Rev. Acad. Canar. Cienc., XVI (Núms. 1-2), 15-27 (2004) (publicado en julio de 2005)

ON FUNCTIONAL RELATIONS OF DIFFERINTEGRALS INVOLVING TRIGONOMETRIC FUNCTIONS

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Abstract

Based on Nishimoto's fractional calculus [7, Vol. 1], the main object in the present paper is to calculate the differintegral to arbitrary order β the some relations functionals involving trigonometric functions of argument z. In addition some illustrative examples are shown.

0. Introduction. (Definition of fractional calculus).

The fractional calculus deals with derivatives and integrals of arbitrary order. The origin of fractional calculus is based in extension of the derivatives concept of integer order n, supposing n real or complex, called fractional differintegrals. Recently, by applying the Nishimoto's definition [3,4,13,14,17,19] of fractional differintegral of order $v \in \mathbb{R}$, many authors have obtained differintegrals of different functions.

Riemann-Liouville [1-2] defines a fractional integration of β order as:

$$ {}_cD_x^{-\beta} f(x)) \frac{1}{\Gamma(\beta)} \int_c^x (x-\tau)^{\beta-1} f(\tau) d\tau \ , \ \beta \geq 0 \quad ; \quad \mathrm{Re}(\beta) \geq 0 $$

$a \leq x < b$ and Γ is the Gamma function.

Nishimoto [7, Vol. 1] defines the fractional differintegration of the function of a variable as follows:

I. Definition.

If $f(x)$ is an analytic function with no internal branching points and on $C\,(C = \{C_-, C_+\})$ is an integral curve along the section joining two points z and $-\infty + i\,\mathrm{Im}(z)$ and C_+ is an integral curve along the section joining two points z and $(+\infty + i\,\mathrm{Im}(z))$

$$f_v = \ {}_c f_v(z) = \frac{\Gamma(v+1)}{2\pi i}\int_c \frac{f(\xi)d(\xi)}{(\xi - z)^{v+1}}, (v \in \mathbb{R} \ \ , \quad v \notin \mathbb{Z}^-) \tag{1}$$

$$f_{-n} = \lim_{y \to -n} {}_c f_v(n \in \mathbb{Z}^+)$$

where $\xi \neq z, -\pi \leq \arg(\xi - z) \leq \pi$ for C_- and $0 \leq \arg(\xi - z) < 2\pi$ for C_+, then $f_v(v > 0)$ is the derivative of fractional order v and $f_v(v < 0)$ is the integral of fractional order $|v|$, if f_v exists.

II. Lemma.

Some properties of the Fractional Calculus [3, Vol. 1, p. 21,22,28,30, 50] and [4, p. 41-46]

1) $(\sin az)_v = a^v \sin\left(az + \frac{\pi}{2}v\right)$ (2)

2) $(\cos az)_v = a^v \cos\left(az + \frac{\pi}{2}v\right)$ (3)

3) $(z^\alpha)_v = e^{-i\pi v}\frac{\Gamma(v-\alpha)}{\Gamma(-\alpha)}z^{\alpha - v} \quad ; \quad \left|\frac{\Gamma(v-\alpha)}{\Gamma(-\alpha)}\right| < \infty$ (4)

4) **a**) $(\mathbf{z})^n_v \ = \ 0 \ \ for n \in \mathbb{Z}^+_0 \quad ; \quad v \notin \mathbb{Z}$ (5)

b) $(z)^n_k \ = \ 0$ for $n \in \mathbb{Z}^+_0 \quad ; \quad k \in \mathbb{Z}^+$ and $k > n$

5) $(u\mathrm{v})_v = \sum_{k=0}^{\infty} \frac{\Gamma(v+1)}{k!\,\Gamma(1+v-k)} u_{v-k}\,\mathrm{v}_k$ (6)

6) $(1)_{-n} = \frac{z^n}{n!} \quad ; \quad n \in \mathbb{Z}^+_0$ (7)

