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AN APPLICATION OF THE DISTRIBUTIONAL GENERALIZED HANKEL-CLIFFORD TRANSFORMATION

S.P.MALGONDE AND B.B.WAPHARE*
Deptt. of Mathematics,College of Engineering,Kopargaon-423 603 (M.S.) India.
***Maeer,s Maharashtra Academy Of Engineering, Alandi (Devachi) (M.S.) India.**

Abstract In this paper a relation between the generalized Hankel-Clifford transformation and the generalized Laplace transformation in the distributional sense is obtained and the theory thus developed is used to solve a distributional integral equation in certain spaces of generalized functions.

KEY WORDS Generalized Hankel-Clifford Transformation , Generalized Laplace Transformation , Generalized Functions, Integral Equation,Application.

1. INTRODUCTION

After Schwartz's [11] extension of the Fourier transform to generalized functions,the extension of classical integral tranformations to generalized functions have comprised an active and interesting area of research (see,for example,Dube and Pandey [3], Koh and Zemanian [4],Zemanian [12] and Mendez [9 and 10]).

Generally there are three approches to extend any classical transform to generalized functions.The first approach is called the method of adjoint in which the transformation can be extended to certain generalized functions through a generalization of a suitable Parseval's equation or through a generalization of a suitable mixed Parseval's equation.This approach has been followed by Schwartz [11], Zemanian [14] and Mendez [9 and 10],amongst others.

The second approach is called the kernel method in which a suitable testing function space is constructed over the range of the integral defining the transformation in such a way that the kernel function be a member of this space.This approach has been followed by Zemanian [14], Koh and Zemanian [4], Dube and Pandey [3] ,and others. The third approach called the method of convolution has been followed by Zemanian [13] and others.

Recently the generalized Hankel-Clifford transformation defined by

$$F_{\alpha,\beta}\{f(x)\}(y) = F(y) = y^{-\alpha-\beta}\int_0^\infty (xy)^{(\alpha+\beta)/2} J_{\alpha-\beta}(2\sqrt{xy})f(x)dx = \int_0^\infty \left(\frac{y}{x}\right)^{-(\alpha+\beta)/2} J_{\alpha-\beta}(2\sqrt{xy})f(x)dx \ ,$$

where $(\alpha-\beta)\geq -\frac{1}{2}$ and $J_{\alpha-\beta}(z)$ is a Bessel function of the first kind of order $(\alpha-\beta)$, has been extended to certain classes of generalized functions by the method of adjoint [6],by the kernel method [7] and by the convolution method [8].The aim of the present paper is to use the theory of distributional generalized Hankel-Clifford transformation recently developed by [7] with the help of one sided distributional generalized Laplace transformation [5] to solve distributional integral equation.

The notations and terminology used are those of [14].Throughout this work I denotes the open interval $(0,\infty)$.D(I) the space of smooth functions whose supports are compact subsets of I.We assign to D(I) the topology that makes its dual $D'(I)$ the space of Schwartz distribution on I [11].E(I) and $E'(I)$ are respectively the space of smooth functions on I and the space of distributions having compact support with respect to I. These spaces have their customary topologies [14].

Following Malgonde [5 and 7] and Zemanian [14] we define the testing function spaces and their duals which we need in solving the distributional integral equations.

2. The testing function spaces $G_{+,a}$; $G_+(w)$ and $G^\alpha_{+,a}$ and $G^\alpha_+(w)$ and their duals

The conventional one –sided Laplace transformation [14] defined by

$$F_1(s) = (L_+f)(s) = \int_0^\infty f(t)\, e^{-st}dt \qquad (2.1)$$

is extended to generalized functions as follows:

Let I be the open interval $(0,\infty)$ and restrict t to I .Let a be any fixed real number.Then , for each a , $G_{+,a}$ denotes the space of all smooth functions $\phi(t)$ on I such that

$$\lambda_{a,k}(\phi) = \sup_{0<t<\infty}\left|e^{at}D^k\phi(t)\right| < \infty \tag{2.2}$$

for k = 0,1,2,…and its topology is generated by $\{\lambda_{a,k}\}_{k=0}^{\infty}$. $G_{+,a}$ is a complete countably multi-normed space. $G_+(w) = \bigcup_{\nu=1}^{\infty} G_{+,a_\nu}$ is countable-union space where $\{a_\nu\}_{\nu=1}^{\infty}$ is a monotonic squence of real numbers such that $a_\nu \to w+$, where w is a real number or $-\infty$.

