Rev.Acad.Canar.Cienc., IV (núms. 1 y 2), 111-124 (1992)

PADE-TYPE APPROXIMANTS FOR A FORMAL LAURENT SERIES

M. Camacho and P. González-Vera
Departamento de Análisis Matemático
Universidad de La Laguna

ABSTRACT

In this paper, we obtain the Laurent Padé Approximats (LPA) to a given formal Laurent series, as a Laurent Padé-Type Aproximant (LPTA) of higher order, in a similar way as the process carried out by Brezinski ([1]), for the classic case. For this purpose a new concept of Padé-Type Approximant to a Laurent series is introduced.

Keywords: Padé-Type Approximants, Formal Laurent series, Laurent polynomials.
AMS Classification: 40G; 41

1. PRELIMINARY CONSIDERATIONS.

Let $\{c_k\}_{-\infty}^{\infty}$ be a bi-infinite sequence of complex numbers, and L the set of all the formal series G(z) of the form,

$$G(z) = \sum_{-\infty}^{\infty} c_k z^k \tag{1.1}$$

We shall denote by Π_n the linear space of the polynomials of degree n at most, and by $\Pi = \bigcup \Pi_n$ the space of all polynomials. For every pair of integers p,q with $p \le q$, we shall denote by $\Lambda_{p,q}$ the linear space of the Laurent polynomials (L-polynomials) or functions of the form

$$L(z) = \sum_{j=p}^{q} \alpha_j z^j$$

and $\Lambda = \bigcup \Lambda_{p,q}$ the linear space of all L-polynomials (obviously $\Pi_n = \Lambda_{0,n}$).

For each $p,q \in \mathbb{Z}$ with $p \le q$, the projection operator is defined by

$$\Pi_{p:q} G(z) = \sum_{j=p}^{q} c_j z^j \tag{1.2}$$

For a formal Laurent series, if the nonzero coefficients extend only to one side, which can be $+\infty$ or $-\infty$, it is necessary to have a notation to indicate which part of the formal Laurent series is zero. So, the notations O_+ and O_- need to be defined as follows

$$G(z) = O_+(z^m) \Leftrightarrow \Pi_{-\infty:m-1} G(z) = 0 \Leftrightarrow G(z) = \sum_{j=m}^{\infty} c_j z^j$$

$$G(z) = O_-(z^m) \Leftrightarrow \Pi_{m+1:\infty} G(z) = 0 \Leftrightarrow G(z) = \sum_{-\infty}^{m} c_j z^j$$

being $G(z) \in L$.

As a natural extension of the classical concept of Padé Approximant to a formal power series, in [5] the so-called Laurent Padé Approximants (LPA) to a formal Laurent series have been defined. On the other hand Bultheel, in [2] makes an algorithmic and algebraic study for such those approximations. Indeed, splitting G(z) in two, that is $G(z) = G^{(0)}(z) + \hat{G}^{(0)}(z)$, being

$$G^{(0)}(z) = c_0/2 + \sum_{k=1}^{\infty} c_k z^k \tag{1.3a}$$

$$\hat{G}^{(0)}(z) = c_0/2 + \sum_{k=1}^{\infty} c_{-k} z^{-k} \tag{1.3b}$$

we have first the following

Definition 1 ([2], p.26)

Let m and n be nonnegative integers and the rational functions $R(z) = \dfrac{P_m(z)}{Q_n(z)}$;

$\hat{R}(z) = \dfrac{\hat{P}_m(z)}{\hat{Q}_n(z)}$, where $P_m(z)$, $Q_n(z)$, $\hat{P}_m(z)$, $\hat{Q}_n(z) \in \Lambda$, such that

1) $Q_n(z) \in \Lambda_{0,n}$ and $\hat{Q}_n(z) \in \Lambda_{-n,0}$

2) $G^{(0)}(z) - R(z) = O_+(z^{m+n+1})$

3) $\hat{G}^{(0)}(z) - \hat{R}(z) = O_-(z^{-(m+n+1)})$

4) $R(z) + \hat{R}(z)$ is equivalent with the quotient of L-polynomials P(z)/Q(z), with $P(z) \in \Lambda_{-m,m}$ and $Q(z) \in \Lambda_{-n,n}$. We shall then call the pair $(R(z),\hat{R}(z))$ a Laurent Padé Approximant (LPA) of type (m,n).

These approximants are obviously an extension of the classic Padé Approximants because the Laurent expansion of $R(z)+\hat{R}(z)$ matches the coefficients of z^k in L for $|k| \le m+n$.

On the other hand, as is well known, Padé-Type Approximants (PTA) are an special interesting case in the study of the rational approximations, because of the free election of the denominator mean, that a lesser number of coefficients in the initial series is required to be known. Such approximants used to provide, in some cases, better estimations than Padé Approximants (see [1] for more details).