1.- Differintegrals of functions relations.
Theorems 1:
We have

$$\textbf{i)}\quad \left(\frac{(\sin z)_\alpha + (\sin z)_{-\alpha}}{2\cos\frac{\pi}{2}\alpha}\right)_\beta = \sin\left(z + \frac{\pi}{2}\beta\right) \tag{8}$$

where $\alpha \neq 2n+1$ and $n = 0, \pm 1, \pm 2, ...$

$$\textbf{ii)}\quad \left(\frac{(\sin z)_\alpha - (\sin z)_{-\alpha}}{2\cos\frac{\pi}{2}\alpha}\right)_\beta = \cos\left(z + \frac{\pi}{2}\beta\right) \tag{9}$$

where $\alpha \neq 2n$ and $n = 0, \pm 1, \pm 2, ...$

Proof of the theorem 1i):
Using the lemma 1, we have

$$(\sin z)_\alpha = \sin\left(z + \frac{\pi}{2}\alpha\right) \tag{10}$$

and

$$(\sin z)_{-\alpha} = \sin\left(z - \frac{\pi}{2}\alpha\right) \tag{11}$$

We know that

$$\sin(A \pm B) = \sin A \cos B \pm \sin B \cos A \tag{12}$$

Using (12), we have

$$(\sin z)_\alpha = \sin\left(z + \frac{\pi}{2}\alpha\right) = \sin z \cos\frac{\pi}{2}\alpha + \sin\frac{\pi}{2}\alpha \cos z \tag{13}$$

$$(\sin z)_{-\alpha} = \sin\left(z - \frac{\pi}{2}\alpha\right) = \sin z \cos\frac{\pi}{2}\alpha - \sin\frac{\pi}{2}\alpha \cos z \tag{14}$$

Adding (13) and (14), we have

$$(\sin z)_\alpha + (\sin z)_{-\alpha} = 2\sin z \cos\frac{\pi}{2}\alpha \tag{15}$$

Then,

$$\frac{(\sin z)_\alpha + (\sin z)_{-\alpha}}{2\cos\frac{\pi}{2}\alpha} = \sin z \tag{16}$$

Afterward, applying lemma 1, we obtain (8).

Proof of the theorem 1ii:

We have this theorem from theorem 1i (see the proof of previous. Subtracting (14) from (13) and applying lemma 2).

Theorem 2:

We have

$$\textbf{i)}\ A(z,\alpha,\beta)=\left(\frac{\{(\sin z)_\alpha+(\sin z)_{-\alpha}\}^2}{4\cos^2\frac{\pi}{2}\alpha}\right)_\beta=-2^{\beta-1}\cos\left(2z+\frac{\pi}{2}\beta\right) \tag{17}$$

where $\alpha\neq 2n+1$ and $n=0,\pm1,\pm2,....,\beta\notin\mathbb{Z}_0^-$.

$$\textbf{ii)}\ B(z,\alpha,\beta)=\left(\frac{\{(\sin z)_\alpha-(\sin z)_{-\alpha}\}^2}{4\sin^2\frac{\pi}{2}\alpha}\right)_\beta=2^{\beta-1}\cos\left(2z+\frac{\pi}{2}\beta\right) \tag{18}$$

where $\alpha\neq 2n$, $n=0,\pm1,\pm2,....$ and $\beta\notin\mathbb{Z}_0^-$.

$$\textbf{iii)}\ A(z,\alpha,\beta)+B(z,\alpha,\beta)=0 \tag{19}$$

Proof of the theorem 2i:

From (15), we have

$$\{(\sin z)_\alpha+(\sin z)_{-\alpha}\}^2=4\sin^2 z\cos^2\frac{\pi}{2}\alpha \tag{20}$$

Then,

$$\left(\frac{\{(\sin z)_\alpha+(\sin z)_{-\alpha}\}^2}{4\cos^2\frac{\pi}{2}\alpha}\right)_\beta=(\sin^2 z)_\beta \tag{21}$$

Using the trigonometric identity

$$\sin^2 z=\frac{1-\cos 2z}{2} \tag{22}$$

we have

$$(\sin^2 z)_\beta=\left(\frac{1}{2}\right)_\beta-\left(\frac{\cos 2z}{2}\right)_\beta \tag{23}$$

Applying lemmas (2) and (4a), we obtain (17).