A generalized function f is L_+-transformable if $f \in G'_+(w)$ for some w, where $G'_+(w)$ is the dual of $G_+(w)$. If σ_f be the infimum of all such w then we define the distributional Laplace transform of f by

$$F_1(s) \overset{\Delta}{=} (L_+ f)(s) \overset{\Delta}{=} \langle f(t), e^{-st} \rangle \text{ , } \operatorname{Re} s > \sigma_f \tag{2.3}$$

In [5],a simple generalization of the two-sided Laplace transformation defined by

$$G(s) = (L^\lambda g)(s) = \int_{-\infty}^{\infty} (st)^\lambda e^{-st} g(t)dt$$

for a suitably restricted function g(t) , where λ is any real number , is extended to a certain class of generalized function space by establishing an isomorphism between the constructed space and the space of two-sided Laplace transformable generalized functions [14].

So following [5] the conventional one-sided generalized Laplace transformation (generalization of (2.1)) defined by

$$F_2(s) = (L^\alpha{}_+ f)(s) = \int_0^\infty f(t)\,(st)^\alpha e^{-st}dt \tag{2.4}$$

is extended to generalized functions as follows:

For any real numbers a and α , $G^\alpha_{+,a}$ denotes the space of all smooth functions ϕ(t) on I such that

$$\rho^k_{a,\alpha}(\phi) \overset{\Delta}{=} \sup_{0<t<\infty}\left|e^{at}t^{k-\alpha}D^k_t\phi(t)\right| < \infty \tag{2.5}$$

for k = 0,1,2,…. and its topology is generated by $\{\rho^k_{a,\alpha}\}_{k=0}^{\infty}$. $G^\alpha_{+,a}$ is a complete countably multinormed space. $G^\alpha_+(w) = \bigcup_{\nu=1}^{\infty} G^\alpha_{+,a}$ is a countable-union space where $\{a_\nu\}_{\nu=1}^{\infty}$ is a monotonic sequence of real numbers such that $a \to w+$, where w is a real number or $-\infty$.

A generalized function f is L^α_+ -transformable if $f \in (G^\alpha_+(w))'$ for some w,where $(G^\alpha_+(w))'$ is the dual of $G^\alpha_+(w)$.We define the distributional generalized Laplace transform of f by

$$F_2(s) \overset{\Delta}{=} (L^\alpha{}_+ f)(s) \overset{\Delta}{=} \langle f(t), (st)^\alpha e^{-st} \rangle \text{ , } \operatorname{Re} s > \sigma_f \tag{2.6}$$

where σ_f is the infimum of all such w. $F_2(s)$ is analytic for $\operatorname{Re} s > \sigma_f$.

Let f be a locally integrable function on I and if $\int_0 |f(t)|\, t^\alpha e^{-at}dt < \infty$ for every a satisfying a>w, then f generates a regular generalized function in $(G^\alpha{}_+(w))'$ through the definition:

$$\langle f, \phi \rangle = \int_0^\infty f(t)\,\phi(t)dt \quad , \quad \phi \in G^\alpha{}_+(w) \tag{2.7}$$

3.The Testing function spaces $H_{\alpha,\beta,a}$, $H_{\alpha,\beta}(\sigma)$ and their duals

