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SOLUTION OF OPERATOR EQUATION AND THE SPACE OF ENTIRE DIRICHLET SERIES

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Abstract

Let Ω be the set of all complex Dirichlet series of the form $f(s) = \sum_{n=1}^{\infty} a_n \exp(-\lambda_n s)$, $a_n \in \mathbf{C}$, $0 \leq \lambda_1 < \lambda_2 < \ldots \lambda_n \to \infty$ as $n \to \infty$, $\lambda_n \in E_n$, $\mathrm{Card}(E_n) = 1_n$ (a fixed integer), $D = \overline{\lim}_{n\to\infty}(\log|a_n|/\lambda_n) \neq 0$. It has been proved that Ω is a nontrivial supermetric space under suitable compositions. Further, conditions have been obtained for the existence of solution of the operator equation $T(f) = \theta$, where T is a Frêchet differentiable operator on Ω and θ is the additive identity of Ω.

1 Introduction

It is known that if $A : \mathbf{R} \to \mathbf{R}$ is a differentiable function, then an approximate solution of $A(X) = 0$ is a obtained by Newton's method/simplified Newton's method. At the very outset, it appears that Newton's method can be developed for an operator equation $A(X) = \theta$, provided $A : X \to X$ is a differentiable operator on X. In fact, Newton's method for the solution of operator equation on a Banach space is well known- details are available in L.V. Kantarovich and

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G.P. Akilov, 1982. It may be remarked that although it is known that idea of differentiability of operators is available even on a supermetric space, still Newton's method can not be developed on a supermetric space, in general, because only differentiability of operators is not enough to discuss convergence of a iterative sequences in Newton's method- Mean valuable inequality involving the differentiable operators is another essential requirement for the purpose. As Mean valuable inequality can be proved only when the supermetric space is a L-supermetric space, so Collatz, 1966 could develop Newton's method on a L-supermetric space. Our purpose, in this paper, is to use the results of Collatz to discuss solution of the equation $Tf = \theta$, $T : \Omega \rightarrow \Omega;\ f \in \Omega$.

Study of Dirichlet series with different exponents as normed linear spaces, was made, for the first time, by R.K. Srivastava, 1990. He considered Dirichlet series of the form $f(s) = \sum_{n=1}^{\infty} a_n \exp(-\lambda_n s)$, abbreviated as

$f = (a_n, \lambda_n)$, $D = \overline{\lim}_{n\to\infty}(\log \mid an \mid / \lambda_n) < \infty$. Naturally, the Dirichlet series for which $D = \infty$ cloud not be considered. But it may be noted that de most interesting type of Dirichlet series are those for which $D = +\infty$, because only this type of Dirichlet can be convergent there. For example,

$f(s) = \sum_{n=2}^{\infty} ((-1)^n/n) \exp(-\log \log ns)$, for which $\sigma_c = -\infty$, $\sigma_a = 1$; where σ_c and σ_a are the abscissa of the ordinary and absolute convergence.

In this paper, $f = (a_n, \lambda_n)$ and $g = (b_n, \mu_n)$ of Ω has been called equivalent

$(f \cong g)$ if $(a_n/\lambda_n) = (b_n/\mu_n), \forall n$. Further, Ω has been taken as the set of all equivalence classes $\Omega_f = \{g \in \Omega / g \cong f\}$.We have denoted by f, the equivalence class Ω_f whose representative is f.

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2 Some known definitions:

Definition 2.1 *The Dirichlet series $f = (a_n, \lambda_n)$ is said to be entire, if*

(i) $\sum_{n=1}^{\infty} a_n$ is convergent

(ii) $\overline{\lim_{n\to\infty}} \frac{\log|\sum_{k=n}^{\infty} a_k|}{\lambda_n} = -\infty$

Definition 2.2 *A metric space (X, ρ) which is also a linear space is called a supermetric space if the linear operations are continuous with respect to the topology induced by the metric ρ and if ρ satisfies an additional property $\rho(x, y+z) = \rho(x-z, y)$,$\forall x, y, z \in X$.*