Thus, we can also consider rational approximants to F(z) with a given denominator in the sense of the following

Definition 2

Let m and n be nonnegative integers and $Q_n(z) \in \Lambda_{0,n}$, $\hat{Q}_n(z) \in \Lambda_{-n,0}$ two given L-polynomials. The pair $(R(z),\hat{R}(z))$ where $R(z)=\dfrac{P_m(z)}{Q_n(z)}$; $\hat{R}(z)= \dfrac{\hat{P}_m(z)}{\hat{Q}_n(z)}$,is a Laurent Padé-Type Approximant (LPTA) of type (m,n) for F(z) if the following conditions are satisfied

1) $P_m(z), \hat{P}_m(z) \in \Lambda$

2) $G^{(0)}(z) - R(z) = O_+(z^{m+1})$

3) $\hat{G}^{(0)}(z) - \hat{R}(z) = O_-(z^{-(m+1)})$

4) $R(z) + \hat{R}(z)$ is equivalent with $\tilde{P}(z)/\tilde{Q}(z)$, where $\tilde{P}(z) \in \Lambda_{-m,m}$ and $\tilde{Q}(z) \in \Lambda_{-n,n}$.

$\tilde{Q}(z)$ will be called the generating L-polynomial of the approximant. Such approximants have been studied by us from an algebraic point of view (see [3], [4]).

In this paper, we shall obtain the LPA as a LPTA of higher order, making use of an analogous process followed in [1] for the classic case. For this purpose it is necessary to define the concept of Padé-Type Approximant to a Laurent series in a similar way as that used by Bultheel in [2].

2. PADE-TYPE APPROXIMANTS TO A FORMAL LAURENT SERIES.

In ([2], p.25) the concept of Padé Approximant for a Laurent series was defined as follows

Definition 3

Let m and n be integers and n≥0, then the pair $(A_m(z)/B_n(z))$ with $B_n(z) \in \Pi_n$ and $A_m(z) = O(z^m)$ is a Padé Approximant of type (m,n) for the formal Laurent series (1.1) and we shall denote by PA_L, if

$$G(z)B_n(z) - A_m(z) = O_+(z^{m+n+1}) \;; B_n(0) \neq 0 \tag{2.1}$$

Observe that $A_m(z)/B_n(z)$ should be understood as a notation for a pair of elements rather as a quotient.

In the particular case in which $c_k = 0$ for k=-1,-2,... (that is, G(z) is a formal power series) and m≥0 then $A_m(z) \in \Pi_m$ and the pair $(A_m(z)/B_n(z))$ coincides with the classic [m/n] PA for G(z).

It can be easily seen that a necessary and sufficient condition for the existence of $A_m(z)$ and $B_n(z)$ (except a multiplicative factor) is $|T_n^{(m)}| \neq 0$, $m \in \mathbb{Z}$; $n \in \mathbb{N}$, $T_n^{(m)}$ being the Toeplitz matrices represented by

$$T_n^{(m)} = \left(c_{m+i-j}\right)_{i,j=0}^{n} = \begin{bmatrix} c_m & c_{m-1} & \cdots\cdots & c_{m-n} \\ c_{m+1} & c_m & \cdots\cdots & c_{m-n+1} \\ \vdots & \vdots & & \vdots \\ c_{m+n} & c_{m+n-1} & \cdots\cdots & c_m \end{bmatrix}$$

In such case it will be said that G(z) is a normal series.

Since we have only taken into account in the correspondence conditions the increasing character (i.e. toward +∞) we might wonder if an alternative definition with decreasing correspondence conditions (toward −∞) could be given. In this way, the rational approximants defined in (2.1) will be called Padé Approximants in an increasing sense. Thus, to define the PA_L in decreasing sense the so-called dual series F(z) to G(z) will be required. Thus we have

$$F(z) = \sum_{-\infty}^{\infty} d_k z^k \quad \text{with } d_k = c_{-k}, \; k \in \mathbb{Z} \; (F(z) = G(z^{-1}))$$

and if we denoted by $S_n^{(m)} = \left(d_{m+i-j}\right)_{i,j=0}^{n} = \left(c_{-m-i+j}\right)_{i,j=0}^{n}$ we have first the following

Proposition 1

G(z) is normal iff F(z) is normal too.

Proof.

One has

$$\left(S_n^{(m)}\right)^T = \left(d_{m+j-i}\right)_{i,j=0}^{n} = T_n^{(-m)}, \text{ henceforth } \left|S_n^{(m)}\right| = \left|\left(S_n^{(m)}\right)^T\right| = \left|T_n^{(-m)}\right|;$$

$$m = 0, \pm 1, \pm 2, \ldots; \; n = 0, 1, 2, \ldots$$

and the proof follows.■

Let $(\bar{A}_m(z)/\bar{B}_n(z))$ be the PA_L of type (m,n) in increasing sense for F(z), with

$$\bar{A}_m(z) = O_-(z^m) = \sum_{-\infty}^{m} \bar{a}_j z^j \text{ and } \bar{B}_n(z) = \sum_{j=0}^{n} \bar{b}_j z^j; \; \bar{B}_n(0) = \bar{b}_0 \neq 0$$

then, by (2.1) we have

$$F(z)\bar{B}_n(z) - \bar{A}_m(z) = O_+(z^{m+n+1})$$

Thus,

$$F(z^{-1})\bar{B}_n(z^{-1}) - \bar{A}_m(z^{-1}) = O(z^{-(m+n+1)})$$

and consequently if we write

$$\tilde{B}_n(z) = \bar{B}_n(z^{-1}) \text{ and } \tilde{A}_m(z) = \bar{A}_m(z^{-1}) = O_+(z^{-m})$$

it follows that

$$G(z)\tilde{B}_n(z) - \tilde{A}_m(z) = O_-(z^{-(m+n+1)})$$

The pair $(\tilde{A}_m(z)/\tilde{B}_n(z))$ it will be called the PA_L in decreasing sense of type (m,n) for G(z) the following immediately holds