Let a denote a positive real number and α and β are any real numbers. Then we define $H_{\alpha,\beta,a}$ [7] as the space of testing functions ϕ(x) which are defined on 0<x<∞ and for which

$$\gamma_k^{\alpha,\beta,a}(\phi) \overset{\Delta}{=} \sup_{0<x<\infty}\left|e^{-ax}x^{-\alpha}\Delta^k_{\alpha,\beta}\phi(x)\right| < \infty \tag{3.1}$$

for k = 0,1,2,…. ,where

$$\Delta_{\alpha,\beta} = \Delta_{\alpha,\beta,x} = x^\beta D_x x^{\alpha-\beta+1} D_x x^{-\alpha} = [xD_x^2 - (\alpha+\beta-1)D_x + \alpha\beta x^{-1}] \tag{3.2}$$

is the generalized Kepinski operator. $H_{\alpha,\beta,a}$ is a complete countably multinormed space.

$H_{\alpha,\beta}(\sigma) = \bigcup_{\nu=1}^{\infty} H_{\alpha,\beta,a_\nu}$ denotes the countable-union space where $\{a_\nu\}_{\nu=1}^{\infty}$ is a monotonocally increasing sequence of positive numbers tending to σ (σ = ∞ is allowed). $H'_{\alpha,\beta,a}$ and $H'_{\alpha,\beta}(\sigma)$ are the duals of $H_{\alpha,\beta,a}$ and $H_{\alpha,\beta}(\sigma)$ respectively.

We now list some of the properties of these spaces.

(i) For any fixed y>0 , $\frac{\partial^m}{\partial y^m}\left\{(y/x)^{-(\alpha+\beta)/2}J_{\alpha-\beta}(2\sqrt{xy})\right\}\in H_{\alpha,\beta}(\sigma)$,$m=0,1,2,...$ *where* $\sigma>0$.

(ii) If a>b>0, then $H_{\alpha,\beta,b}\subset H_{\alpha,\beta,a}$, and the topology of $H_{\alpha,\beta,b}$ is stronger than that induced on it by $H_{\alpha,\beta,a}$.

(iii) For each $f\in H'_{\alpha,\beta,a}$, there exist a non-negative integer r and a positive constant c such that, for all $\phi\in H_{\alpha,\beta,a}$,

$$\left|<f,\phi>\right|\le c\max_{0\le k<r}\gamma_k^{\alpha,\beta,a}(\phi) \tag{3.3}$$

(iv) Let f(x) be locally integrable on 0<x<∞ and such that $\int_0^\infty \left|f(x)e^{ax}x^{\alpha}\right|dx<\infty$.

Then f(x) generates a regular generalized function in $H'_{\alpha,\beta,a}$ defined by

$$<f,\phi>=\int_0^\infty f(x)\,\phi(x)dx,\ \phi\in H_{\alpha,\beta,a} \tag{3.4}$$

Let α,β be real numbers restricted to $-1/2<(\alpha-\beta)<\infty$.In view of note (ii) (see also [7],§ 3), to every $f\in H'_{\alpha,\beta,a}$ there exists a unique real number σ_f (possibly $\sigma_{f=\infty}$) such that $f\in H'_{\alpha,\beta,b}$ if $b<\sigma_f$ and $f\notin H'_{\alpha,\beta,b}$ if $b>\sigma_f$. Therefore $f\in H'_{\alpha,\beta}(\sigma_f)$. We define the distributional generalized Hankel-Clifford transform $F'_{\alpha,\beta}$ f of f as the application of f to the kernel

$$k(x,y)=y^{-\alpha-\beta}(xy)^{(\alpha+\beta)/2}J_{\alpha-\beta}(2\sqrt{xy})=(y/x)^{-(\alpha+\beta)/2}J_{\alpha-\beta}(2\sqrt{xy}) \tag{3.5}$$

That is ,

$$F'(y)=(F'_{\alpha,\beta}f)(y)=<f(x),k(x,y)> \tag{3.6}$$

where y>0 and σ_f>0.