Definition 2.3 *A paranorm p on a linear space X is the function $p : X \to R^+ \cup \{0\}$ such that*

(i) $p(x) = 0$ *iff* $x = \theta$

(ii) $p(-x) = p(x)$

(iii) $p(x+y) \leq p(x) + p(y)$

(iv) *If* $\{\lambda_n\} \subset C, \lambda \in \mathbf{C}$ *such that* $|\lambda_n - \lambda| \to 0$, *as* $n \to \infty$ *and* $\{x_n\} \subset X,\ x \in X,$ *be such that* $p(x_n - x) \to 0$ *as* $n \to \infty$, *then* $p(\lambda_n x_n - \lambda x) \to \infty$.

Remark 2.1 *A supermetricspace/linear metric space/metric linear space (X, ρ) is a paranormed space (X, p) where $p(x) = \rho(x, \theta), \theta$ is the additive identity of X. A paranormed space (X, p) is always a supermetric space (X, ρ) where $\rho(x, y) = p(x - y), x, y \in X$.*

Definition 2.4 *A supermetric space (X, ρ) is called a L-supermetric space if for every element $x \in X$, there exists a bounded linear functional L on X such that $\| L \| = 1, Lx = \rho(x, \theta)$, where $\| L \| = \inf K(> 0)$ such that $\nabla(Lx, Ly) \leq K\rho(x, y)$, $\forall x, y \in X$; σ being the metric on the scalar field.*

Obviously, every Banach space is a L-Banach space. But there may be L-spaces which are not L-Banach spaces. An example of such space is given by theorem 3.1.

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Definition 2.5 *Let (R, ρ) and (S, σ) be two complete linear supermetric spaces and let T be an operator with a domain D in R and a range in S.*

The operator T is called Frêchet differentiable at the point $f \in D$ if there exists a bounded linear operator H with a domain in D and a range in S and a real valued function $\varepsilon(\delta)$ with the property $\sigma(Th - tf, H(h-f)) \leq \rho(h,f)\varepsilon(\rho(h,f))$ for a certain neighbourhood $\rho(h,f) < K_0$, where $\lim\limits_{\delta \to 0} \varepsilon(\delta) = 0$. The operator $H = T'_{(f)}$ is called the Frêchet derivative of T at the point f.

Theorem 2.1 *Let (R, ρ) be a L-supermetric space. Let D be a convex domain of R. Let $T : D \rightarrow$ be Frêchet differentiable having T' as the Frêchet derivative on D. Let the iterative sequence $\{u_n\}$ of approximate values for the solution of the equation $Tu = \theta$ be given by simplified Newton's method as $u_{n+1} = u_n - T'^{-1}_{u_o} u_n,$ where $T'^{-1}_{u_0}$ exists at $u_o \in D$.*

If $p = \parallel T'^{-1}_{u_o} \parallel \sup\limits_{f \in D} \parallel T' - T'_f \parallel < 1$ and if with first two members $u_{o,}$ u_1 of $\{u_n\}$,the sphere $K = \{v \in D / \rho(v, u_1) \leq (p/1-p)\rho(u_0, u_1)\} \subset D$, then there exists in D a solution of the equation $Tu = \theta$, where u lies also in K, u_n remains in D and $u_n \to u$ as $n \to \infty$.

3 L-supermetric and the space of entire Dirichlet series.

Theorem 3.1 *Let Ω be the set of all entire Dirichlet series for which $D \neq 0$. Then Ω is a $L-$supermetric space (Ω, ρ) if for $f = (a_n, \lambda_n)$ and $g = (b_n, \mu_n)$ of Ω, $z \in C$, $f+g$ and zf are defined as $f + g = (((a_n/\lambda_n) + (b_n/\mu_n)x_n), x_n)$, $zf = ((za_n/\lambda_n)x_n, x_n)$, where $x_n = \min\limits_{\lambda}(\lambda_n)$ and ρ is defined as $\rho(f,g) = \sup\limits_{\xi \in C} \Phi(\xi) \frac{|f(\xi)-g(\xi)|}{1+|f(\xi)-g(\xi)|}$ where Φ is a bounded function on $\mathbf{C}$ with $\Phi_0 = \max\limits_{\xi \in C} \Phi(\xi) = \Phi(\xi_0)$.*