Proposition 2

The pair $(\tilde{A}_m(z^{-1})/\tilde{B}_n(z^{-1}))$ coincides with the PA_L in increasing sense for the dual series F(z).

Proof.

It is enough to take into account the unicity property of the PA_L for G(z).■

In [2], connection between the LPA and PA_L (see [2], Theorems 4.2 and 4.3) is exhibited. On the other hand, if $(R(z),\hat{R}(z))$ represents the LPA of type (m,n) for G(z) and $(S(z),\hat{S}(z))$ is the LPA of type (m,n) for the dual series F(z), then we have

Proposition 3
$(\hat{R}(z^{-1}),R(z^{-1}))$ coincides with $(S(z),\hat{S}(z))$

Proof.

Since $(R(z),\hat{R}(z))$ is the LPA for G(z), and from conditions 2 and 3 in the Definition 1 we have,

$$G^{(0)}(z^{-1}) - R(z^{-1}) = O_+((z^{-1})^{m+n+1}) = O_-(z^{-(m+n+1)}) \quad (2.2a)$$

$$\hat{G}^{(0)}(z^{-1}) - \hat{R}(z^{-1}) = O_-((z^{-1})^{-(m+n+1)}) = O_+(z^{m+n+1}) \quad (2.2b)$$

Now, taking $F(z) = F^{(0)}(z) + \hat{F}^{(0)}(z)$, with

$$F^{(0)}(z) = c_0/2 + \sum_{k=1}^{\infty} c_{-k}z^k \quad \text{and} \quad \hat{F}^{(0)}(z) = c_0/2 + \sum_{k=1}^{\infty} c_k z^{-k}, \text{ we have}$$

$G^{(0)}(z^{-1})=\hat{F}^{(0)}(z)$ and $\hat{G}^{(0)}(z^{-1})=F^{(0)}(z)$ and replacing in (2.2) one has that the pair $(\hat{R}(z^{-1}),R(z^{-1}))$ constitutes the LPA for the dual series F(z).■

From the two last propositions we can conclude that it does not matter which definition of PA_L (increasing or decreasing) can be taken. We shall consider throughout the paper the PA_L in the increasing sense.

Now, as a new extension of the classical concept of Padé-Type Approximant we shall define this one for a Laurent series, as follows

Definition 4
Let m and n be integers with n≥0, and the polynomial $B_n(z) \in \Pi_n$ with $B_n(0)\neq 0$. The pair $(A_m(z)/B_n(z))$, being $A_m(z) = O_-(z^m)$, is said to be a Padé-Type Approximant of type (m,n) for the formal Laurent series (1.1) with generator polynomial $B_n(z)$ and we shall represent it by PTA_L, if

$$G(z)B_n(z) - A_m(z) = O_+(z^{m+1}) \; ; \quad (2.3)$$

It is readily seen that the coefficients of $A_m(z)$ can be obtained from the coefficients of the series and from the generator polynomial. Indeed, if we write $A_m(z) = \sum_{-\infty}^{m} a_j z^j$ and $B_n(z) = \sum_{j=0}^{n} b_j z^j$ by (2.3), then the coefficients are given by

$$a_j = \sum_{k=0}^{n} c_{j-k} b_k, \qquad j=\ldots,m-2,m-1,m \quad (2.4)$$

When m≥0 and G(z) is a formal power series (c_k=0 if k≤-1), then by (2.4), one has, a_j=0 for j≤-1, that is, $A_m(z) \in \Pi_m$ and the pair $(A_m(z)/B_n(z))$ takes sense as a quotient, and represents the (m/n) PTA for the formal power series

$G(z)$ with denominator $B_n(z)$.

On the sequel we shall show how an LPTA can be deduced from two PTA_L. For that, we shall use a more general decomposition than (1.3) for $G(z)$. So if $m \in \mathbb{Z}$, then we write

$$G(z) = G^{(m)}(z) + \hat{G}^{(m)}(z) \tag{2.5a}$$

with $G^{(m)}(z) = 1/2c_m z^m + \sum_{k=m+1}^{\infty} c_k z^k$ and $\hat{G}^{(m)}(z) = 1/2c_m z^m + \sum_{-\infty}^{m-1} c_k z^k$ (2.5b)

Theorem 1.