<u>**THEOREM 1**</u> Let F'(y) be defined by (3.6).Then F'(y) is bounded according to

$$|F'(y)|\le\begin{cases} c\,y^{-\beta} & as\ y\to 0+ \\ c\,y^{r-\beta} & as\ y\to\infty\end{cases}$$

where c is a positive constant and r is some non-negative integer depending on f.

PROOF: Proof follows from [7,p.63,Theorem 8] .

<u>4. APPLICATION</u>

The classical problem is that of solving the integral equation

$$f(x)+k\int_0^\infty(\frac{x}{y})^{-(\alpha+\beta)/2}J_{\alpha-\beta}(2\sqrt{xy})f(y)dy=g(x)$$

$$or\quad f(x)+k(F_{\alpha,\beta}\ f)(x)=g(x),$$

where g(x) is a prescribed function and (α−β)≥-1/2.In general problem, g(x) is a prescribed distribution from certain space and we seek a distribution f(x) such that

$$f(x)+k(F'_{\alpha,\beta}\ f)(x)=g(x)$$

in the sense of equality in certain space.

In the end ,we shall use the preceding theory so far developed in solving the above distributional integral equation.To formulate and solve this we require the following lemmas concerning the relation between generalized Laplace transformation and generalized Hankel-Clifford transformation .Thus taking into account [2,p.185(30)]

$$L_+^{\alpha}[k(x,y)](p)=\int_0^\infty\left(\frac{y}{x}\right)^{-(\alpha+\beta)/2}J_{\alpha-\beta}\left(2\sqrt{xy}\right)(py)^{\alpha}e^{-py}dy=p^{\beta-1}x^{\alpha}e^{-x/p},\mathrm{Re}(p)>0\ \text{and}\ \mathrm{Re}(\beta)<1$$

and proceeding as in [1,pages 62-63], we obtain

<u>LEMMA 1</u> If Rep > 0 and Re(a + b) < 1, then

$$L_+^{\alpha}\{(F_{\alpha,\beta}f(x))(y)\}(p)=p^{\alpha+\beta-1}F_2\left(\frac{1}{p}\right) \tag{4.1}$$

where

$$F_2(p)=L_+^{\alpha}[f(x)](p)=\int_0^\infty(px)^{\alpha}e^{-px}f(x)dx$$

LEMMA 2 Let $(\alpha-\beta)\geq-1/2$ and $0<w<1$. Let $f\in H'_{\alpha,\beta}(\sigma)\cap(G^{\alpha}_{+}(w))'$. If $F'_{\alpha,\beta}(f)$ is as defined by (3.6), then

$$L^{\alpha}_{+}\{(F'_{\alpha,\beta}f)(y)\}(p)=p^{\alpha+\beta-1}F_2(1/p) \tag{4.2}$$

for $w<\text{Re}p<1/w$ where $F_2(p)$ is as defined by (2.6).

PROOF: In view of Theorem 1, and for $w>0$, $\int_0^{\infty} y^{a}e^{-ay}\left|(F'_{\alpha,\beta}f)(y)\right|dy<\infty$

for every a satisfying $a>w$, which shows that $(F'_{\alpha,\beta}f)(y)$ generates a regular generalized function in $(G^{\alpha}_{+}(w))'$ and therefore, if $\text{Re}p>w$, the generalized Laplace transform of $(F'_{\alpha,\beta}f)(y)$ is given by

$$\begin{aligned}L^{\alpha}_{+}\{(F'_{\alpha,\beta}f)(y)\}(p)&=<(F'_{\alpha,\beta}f)(y),(py)^{\alpha}e^{-py}>=\int_0^{\infty}(F'_{\alpha,\beta}f)(y)(py)^{\alpha}e^{-py}dy\\&=\int_0^{\infty}<f(x),k(x,y)>(py)^{\alpha}e^{-py}dy\\&=<f(x),\int_0^{\infty}k(x,y)(py)^{\alpha}e^{-py}dy> \qquad (4.3)\\&=<f(x),p^{\beta-1}x^{\alpha}e^{-x/p}>=<f(x),p^{\alpha+\beta-1}(x/p)^{\alpha}e^{-x/p}>\\&=p^{\alpha+\beta-1}<f(x),(x/p)^{\alpha}e^{-x/p}>\\&=p^{\alpha+\beta-1}F_2(1/p)\;,\; if\;\; w<\text{Re}\,p<1/w\end{aligned}$$