Proof: We first show that $f+g$ is entire if f and g are so. In fact,

$$\overline{\lim_{n \to \infty}} \frac{\log \left| \sum\limits_{k=n}^{\infty} ((a_k/\lambda_k) + (b_k/\mu_k))x_k \right|}{x_n} \leq$$

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$$\leq \overline{\lim_{n\to\infty}} \frac{\log\left|\sum_{k=n}^{\infty}(a_k/\lambda_k)x_k + \sum_{k=n}^{\infty}(b_k/\mu_k)x_k\right|}{x_n} \leq$$

$$\leq \overline{\lim_{n\to\infty}} \frac{\log 2}{x_n} + \overline{\lim_{n\to\infty}} \frac{\log\left|\sum_{k=n}^{\infty}(a_k/\lambda_k)x_k\right|}{x_n}$$

if $\left|\sum_{k=n}^{\infty}(a_k/\lambda_k)\right| \geq \left|\sum_{k=n}^{\infty}(b_k/\mu_k)\right|$

$= -\infty$, as $(a_n/\lambda_n) \cong ((a_n/\lambda_n)x_n, x_n)$.

Again if $\left|\sum_{k=n}^{\infty}(b_k/\mu_k)\right| > \left|\sum_{k=n}^{\infty}(a_k/\lambda_k)\right|$, then also the same result follows. Hence $f+g$ is entire if f and g are so. Next zf is entire, since

$$\overline{\lim_{n\to\infty}}(1/x_n)\log\left|\sum_{k=n}^{\infty}(za_k/\lambda_k)x_k\right| = |z|\overline{\lim_{n\to\infty}}(1/x_n)\log\left|\sum_{k=n}^{\infty}(a_k/\lambda_k)x_k\right| =$$

$= -\infty$, as $(a_n/\lambda_n) \cong ((a_n/\lambda_n)x_n, x_n)$.

By standard inequality it follows that $p(f+g) \leq p(f) + p(g)$, $p(x) = \rho(x,\theta)$. $p(f) \geq 0$, $p(f) = 0$ iff $f = \theta$. But $p(zf) \neq |z|p(f)$. Hence p is not homogeneous. In order to prove that ρ is a supermetric, it therefore, remains to show that scalar operation is continuous with respect to p. Let $\{f_i\} \subset \Omega$, $f \in \Omega$ such that $f_i \to f$ as $i \to \infty$. We show that $zf_i \to zf$ for each $z \in \mathbf{C}$. Now

$$\begin{aligned} p(zf) &= \rho(zf,\theta) = \sup_s \Phi(s)|zf(s)|/(1+|zf(s)|) \\ &= |z|\sup_s \Phi(s)|f(s)|/(1+|zf(s)|) \\ &\leq |z|\; p(f), \qquad |z| \geq 1. \end{aligned}$$

So, $p(z(f_i - f)) \leq |z|\; p(f_i - f)$, $\qquad |z| \geq 1$.

Hence $zf_i \to zf$ when $f_i \to f$, as $i \to \infty$, for each $z \in \mathbf{C}$.

When $|z| < 1$, $p(z(f_i - f)) < p(f_i - f)$. Hence, for $f_i \to f$, $zf_i \to zf$, as $i \to \infty$.

Thus $f_i \to f \Rightarrow zf_i \to zf$ as $i \to \infty$, for each $z \in \mathbf{C}$.

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Next we show that if $\{z_i\} \subset \mathbf{C}$, $z \in \mathbf{C}$ be such that $z_i \to z$ as $i \to \infty$, then $z_i f \to zf$ as $i \to \infty$ for each $f \in \Omega$. For $|z| \geq 1$, the result is obvious, as $p((z_i - z)f) \leq |z_i - z| p(f)$ when $|z| \geq 1$. When $|z| < 1$,

$$\begin{aligned} p(zf) &= \sup_s \Phi(s)|zf(s)|/(1 + |zf(s)|) = |z| \sup_s \Phi(s)|f(s)|/(1 + |zf(s)|) \\ &\leq |z| \sup_s \{\Phi(s)|f(s)|\} \leq |z| \sup_s \Phi(s) \sup_s |f(s)| = |z|\Phi(\xi_o) \sup_s |f(s)| \end{aligned}$$

So $p((z_i - z)f) \leq |z|\Phi(\xi_o) \sup_s |f(s)|$. Hence $z_i f \to zf$ if $z_i \to z$, as $i \to \infty$.