a) Let $(P_n^{(m)}(z)/Q_n^{(m)}(z))$ be a PTA_L of type $(m+n,n)$ for the normal series (1.1) being $Q_n^{(m)}(z) \in \Pi_n$ its generator polynomial such that $Q_n^{(m)}(0) = 1$, and define $A_n^{(m)}(z)$ by

$$z^m A_n^{(m)}(z) = P_n^{(m)}(z) - \hat{G}^{(m)}(z) Q_n^{(m)}(z) \tag{2.6a}$$

Then $A_n^{(m)}(z)$ is a polynomial of degree n and $R_n^{(m)}(z) = \dfrac{A_n^{(m)}(z)}{Q_n^{(m)}(z)}$ is a (n/n) PTA to the formal series $z^{-m}G^{(m)}(z)$.

b) If $(\hat{P}_n^{(m)}(z)/\hat{Q}_n^{(m)}(z))$ is a PTA_L of type $(m-1,m)$ for $G(z)$ with generator polynomial $\hat{Q}_n^{(m)}(z)$, then $\hat{R}_n^{(m)}(z) = \dfrac{z^{-n}\hat{A}_n^{(m)}(z)}{z^{-n}\hat{Q}_n^{(m)}(z)}$ represents a (n/n) PTA to the formal power series $z^{-m}\hat{G}^{(m)}(z)$. Now $\hat{A}_n^{(m)}(z)$ is given by

$$z^m \hat{A}_n^{(m)}(z) = \hat{G}^{(m)}(z)\hat{Q}_n^{(m)}(z) - \hat{P}_n^{(m)}(z) \tag{2.6b}$$

Proof.

a) If we denote by $E_n^{(m)}$ the residual of the PTA_L $(P_n^{(m)}(z)/Q_n^{(m)}(z))$, then

$$E_n^{(m)}(z) = G(z)\, Q_n^{(m)}(z) - P_n^{(m)}(z) = O_+(z^{m+n+1})$$

hence, by (2.5),

$$E_n^{(m)}(z) = [G^{(m)}(z) + \hat{G}^{(m)}(z)]\, Q_n^{(m)}(z) - P_n^{(m)}(z)$$

and consequently,

$$G^{(m)}(z)\, Q_n^{(m)}(z) - E_n^{(m)}(z) = P_n^{(m)}(z) - \hat{G}^{(m)}(z)\, Q_n^{(m)}(z) \tag{2.7a}$$

Now, taking into account that $P_n^{(m)}(z) = O_+(z^{m+n+1})$, $G^{(m)}(z) = O_+(z^m)$, the left-hand side of (2.7a) is $O_+(z^m)$, meanwhile the right-hand side is $O_-(z^{m+n})$, thus (2.7a) represents a polynomial of degree m+n, denoted by $z^m A_n^{(m)}(z)$, with $A_n^{(m)}(z) \in \Pi_n$, hence

$$z^m A_n^{(m)}(z) = G^{(m)}(z)\, Q_n^{(m)}(z) - E_n^{(m)}(z) \tag{2.7b}$$

and

$$G^{(m)}(z)\,Q_n^{(m)}(z) - z^m A_n^{(m)}(z) = E_n^{(m)}(z) = O_+(z^{m+n+1}). \qquad (2.8)$$

After dividing by $z^m Q_n^{(m)}(z)$ we get

$$z^{-m}G^{(m)}(z) - \frac{A_n^{(m)}(z)}{Q_n^{(m)}(z)} = O_+(z^{n+1}) \qquad (2.9)$$

which proves that $\dfrac{A_n^{(m)}(z)}{Q_n^{(m)}(z)}$ is an (n/n) PTA to the series $z^{-m}G^{(m)}(z)$

b) Let now $(\hat{P}_n^{(m)}(z)/\hat{Q}_n^{(m)}(z))$ be a PTA_L for G(z) of type (m-1,n), and denote by $\hat{E}_n^{(m)}(z)$ the residual, then we have

$$\hat{E}_n^{(m)}(z) = [G^{(m)}(z) + \hat{G}^{(m)}(z)]\,\hat{Q}_n^{(m)}(z) - \hat{P}_n^{(m)}(z) = O_+(z^m). \qquad (2.10a)$$

Hence,

$$\hat{E}_n^{(m)}(z) - G^{(m)}(z)\,\hat{Q}_n^{(m)}(z) = \hat{G}^{(m)}(z)\,\hat{Q}_n^{(m)}(z) - \hat{P}_n^{(m)}(z)$$

being the left-hand side $O_+(z^m)$ and the right-hand side $O_-(z^{m+n})$, and therefore

$$z^m\hat{A}_n^{(m)}(z) = \hat{G}^{(m)}(z)\,\hat{Q}_n^{(m)}(z) - \hat{P}_n^{(m)}(z) \qquad (2.10b)$$

with $\hat{A}_n^{(m)}(z) \in \Pi_n$. Consequently

$$\hat{G}^{(m)}(z)\,\hat{Q}_n^{(m)}(z) - z^m\hat{A}_n^{(m)}(z) = \hat{P}_n^{(m)}(z) = O_-(z^{m-1}) \qquad (2.11)$$

and

$$z^{-m}\hat{G}^{(m)}(z) - \frac{z^{-n}\hat{A}_n^{(m)}(z)}{z^{-n}\hat{Q}_n^{(m)}(z)} = O_-(z^{-n-1}) \qquad (2.12)$$

and the proof follows.■

For the sake of simplicity, if we set m=0 then one has

Corollary 1

Under the same conditions as in Theorem 1, the pair

$$\left(\frac{A_n^{(0)}(z)}{Q_n^{(0)}(z)}, \frac{z^{-n}\hat{A}_n^{(0)}(z)}{z^{-n}\hat{Q}_n^{(0)}(z)}\right)$$

represents a LPTA of type (n,n) for the series G(z).