Proof of the theorem 2ii:

We have this theorem from theorem 2i (see the proof of previous). Subtracting (14) from (13) and using the trigonometric identity

$$\left(\cos^2 z = \frac{1+\cos 2z}{2}\right) \tag{24}$$

we obtain (19) adding (17) and (18).

Theorem 3:

If $\beta \notin \mathbb{Z}_0^-$, we have

i) $C(z,\alpha,\beta) = ((\sin z)_\alpha (\sin z)_{-\alpha})_\beta = -2^{\beta-1}\cos\left(2z + \frac{\pi}{2}\beta\right)$ (25)

ii) $D(z,\alpha,\beta) = ((\cos z)_\alpha (\cos z)_{-\alpha})_\beta = 2^{\beta-1}\cos\left(2z + \frac{\pi}{2}\beta\right)$ (26)

iii) $C(z,\alpha,\beta) + D(z,\alpha,\beta) = 0$ (27)

Proof of the theorem 3i:

Multiplying (13) and (14), we have

$$(\sin z)_\alpha (\sin z)_{-\alpha} = \cos^2\frac{\pi}{2}\alpha \sin^2 z - \sin^2\frac{\pi}{2}\alpha \cos^2 z \tag{28}$$

Then

$$((\sin z)_\alpha (\sin z)_{-\alpha})_\beta = \cos^2\frac{\pi}{2}\alpha \left(\sin^2 z\right)_\beta - \sin^2\frac{\pi}{2}\alpha (\cos^2 z)_\beta$$

Using (22); (23) and applying lemmas 2 and 4a, we obtain (25).

Proof of the theorem 3ii:

Applying lemma 2, we have

$$(\cos z)_\alpha = \cos\left(z + \frac{\pi}{2}\alpha\right) \tag{29}$$

$$(\cos z)_{-\alpha} = \cos\left(z - \frac{\pi}{2}\alpha\right) \tag{30}$$

Using the trigonometric identity

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B \tag{31}$$

Applying (31) in (29) and (30), we have

$$(\cos z)_\alpha = \cos z \cos \frac{\pi}{2}\alpha - \sin z \sin \frac{\pi}{2}\alpha \tag{32}$$

$$(\cos z)_{-\alpha} = \cos z \cos \frac{\pi}{2}\alpha + \sin z \sin \frac{\pi}{2}\alpha \tag{33}$$

Multiplying (32) and (33), we have

$$(\cos z)_\alpha (\cos z)_{-\alpha} = \cos^2 z \cos^2 \frac{\pi}{2}\alpha - \sin^2 z \sin^2 \frac{\pi}{2}\alpha \tag{34}$$

Afterward, using (22) and (24), calculating the differintegral from (34) applying lemmas 2 and 4a, we obtain (26).

Proof of the theorem 3iii: Adding (25) and (26), we obtain (27).