This proves that p is a paranorm and so ρ is a supermetric on Ω. Now we prove that Ω is complete with respect to ρ. Let $\{f_i\} = \{a_{in}, \lambda_{in}\}$ be a Cauchy sequence in Ω. Then

$$\sup_s \Phi(s)\,|f_i(s) - f_j(s)| < \varepsilon\ ,\ i,\ j \geq N \tag{3.1}$$

where $\varepsilon > 0$ is arbitrary and N is a positive integer depending on ε. So

$$|f_i(s) - f_j(s)| < \varepsilon',\ \text{for } i,\ j \geq N \tag{3.2}$$

Hence $\{f_i(s)\}$ is a Cauchy sequence in $\mathbf{C}$. Let $f_i(s) \to f(s)$ in $\mathbf{C}$, as $i \to \infty$, where $f = (a_n, \lambda_n)$, say. We show that $f_i \to f$ as $i \to \infty$ and that $f \in \Omega$.

Now keeping i fixed and talking $j \to \infty$ in (3.2), We have $\sup_s |f_i(s) - f(s)| < \varepsilon'$, if $i \geq N$.So ultimately we have $\rho(f_i,\ f) \to 0$ as $i \to \infty$. Thus $f_i \to f$ as $i \to \infty$. So

$$\sup_s \Phi(s)\,|(f_i - f)(s)| < \varepsilon_o\ ,\ i \geq M \tag{3.3}$$

where ε_o is arbitrary and the positive integer M depends on ε_o. Now

$$|(f_i - f)(s)| = \left|\sum_{n=1}^{\infty}((a_{in}/\lambda_{in}) - (a_n/\lambda_n))x_n \exp(-x_n s)\right|$$

So it follows from (3.3) that $\left|\sum_{n=1}^{\infty}((a_{in}/\lambda_{in}) - (a_n/\lambda_n))x_n \exp(-x_n s)\right| < \varepsilon''\ , i \geq M$, say. Thus $|(a_{in}/\lambda_{in}) - (a_n/\lambda_n)| \to 0$ as $i \to \infty$. Hence, $(a_n/\lambda_n) = (a_{in}/\lambda_{in}) + \delta_{in}$, $\delta_{in} \in \mathbf{C}, |\delta_{in}| \to 0$ as $i \to \infty$. So $a_n(\lambda_{in}/\lambda_n) = a_{in} + \delta'_{in}, \left|\delta'_{in}\right| = |\lambda_{in}\delta_{in}| \to 0$ as $i \to \infty$.

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Now putting $a_1 = a_2 = 0$ and writing $f(x,y,z) = L_{a,b,m,n}(x) y^m z^n$ in (2.5), we get

$$e^{a_6 A_6} e^{a_5 A_5} e^{a_4 A_4} e^{a_3 A_3} (L_{a,b,m,n}(x) y^m z^n) =$$

$$(3.1) \qquad = \left(1 + \frac{a_4}{y} + \frac{a_5 bz}{y}\right)^{m-1} \exp\left[-\frac{a_5 az}{y}\left(x + \frac{a_6}{a}\frac{y}{z}\right) - a_3 y \left(1 + \frac{a_4}{y} + \frac{a_5 bz}{y}\right)\right] \cdot$$

$$\cdot L_{a,b,m,n}\left(\left(x + \frac{a_6}{a}\frac{y}{z} + \frac{a_3 by}{a}\right)\left(1 + \frac{a_4}{y} + \frac{a_5 bz}{y}\right)\right) y^m z^n .$$