Proof.

Use Theorem 1 and Definition 2.■

As a result, we have seen how from two PTA_L of types (-1,n) and (n,n) we can obtain an LPTA of type (n,n) for the same formal Laurent series G(z).

3. HIGHER ORDER APPROXIMANTS

Since to obtain the LPTA for a formal Laurent series G(z) we need two arbitrary polynomials $Q_n^{(0)}(z)$ and $\hat{Q}_n^{(0)}(z)$, we might think how to choose these ones, so that the order of correspondence were increased as much as possible. Thus, the so-called higher order approximant arise (see [1]). On the sequel, when m=0, upperindex (0) will be deleted.

Let us suppose that the pair $(P_n(z)/Q_n(z))$ is a PTA_L of type (n,n), where

$$P_n(z) = \sum_{-\infty}^{n} a_j z^j, \quad Q_n(z) = \sum_{j=0}^{n} b_j z^j, \quad b_0 \neq 0$$

By (2.4), one has

$$a_j = \sum_{k=0}^{n} c_{j-k} b_k, \qquad j=\ldots,n-2,n-1,n$$

Moreover, because of $(\hat{P}_n(z)/\hat{Q}_n(z))$ is a PTA_L of type (-1,n), with

$$\hat{P}_n(z) = \sum_{-\infty}^{-1} \hat{a}_j z^j = O_-(z^{-1}), \quad \hat{Q}_n(z) = \sum_{j=0}^{n} \hat{b}_j z^j, \quad \hat{b}_0 \neq 0$$

then,

$$\hat{a}_j = \sum_{k=0}^{n} c_{j-k} \hat{b}_k, \qquad j=-1,-2,\ldots$$

Now, if $E_n(z)$ is the residual of the PTA_L $(P_n(z)/Q_n(z))$, one has

$$E_n(z) = G(z)\, Q_n(z) - P_n(z) = O_+(z^{n+1}) = \sum_{j=n+1}^{\infty} e_j z^j$$

and the following Theorem holds.

Theorem 2

With the above notations if the coefficients b_j and $\hat{b}_j$ of the denominator polynomials satisfy the conditions

$$\sum_{j=0}^{n} c_{-i-j} \hat{b}_j = \hat{a}_{-i} = 0 \qquad i=1,2,..,p \ (1 \leq p \leq n) \tag{3.1a}$$

$$\sum_{j=0}^{n} c_{i-j} b_j = e_i = 0 \qquad i=n+1,n+2,..,n+q \tag{3.1b}$$

then, for the pair $\left(\dfrac{A_n(z)}{Q_n(z)}, \dfrac{z^{-n}\hat{A}_n(z)}{z^{-n}\hat{Q}_n(z)} \right)$ one has,

$$G^{(0)}(z) - \frac{A_n(z)}{Q_n(z)} = O_+(z^{n+q+1}) \text{ and } \hat{G}^{(0)}(z) - \frac{z^{-n}\hat{A}_n(z)}{z^{-n}\hat{Q}_n(z)} = O_-(z^{-n-p-1}) \tag{3.2}$$

Proof.

Let us first suppose that (3.1a) holds, then $\hat{P}_n(z) = O(z^{-p-1})$ and by (2.11) one has

$$\hat{G}^{(0)}(z)\, \hat{Q}_n(z) - \hat{A}_n(z) = \hat{P}_n(z) = O_-(z^{-p-1})$$

and henceforth

$$\hat{G}^{(0)}(z) - \frac{z^{-n}\hat{A}_n(z)}{z^{-n}\hat{Q}_n(z)} = O(z^{-n-p-1})$$

In other words, we see that the order of correspondence for the PTA_L to $\hat{G}^{(0)}(z)$ has been increased up to n+p+1.

From (3.1b), we have $E_n(z) = O_+(z^{n+q+1})$ so, by using (2.8), it results

$$G^{(0)}(z)\, Q_n(z) - A_n(z) = E_n(z) = O_+(z^{n+q+1})$$

and consequently

$$G^{(0)}(z) - \frac{A_n(z)}{Q_n(z)} = O_+(z^{n+q+1}) \quad \blacksquare$$

In such that case, it will be said that

$$\left(\frac{A_n(z)}{Q_n(z)}, \frac{z^{-n}\hat{A}_n(z)}{z^{-n}\hat{Q}_n(z)} \right)$$

represents an LPTA of higher order.