Theorem 4: We have

i) $$\left(\frac{(\cos z)_\alpha + (\cos z)_{-\alpha}}{2\cos\frac{\pi}{2}\alpha}\right)_\beta = \cos\left(z + \frac{\pi}{2}\beta\right) \tag{35}$$

where $\alpha \neq 2n+1$, and $n = 0, \pm 1, \pm 2, ...$

ii) $$\left(\frac{(\cos z)_{-\alpha} - (\cos z)_\alpha}{2\sin\frac{\pi}{2}\alpha}\right)_\beta = \sin\left(z + \frac{\pi}{2}\beta\right) \tag{36}$$

where $\alpha \neq 2n$ and $n = 0, \pm 1, \pm 2, ...$

iii) $$E(z,\alpha,\beta) = \left(\frac{\{(\cos z)_\alpha + (\cos z)_{-\alpha}\}^2}{4\cos^2\frac{\pi}{2}\alpha}\right)_\beta = 2^{\beta-1}\cos\left(2z + \frac{\pi}{2}\beta\right) \tag{37}$$

where $\alpha \neq 2n+1, n = 0, \pm 1, \pm 2, ...,$ and $\beta \notin \mathbb{Z}_0^-$.

iv) $$F(z,\alpha,\beta) = \left(\frac{\{(\cos z)_{-\alpha} - (\cos z)_\alpha\}^2}{4\cos^2\frac{\pi}{2}\alpha}\right)_\beta = -2^{\beta-1}\cos\left(2z + \frac{\pi}{2}\beta\right) \tag{38}$$

where $\alpha \neq 2n, n = 0, \pm 1, \pm 2, ...,$ and $\beta \notin \mathbb{Z}_0^-$.

v) $E(z,\alpha,\beta)+F(z,\alpha,\beta) = 0$ (39)

Proof of the theorem 4i:
Adding (32) and (33), we have

$$(\cos z)_\alpha + (\cos z)_{-\alpha} = 2\cos\frac{\pi}{2}\alpha\cos z \tag{40}$$

Then

$$\left(\frac{(\cos z)_\alpha + (\cos z)_{-\alpha}}{2\cos\frac{\pi}{2}\alpha}\right)_\beta = (\cos z)_\beta \tag{41}$$

Applying lemma 2, we obtain (35).

Proof of the theorem 4ii:
We have this theorem from theorem 4i (see the proof of previous. Subtracting (32) from (33) and using lemma 1.)
Using (40), (24) and lemmas 2 and 4a, we obtain (37).
The proof of (38) is similar from theorem (4iii) using (22) and lemmas 2 and 4a.

Theorem 5: We have

i) $$((\cos z)_\alpha(\cos z)_\alpha - (\sin z)_\alpha(\sin z)_{-\alpha})_\beta = 2^\beta\cos\left(2z+\frac{\pi}{2}\beta\right) \tag{42}$$

ii) $$((\sin \mathbf{z})_\alpha(\cos z)_{-\alpha} + (\cos z)_\alpha(\sin z)_{-\alpha})_\beta = 2^\beta\sin\left(2z+\frac{\pi}{2}\beta\right) \tag{43}$$

iii. a) $$((\sin z)_\alpha(\cos z)_{-\alpha} + (\cos z)_\alpha(\sin z)_{-\alpha})_{-n} = (\sin^2 z)_{-(n-1)} \tag{44}$$

b) $$((\sin z)_\alpha(\cos z)_{-\alpha} + (\cos z_\alpha(\sin z)_{-\alpha})_{-n} =$$

$$= 2^{-n}\sin\left(2z - n\frac{\pi}{2}\right) + \frac{1}{2}\frac{Z^{n-1}}{(n-1)} \tag{45}$$

where $n \in \mathbb{Z}^+$.

Proof of the theorem 5i:
Subtracting (28) from (34), we have

$$((\cos z)_\alpha(\cos z)_{-\alpha} - (\sin z)_\alpha(\sin z)_{-\alpha})_\beta = (\cos^2 z)_\beta - (\sin^2 z)_\beta$$

Afterward, using (22), (24) and lemmas 2 and 4a obtain (42).

Proof of the theorem 5ii:
Multiplying (13) for (33) and (32) for (14), we have

$$(\sin z)_\alpha(\cos z)_{-\alpha} + (\cos z)_\alpha(\sin z)_{-\alpha} = 2\cos z\sin z \tag{46}$$

Using the trigonometric identity

$$\sin 2z = 2\sin z\cos z \tag{47}$$

and lemma 1, we obtain (43).