which proves the lemma if we justify the equation (4.3). It can justified as follows:
For any $R>0$, if $(\alpha-\beta)\geq-1/2$

$$\int_0^{R}k(x,y)(py)^{\alpha}e^{-py}dy\in H_{\alpha,\beta}(\sigma)$$

and using the technique of Riemann sums it can be shown that

$$\int_0^{R}<f(x),k(x,y)>(py)^{\alpha}e^{-py}dy=<f(x),\int_0^{R}k(x,y)(py)^{\alpha}e^{-py}dy> \tag{4.4}$$

Again it can be shown that when $(\alpha-\beta)\geq-1/2$ and $\sigma>0$

$$\int_0^{R}k(x,y)(py)^{\alpha}e^{-py}dy\to\int_0^{\infty}k(x,y)(py)^{\alpha}e^{-py}dy \;\; in\;\; H_{\alpha,\beta}(\sigma)\;\; as\;\; R\to\infty.$$

Therefore equation (4.3) now follows by letting $R\to\infty$ in (4.4). Thus the proof is complete.

Now we use these facts to solve the following distributional integral equation given by

$$f(y)+k\{F'_{\alpha,\beta}f(x)\}(y)=g(y) \tag{4.5}$$

in the space $H'_{\alpha,\beta}(\sigma)\cap(G^{\alpha}_{+}(w))'$ where $\sigma>0$, $0<w<1$ and α is an integer and $g(y)$ is a known distribution in $H'_{\alpha,\beta}(\sigma)\cap(G^{\alpha}_{+}(w))'$ with $k\neq-1$.

By applying the distributional generalized Laplace transform L_{+}^{α}, (4.5) can be rewritten as

$L^{\alpha}_{+}[f(y)]+kL^{\alpha}_{+}\left[\{F'_{\alpha,\beta}f\}(y)\right]=L^{\alpha}_{+}[g(y)]$.

Using (4.2), we get

$$(L^{\alpha}_{+}f)(p)+k\,p^{\alpha+\beta-1}(L^{\alpha}_{+}f)(1/p)=(L^{\alpha}_{+}g)(p) \tag{4.6}$$

for $w<\text{Re}p<1/w$. Now replacing p by 1/p in (4.6), we have

$$(L^{\alpha}_{+}f)(1/p)+k\,p^{-\alpha-\beta+1}(L^{\alpha}_{+}f)(p)=(L^{\alpha}_{+}g)(1/p) \tag{4.7}$$

for $w<\text{Re}1/p<1/w$.

Eliminating $((L^{\alpha}_{+}f)(1/p)$ from (4.6) and (4.7), we get

$$\left(L_+^\alpha f\right)(p) = \frac{1}{(1-k^2)}\left[\left(L_+^\alpha g\right)(p) - kp^{\alpha+\beta-1}\left(L_+^\alpha g\right)(1/p)\right]$$

$$= \frac{1}{(1-k^2)}\left[\left(L_+^\alpha g\right)(p) - kL_+^\alpha\left[\left\{F'_{\alpha,\beta} g(x)\right\}(y)\right](p)\right]$$

Therefore

$$L_+^\alpha f = \frac{1}{(1-k^2)} L_+^\alpha \left\{g - kF'_{\alpha,\beta}\ g(x)\right\}.$$

Taking inverse distributional generalized Laplace transform we get

$$f(y) = \frac{1}{(1-k^2)}\left\{g(y) - k(F'_{\alpha,\beta}\ g(x))(y)\right\}$$

the required formal solution of the distributional integral equation (4.5).

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