Obviously, the highest order of correspondence will be reached when p=q=n, and the coefficients of the polynomial $Q_n(z) = \sum_{j=0}^{n} b_j z^j$, will be determined from the linear system of equations

$$\sum_{j=0}^{n} c_{i-j} b_j = 0 \quad i=n+1,n+2,..,2n \tag{3.3}$$

Since G(z) is a normal series, then the polynomial $Q_n(z)$ satisfying the relations (3.3) is uniquely given by

$$Q_n(z) = K \begin{vmatrix} 1 & z & & z^n \\ c_{n+1} & c_n & \cdots & c_1 \\ c_{n+2} & c_{n+1} & \cdots & c_2 \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ c_{2n} & c_{2n-1} & \cdots & c_n \end{vmatrix} \qquad (K \neq 0)$$

In short, $Q_n(z)$ coincides apart a multiplicative factor with the denominator polynomial of the [n/n] PA to the formal power series $G^{(0)}(z)$ (see [1]).

On the other hand, if $\hat{Q}_n(z)$ satisfies the conditions (3.1a) with p=n, then the coefficients $\hat{b}_j$ are now obtained by solving the linear system

$$\sum_{j=0}^{n} c_{-i-j} \hat{b}_j = 0 \qquad i=1,2,..,n \tag{3.4}$$

Taking into account that

$$z^{-n}\hat{Q}_n(z) = \sum_{j=0}^{n} \hat{b}_{n-j} z^{-j}$$

$$F_1(z) - \frac{A_n(z)}{Q_n(z)} = O_+(z^{n+1})$$

hence

$$z^{-1}F_1(z) - \frac{z^{-1}A_n(z)}{Q_n(z)} = O_+(z^n)$$

but $z^{-1}F_1(z) = G^{(0)}(z)$ and $z^{-1}A_n(z) \in \Pi_{n-1}$, that is,

$$G^{(0)}(z) - \frac{z^{-1}A_n(z)}{Q_n(z)} = O_+(z^n) \tag{3.5}$$

Let $(\hat{P}_n(z)/\hat{Q}_n(z))$ be now the PTA_L to F(z) of type (-1,n), being $\hat{Q}_n(z)$ the generator polynomial, then by (b) in Theorem 1, there exists $\hat{A}_n(z) \in \Pi_n$ such that

$$\hat{F}_1(z) - \frac{z^{-n}\hat{A}_n(z)}{z^{-n}Q_n(z)} = O_-(z^{-n-1})$$

Moreover, the highest degree coefficient in $\hat{A}_n(z)$ is zero, because of $d_0=0$, that is, $\hat{A}_n(z) \in \Pi_{n-1}$, and then

$$z\,\hat{F}_1(z) - \frac{z^{-(n-1)}\hat{A}_n(z)}{z^{-n}\hat{Q}_n(z)} = O_-(z^{-n})$$

and due to the fact that $z\hat{F}_1(z) = \hat{G}^{(0)}(z)$, one has

$$\hat{G}^{(0)}(z) - \frac{z^{-(n-1)}\hat{A}_n(z)}{z^{-n}\hat{Q}_n(z)} = O_-(z^{-n}) \tag{3.6}$$

from (3.5) and (3.6) the proof follows.■

REMARK 1:

According to Definition 2, the pair $\left(\frac{z^{-1}A_n(z)}{Q_n(z)}, \frac{z^{-(n-1)}\hat{A}_n(z)}{z^{-n}\hat{Q}_n(z)}\right)$ is not estrictly an LPTA of type (n-1,n), because when adding both components, the resulting rational function has as numerator and denominator L-polynomials both in the linear space $\Lambda_{-n,n}$. However, taking into account that $Q_n(0)\hat{Q}_n(0)\neq 0$ then by ([4],p.11), we can obtain two constants α_n^+ and α_n^- such that the pair

$$\left(\frac{\alpha_n^+ z^n + z^{-1}A_n(z)}{Q_n(z)}, \frac{\alpha_n^- z^{-n} + z^{-(n-1)}\hat{A}_n(z)}{z^{-n}\hat{Q}_n(z)}\right)$$

is in effect an LPTA of type (n-1,n).

In the next two Theorems, we shall see how obtain LPTA of arbitrary type (m,n) with m and n nonnegative integers. For this purpose, we first define the is the denominator of the second component of the LPTA of type (n,n), and because of $\hat{Q}_n(z) = \sum_{j=0}^{n} \hat{b}_{n-j} z^j$ ($\hat{b}_j$ solutions of (3.4)), represents the denominator of the PA for the series

$$\hat{G}^{0)}(z^{-1}) = c_0/2 + \sum_{k=1}^{\infty} c_{-k} z^k,$$

then $\tilde{Q}_n(z^{-1}) = z^{-n}\hat{Q}_n(z)$ will be the denominator of the classic PA to $G^{(0)}(z)$. In other words, we have proved the following

Theorem 3

Let $\left(\dfrac{A_n(z)}{Q_n(z)}, \dfrac{z^{-n}\hat{A}_n(z)}{z^{-n}\hat{Q}_n(z)} \right)$ be a pair so that conditions (3.3) and (3.4) are satisfied by the coefficients of the polynomials $Q_n(z) = \sum_{j=0}^{n} b_j z^j$ and $\hat{Q}_n(z) = \sum_{j=0}^{n} \hat{b}_j z^j$ respectively then, such pair represents the LPA of type (n,n) to the Laurent series G(z).