Proof of the theorem 5iiia and 5iiib:
Calculating the differintegral of integer order $-n$ in (46), we have

$$((\sin z)_\alpha(\cos z)_{-\alpha} + (\cos z)_\alpha(\sin z)_{-\alpha})_{-n} = (2\sin z\cos z)_{-n}$$

We know that

$$(\sin z)_1 = 2\sin z\cos z$$

Then

$$(\sin z)_{1-n} = (\sin^2 z)_{-(n-1)} \tag{48}$$

Afterward, using (22) and lemmas 2 and 6 in (48) we obtain (45).

Theorem 6: We have

i) $$\left(\frac{((\sin z)z)_\alpha + ((\sin z)z)_{-\alpha}}{2\left\{z\cos\frac{\pi}{2}\alpha + \alpha\sin\frac{\pi}{2}\alpha\right\}}\right)_\beta = \sin\left(z + \frac{\pi}{2}\beta\right) \tag{49}$$

ii) $$\left(\frac{((\cos z)z)_\alpha + ((\cos z)z)_{-\alpha}}{2\left\{z\cos\frac{\pi}{2}\alpha + \alpha\sin\frac{\pi}{2}\alpha\right\}}\right)_\beta = \cos\left(z + \frac{\pi}{2}\beta\right) \tag{50}$$

iii) $$\frac{((\sin z)z)_\alpha + ((\sin z)z)_{-\alpha}}{((\cos z)z)_\alpha + ((\cos z)z)_{-\alpha}} = \tan z \tag{51}$$

Proof of the theorem 6i:

Using lemma 5 with $u = \sin z$ and $v = z$ (52)

Afterward, developing the serie and using lemmas 1 and 4b, we have

$$((\sin z)z)_\alpha = z \sin\left(z + \frac{\pi}{2}\alpha\right) + \alpha \sin\left(z + \frac{\pi}{2}(\alpha - 1)\right) \tag{53}$$

Then

$$((\sin z)z)_\alpha = z \sin\left(z + \frac{\pi}{2}\alpha\right) - \alpha \cos\left(z + \frac{\pi}{2}\alpha\right) \tag{54}$$

Using lemmas 1 and 5, we have

$$((\sin z)z)_{-\alpha} = z \sin\left(z - \frac{\pi}{2}\alpha\right) + \alpha \cos\left(z - \frac{\pi}{2}\alpha\right) \tag{55}$$

Afterward, adding (54) and (55), we have

$$((\sin z)z)_\alpha + ((\sin z)z)_{-\alpha} = 2 \sin z \left(z \cos \frac{\pi}{2}\alpha + \alpha \sin \frac{\pi}{2}\alpha\right) \tag{56}$$

Calculating the differintegral to order β in (56) and using lemma 1, we have obtain (49).

Proof of the theorem 6ii:

Using lemma 2 and 5, we have

$$((\cos z)z)_\alpha = z \cos\left(z + \frac{\pi}{2}\alpha\right) + \alpha \sin\left(z + \frac{\pi}{2}\alpha\right) \tag{57}$$

and

$$((\cos z)z)_{-\alpha} = z \cos\left(z - \frac{\pi}{2}\alpha\right) - \alpha \sin\left(z - \frac{\pi}{2}\alpha\right) \tag{58}$$

Adding (57) and (58), we have

$$((\cos z)z)_\alpha + ((\cos z)z)_{-\alpha} = 2\left\{z \cos \frac{\pi}{2}\alpha + \alpha \sin \frac{\pi}{2}\alpha\right\} \cos z \tag{59}$$

Afterward, calculating the differintegral of (59), using lemma 2, we obtain (50).

Obtain (51) using (56) and (59).

