q}\frac{(-1)^q(-v+1)_q}{q!}\,{}_2F_1(-n,\beta-q;v-q;x)v^{n-q}.$$

Simplifying the above result finally we get.

$$(3.2) \qquad \{1+w(1-x)\}^{v-1}(1-w)^n(1-wx)^{\beta-v} G\left(x(1+w(1-x),\frac{wv}{(1-w)(1-wx)}\right)$$

$$= \sum_{n=0}^{\alpha} w^n \sigma_n(x,v)$$

where

$$\sigma_n(x,v) = \sum_{q=0}^{n} a_q \frac{(-1)^{n-q}(-v+1)_{n-q}}{(n-q)!}\,{}_2F_1(-n,\beta-n+q;v-n+q;x)v^q.$$

which is believed to be new.

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