Nextly, we shall concentrate on how obtaining the LPTA of order (n-1,n) for G(z). For this purpose we define $F(z) = \sum_{-\infty}^{\infty} d_k z^k$, with $d_0=0$; $d_1=d_{-1}=c_0/2$; $d_i=c_{i-1}$ if i=2,3,... ; and $d_i=c_{i+1}$ if i=-2,-3,... Then one has

Proposition 4

Let $\left(\dfrac{A_n(z)}{Q_n(z)}, \dfrac{z^{-n}\hat{A}_n(z)}{z^{-n}\hat{Q}_n(z)} \right)$ be a LPTA of type (n,n) for F(z), with n≥1 then, the pair $\left(\dfrac{z^{-1}A_n(z)}{Q_n(z)}, \dfrac{z^{-(n-1)}\hat{A}_n(z)}{z^{-n}\hat{Q}_n(z)} \right)$ represents the LPTA of type (n-1,n) for G(z) with the same generator polynomial, being $Q_n(0)\,\hat{Q}_n(0) \neq 0$.

Proof.

Set

$$F_1(z) = 0 + \frac{c_0}{2} z + c_1 z^2 + \ldots$$
$$\hat{F}_1(z) = 0 + \frac{c_0}{2} z^{-1} + c_1 z^{-2} + \ldots$$

so that $F(z) = F_1(z) + \hat{F}_1(z)$.

Let $(P_n(z)/Q_n(z))$ be the PTA_L for F(z) with $Q_n(0)=1$. If we denote by $E_n(z)$ the residual, that is,

$$E_n(z) = F(z)\, Q_n(z) - P_n(z) = O_+(z^{n+1})$$

using (a) in Theorem 1, we can obtain $A_n(z) = F_1(z)Q_n(z) - E_n(z)$, $A_n(z)$ being a polynomial of degree n. Furthermore, since $d_0=0$, one has $A_n(0)=0$ and formal Laurent series $F(z) = \sum_{-\infty}^{\infty} d_k z^k$, with $d_l=0$; $-(k-1)\le i\le k-1$

$d_{\pm k}=c_0/2$ and $d_l=c_{l+k}$ if $i=-(k+1),-(k+2),\ldots$

$d_l=c_{l-k}$ if $i=k+1,k+2,\ldots$

clearly, taking

$$F_k(z) = \frac{c_0}{2} z^k + c_1 z^{k+1}+\ldots$$

$$\hat{F}_k(z) = \frac{c_0}{2} z^{-k}+ c_{-1} z^{-k-1}+\ldots$$

then $F(z) = F_k(z) + \hat{F}_k(z)$ $(1\le k\le n)$, and

$$z^{-k}F_k(z) = G^{(0)}(z) \text{ and } z^kF_k(z) = \hat{G}^{(0)}(z) \tag{3.7}$$

one has

<u>Theorem 4</u>

If the pair $\left(\frac{A_n^{(k)}(z)}{Q_n(z)}, \frac{z^{-n}\hat{A}_n^{(k)}(z)}{z^{-n}\hat{Q}_n(z)}\right)$ represents the LPTA for $F(z)$ of type (n,n), with denominator polynomials $Q_n(z)$ and $\hat{Q}_n(z)$, then the pair

$$\left(\frac{z^{-k}A_n^{(k)}(z)}{Q_n(z)}, \frac{z^{-(n-k)}\hat{A}_n^{(k)}(z)}{z^{-n}\hat{Q}_n(z)}\right) \tag{3.8}$$

represents an LPTA of type $(n-k,n)$ for $G(z)$.

Proof.

Taking into account the correspondence conditions for the LPTA $\left(\frac{A_n^{(k)}(z)}{Q_n(z)}, \frac{z^{-n}\hat{A}_n^{(k)}(z)}{z^{-n}\hat{Q}_n(z)}\right)$ and by (3.7), it is easy to see that

$$G^{(0)}(z) - \frac{z^{-k}A_n^{(k)}(z)}{Q_n(z)} = O_+(z^{n-k+1}) \tag{3.9}$$

$$\hat{G}^{(0)}(z) - \frac{z^{-n+k}\hat{A}_n^{(k)}(z)}{z^{-n}\hat{Q}_n(z)} = O_-(z^{-(n-k)-1}) \tag{3.10}$$

Now, from (2.7b) one has

$$A_n^{(k)}(z) = F_k(z)\,Q_n(z) - E_n(z), \text{ being } E_n(z) = O_+(z^{n+1})$$

and since the k-1 terms for $F_k(z)$ are zeros, $A_n^{(k)}(z) = \sum_{j=k}^{n} a_j z^j$, that is, $z^{-k}A_n^{(k)}(z)$ is a polynomial of degree $\le n-k$.