Theorem 7: We have

$$\textbf{i)}\quad \left(\{((\sin z)z)_\alpha\}^2 + \{((\cos z)z)_\alpha\}^2\right)_\beta = 0 \tag{60}$$

where $\beta \notin \mathbb{Z}$.

$$\textbf{ii)}\quad \left(\{((\sin z)z)_\alpha\}^2 + \{((\cos z)z)_\alpha\}^2\right)_{-n} =$$

$$= (-1)^n \frac{\Gamma(-2-n)}{\Gamma(-2)} z^{2+n} + \alpha^2 \frac{z^n}{n!} \tag{61}$$

where $n \in \mathbb{Z}_0^+$.

Proof of the theorem 7i:
Adding (54) and (57), we have

$$\{((\sin z)z)_\alpha\}^2 + \{((\cos z)z)_\alpha\}^2 = z^2 + \alpha^2 \tag{62}$$

Then

$$\left(\{((\sin z)z)_\alpha\}^2 + \{((\cos z)z)_\alpha\}^2\right)_\beta = (z^2)_\beta + (\alpha^2)_\beta \tag{63}$$

Afterward, applying lemma (4a) in (63), we obtain (60).

Proof of the theorem 7ii:
Calculating the differintegral of integer order $-n$ in (62), we have

$$\left(\{((\sin z)z)_\alpha\}^2 + \{((\cos z)z)_\alpha\}^2\right)_{-n} = (z^2)_{-n} + (\alpha^2)_{-n} \tag{64}$$

Afterward, applying lemmas (3) and (6) in (64) we obtain (61).

EJERCICIOS

1) $$\left(\frac{(\sin z)_{\sqrt{3}} + (\sin z)_{-\sqrt{3}}}{2\cos\frac{\sqrt{3}}{2}\pi}\right)_6 = \sin(z+3\pi) = -\sin z$$

2) $$\left(\frac{(\sin z)_5 - (\sin z)_{-5}}{2\sin 5\frac{\pi}{2}}\right)_{-1/2} = \cos\left(z - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(\sin z + \cos)$$

3) $$\left(\frac{\left\{(\sin z)_{-1/3}-(\sin z)_{1/3}\right\}^2}{4\cos^2\frac{\pi}{6}}\right)_{2/3} = -2^{-1/3}\cos\left(2z+\frac{\pi}{3}\right)$$
$$= -2^{-5/3}(\sqrt{3}\cos 2z-\sin 2z)$$

4) $$\left((\cos z)_{\sqrt{2}}(\cos z)_{-\sqrt{2}}\right)_{-1/3} = 2^{-4/3}\cos\left(2z-\frac{\pi}{6}\right)$$
$$= 2^{-7/2}\left(\cos 2z+\sqrt{3}\sin 2z\right)$$

5) $$\left((\sin z)_{1/2}(\cos z)_{-1/2}+(\cos z)_{1/2}(\sin z)_{-1/2}\right)_{-7} = (\sin^2 z)_{-6}$$
$$= 2^{-7}\cos 2z+\frac{1}{2}\frac{z^6}{6!}$$

6) $$\left(\frac{((\sin z)z)_{\alpha}+(\sin z)z)_{-\alpha}}{2\left\{\cos\frac{\pi}{2}\alpha+\alpha\sin\frac{\pi}{2}\alpha\right\}}\right)_{4/3} = \sin\left(z+\frac{2}{3}\pi\right)$$
$$= \frac{1}{2}\left(\sqrt{3}\cos z-\sin z\right)$$

7) $$\left(\left\{((\sin z)z)_{\sqrt{2}}\right\}^2+\left\{((\cos z)z)_{\sqrt{2}}\right\}^2\right)_{-8} =$$
$$= \frac{\Gamma(-10)}{\Gamma(-2)}z^{10}+2\frac{z^8}{8!}$$

ACKNOWLEDGEMENT

The authors are thankful to CONDES, Universidad del Zulia, for providing a research grant.

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