But

$$e^{a_6 A_6} e^{a_5 A_5} e^{a_4 A_4} e^{a_3 A_3} (L_{a,b,m,n}(x) y^m z^n) =$$

$$(3.2) \qquad = \sum_{s=0}^{n+r} \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \frac{(-a_6)^s}{s!} \frac{a_5^r}{r!} \frac{(-a_4)^k}{k!} \frac{(-a_3)^p}{p!} (n+1)_r (-n-p-m+1)_k \cdot$$

$$\cdot L_{a,b,m+p-k-r+s,n+r-s}(x) y^{m+p-k-r+s} z^{n+r-s} .$$

Equating the two results (3.1) and (3.2), we get

$$\left(1 + \frac{a_4}{y} + \frac{a_5 bz}{y}\right)^{m-1} \exp\left[-\frac{a_5 az}{y}\left(x + \frac{a_6}{a}\frac{y}{z}\right) - a_3 y \left(1 + \frac{a_4}{y} + \frac{a_5 bz}{y}\right)\right] \cdot$$

$$(3.3) \qquad \cdot L_{a,b,m,n}\left(\left(x + \frac{a_6}{a}\frac{y}{z} + \frac{a_3 by}{a}\right)\left(1 + \frac{a_4}{y} + \frac{a_5 bz}{y}\right)\right) =$$

$$= \sum_{s=0}^{n+r} \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \frac{(-a_6)^s}{s!} \frac{a_5^r}{r!} \frac{(-a_4)^k}{k!} \frac{(-a_3)^p}{p!} (n+1)_r (-n-p-m+1)_k \cdot$$

$$\cdot L_{a,b,m+p-k-r+s,n+r-s}(x) y^{p-k-r+s} z^{r-s} .$$

The above generating function does not seem to appear before. We can get a large number of generating functions from (3.3) by attributing different values to a_i $(i = 3,4,5,6)$.

Before discussing the particular cases of (3.3), it may be pointed out that if at least one of the all possible pairs of the operators A_i $(i = 3,4,5,6)$ be non-commutative then

$$e^{a_6 A_6} e^{a_5 A_5} e^{a_4 A_4} e^{a_3 A_3} \neq e^{a_6 A_6 + a_5 A_5 + a_4 A_4 + a_3 A_3} .$$

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Theorem 3.2 *Let Ω be the set of all entire Dirichlet series of the form $f = (a_n, f_n)$ where $D = \overline{\lim}_{n\to\infty}(\log n/\lambda_n) \neq 0$. Let for $g = (b_n, \mu_n) \in \Omega$, $z \in \mathbf{C}$ and a bounded function Φ on $\mathbf{C}$, $f + g$, zf, $\rho(f,g)$ be defined as in the theorem 3.1. Let D be a convex domain of Ω. Let $T : D \to \Omega$ be Frêchet differentiable having T' as the Frêchet derivative on D. Let the iterative sequence of approximate values for the solution of the equation $Tf = \theta$ be given by simplified Newton's method as $f_{n+1} = f_n - T_{f_o}^{\prime -1} f_n$, where $T_{f_o}^{\prime -1}$ exists at $f_o \in D$. If $P = \left\|T_{f_o}^{\prime -1}\right\| \sup_{f\in D} \left\|T'_{u_o} - T'_f\right\| < 1$ and if with f_o, f_1 as the first two members of $\{f_n\}$, the sphere $K = \{g \in \Omega \ / \ \rho(g, f_1) < (p/1-p)\rho(f_o, f_1)\} \subset D$, then there exists in D a solution of the equation $Tf = \theta$, where f lies also in K, f_n remains in D and $f_n \to f$ as $n \to \infty$.*

Concluding Remarks: Although a Banach space is a L-Banach space, but the construction of the functional L may not be possible, as its existence is proved by Hahn Banach theorem which depends on Zorn's lemma. Even when the Banach space is separable, it is very difficult to find out L. To find out L corresponding to $h \neq \theta$ adds further difficulty. In fact, its existence does not follow directly from Hahn Banach theorem. In case of a supermetric space existence of L is not assured as Hahn Banach theorem is not available. However, construction of L may be tried directly. Even when the supermetric space is not separable, it may be impossible to find out L.

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