Moreover, from (2.10b), the polynomial $\hat{A}_n^{(k)}(z) = \hat{F}_k(z)\,\hat{Q}_n(z) - \hat{P}_n(z)$ will be of degree $\le n-k$, and by this, $z^{-(n-k)}\hat{A}_n^{(k)}(z) \in \Lambda_{-(n-k),0}$ and the proof follows.■

In order to determine an LPTA of type (n+k,n), with $k \geq 1$, we must introduce the formal Laurent series $H(z) = H_k(z) + \hat{H}_k(z)$ being

$$H_k(z) = c_k + c_{k+1}z + \ldots \qquad (3.11a)$$

$$\hat{H}_k(z) = c_{-k} + c_{-k-1}z^{-1} + \ldots \qquad (3.11b)$$

Let $\left(\dfrac{A_n^{(k)}(z)}{Q_n(z)}, \dfrac{z^{-n}\hat{A}_n^{(k)}(z)}{z^{-n}\hat{Q}_n(z)}\right)$ be an LPTA of type (n,n) for H(z) obtained from the PTA_L of types (n,n) and (-1,n) for $H_k(z)$ and $\hat{H}_k(z)$ and with denominator polynomials $Q_n(z)$, $\hat{Q}_n(z)$ respectively Let

$$\frac{A_{n+k}(z)}{Q_n(z)} = \sum_{j=0}^{k-1} c_j z^j + z^k \frac{A_n^{(k)}(z)}{Q_n(z)} \qquad (3.12)$$

be a rational function, being $A_{n+k}(z) \in \Pi_{n+k}$ and consider

$$\frac{z^{-(n+k)}\hat{A}_{n+k}(z)}{z^{-n}\hat{Q}_n(z)} = \sum_{j=0}^{k-1} c_{-j} z^{-j} + \frac{z^{-(n+k)}\hat{A}_n^{(k)}(z)}{z^{-n}\hat{Q}_n(z)} \qquad (3.13)$$

and $\hat{A}_{n+k}(z) = z^k \hat{Q}_n(z) \sum_{j=0}^{k-1} c_{-j} z^{-j} + \hat{A}_n^{(k)}(z) \in \Pi_{n+k}$

With these notations we have the next

Theorem 5.-

Let $\left(\dfrac{A_n^{(k)}(z)}{Q_n(z)}, \dfrac{z^{-n}\hat{A}_n^{(k)}(z)}{z^{-n}\hat{Q}_n(z)}\right)$ be the LPTA of type (n,n) for H(z) with generator polynomials $Q_n(z)$ and $\hat{Q}_n(z)$, then the pair

$$\left(\frac{A_{n+k}(z)}{Q_n(z)}, \frac{z^{-(n+k)}\hat{A}_{n+k}(z)}{z^{-n}\hat{Q}_n(z)}\right)$$

represents a LPTA for G(z) of type (n+k,n) (with the same generator polynomials), with $A_{n+k}(z)$ and $\hat{A}_{n+k}(z)$ given by (3.12) and (3.13) respectively.

Proof.

By using the definition of LPTA, one has

$$z^k H_k(z) - \frac{z^k A_n^{(k)}(z)}{Q_n(z)} = O_+(z^{n+k+1}) \qquad (3.14a)$$

$$z^{-k}\hat{H}_k(z) - \frac{z^{-(n+k)}\hat{A}_n^{(k)}(z)}{z^{-n}\hat{Q}_n(z)} = O_-(z^{-n-k-1}) \qquad (3.14b)$$

Now, because of

$$G^{(0)}(z) = \sum_{j=0}^{k-1} c_j z^j + z^k H_k(z) \text{ and } \hat{G}^{(0)} = \sum_{j=0}^{k-1} c_{-j} z^{-j} + z^{-k}\hat{H}_k(z)$$

from (3.12), (3.13) and (3.14), it yields

$$G^{(0)}(z) - \frac{A_{n+k}(z)}{Q_n(z)} = O_+(z^{n+k+1}) \quad \text{and} \quad \hat{G}^{(0)} - \frac{z^{-(n+k)}\hat{A}_{n+k}(z)}{z^{-n}\hat{Q}_n(z)} = O_-(z^{-n-k-1})$$

and the Theorem follows.■

REFERENCES

[1] **C. Brezinski,** *Padé-Type approximants and general orthogonal polynomials,* (Birkhaüser, Basel, 1980).

[2] **A. Bultheel,** *Laurent series and their Padé Approximations* (OT, Vol. 27 Birkhaüser, Basel 1987)

[3] **M. Camacho, P. González-Vera,** "Aproximaciones racionales a series doblemente infinitas", *XIII Jornadas Hispanolusas de Matemáticas* (Universidad de Valladolid, 1988) to appear.

[4] **M. Camacho,** *Sobre funciones racionales asociadas a sucesiones doblemente infinitas,* (Doctoral Thesis; Universidad de La Laguna, 1991)

[5] **W.B. Gragg, G.D. Johnson,** "The Laurent Padé Table". *Information Processing* 74, (North Holland, Amsterdam, 1974) pp.78-89.