SOME EXTREMAL PROPERTIES OF ORTHOGONAL POLYNOMIALS

Francisco Pérez Acosta

Departamento de Análisis Matemático

Universidad de La Laguna

38271 La Laguna, Tenerife, Canary Islands

Spain

Abstact

Extremal properties of the Chebychev polynomials are stablished. Also extremal properties of general orthogonal polynomials are proved.

1. Introduction

let f be a continuous function, $f : [a,b] \longrightarrow \mathbb{R}$. We denote by P_n the space of all polynomials of degree n at most. Let p_n be the best approximation of f in the uniform norm $\| \ \|_\infty$,

$$\|f\|_\infty = \max_{a \leq x \leq b} |f(x)|$$

The minimax of f is

$$E_n(f) = \|f-p_n\|_\infty = \inf_{p \in P_n} \|f-p\|_\infty$$

Consider now the infinite sum of all the errors $E_n(f)$, such series can be seen as a "measure" of "how good" the function f is approximated by polynomials. By this reason, the study of the called minimax series

$$S(f) \equiv \sum_{k=0}^{\infty} E_k(f) \ ,$$

turns into important. As a consequence of the Weierstrass Approximation Theorem, one has

$$\lim_{n \to \infty} E_n(f) = 0 \ .$$

Some aspects concerning with this series can be seen in [4]. There exist continuous functions f on [a,b] such that $S(f)=\infty$, for example in the case $[-1,1]=[a,b]$, we have $S(|x|)=\infty$, because $E_n(|x|) \geq \frac{1}{2\pi(n-1)}$ (see [5] pp.148).

We also we consider the operator T, defined by

$$T(f)=S\left(\int_{x_0}^{x} f(t)dt\right) \quad , \ (x_0 \in [a,b] \text{ fixed})$$

on the space of all continuous functions on [a,b] such that $S(f)<\infty$.

The Chebychev polynomials of first kind $T_n(x)=\cos n\vartheta$, $\cos\vartheta=x$, $x\in[-1,1]$ have a minimal property related to S. Furthermore, the Chebychev polynomials of the sencond kind $U_n(x)=\frac{\operatorname{sen}(n+1)\vartheta}{\operatorname{sen}\vartheta}$ have a minimal property respect to T.

Analogously we costruct other operator in connection with L_p-norms,

$$\|f\|_p = \left(\int_a^b |f|^p\right)^{1/p} .$$

If p_n denotes a best approximation of a function f on P_n with respect to the L_p-norm, and we can to consider the series

$$s_p(f) = \sum_{k=0}^{\infty} \|f-p_k\|_p .$$

Special treatment requires the case p=2, when a function weight w on [a,b] is taken, so that we have:

$$\|f\|_{2w} = \left(\int_a^b |f|^2 w\right)^{1/2}$$

The series $S_{2w}(f) = \sum_{k=0}^{\infty} \|f-\tilde{p}_k\|_{2w}$ where $\tilde{p}_n$ is the best approximation of f on P_n, i.e. $\tilde{p}_n = \sum_{k=0}^{n} \langle f,p_k\rangle p_k$ where $\{p_k\}$ is the sequence of orthogonal polynomials with respect to w and

$$\langle f,p_k\rangle=\int_a^b f(x)p_k(x)w(x)dx \quad ,$$

In this cases the orthogonal polynomials respect to w have extremal properties related to the series S_{2w}.

2. Extremal properties of the Chebychev polynomials related to S and T

2.1 Extremal property of the Chebychev polynomial of the first kind

The nth Chebychev polynomial of first kind is $T_n(x) = \cos n\vartheta$, where $\cos\vartheta=x$, $x\in[-1,1]$. As it is well known such polynomial has extremal properties (see e.g. [4]).

Next, we stablish an extremal property for $T_n(x)$ related to S.

In this case $[a,b]=[-1,1]$, and define $\tilde{T}_n(x) = \dfrac{T_n(x)}{2^{n-1}}$ which is monic.

We denote by $\tilde{P}_n$ the set of all monic polynomials of degree n.

We have the next extremal property

Theorem 1

(a) $S(\tilde{T}_n) = \dfrac{n}{2^{n-1}}$ and

(b) $S(p) \geq S(\tilde{T}_n)$, $\forall\ p \in \tilde{P}_n$

Proof

This result can be found in [3], in a different context. The best approximating polynomial p_k for $\tilde{T}_n$ in P_k is $p_k\equiv 0$ for $0\leq k\leq n-1$ and $\tilde{T}_n$ for $k\geq n$.

Then $\quad E_k(\tilde{T}_n) = \|\tilde{T}_n\|_\infty = \dfrac{1}{2^{n-1}}$ $(k<n)$ and $E_k(\tilde{T}_n)=0$ $(k\geq n)$,

hence,
$$S(\tilde{T}_n) = \sum_{k=0}^{n-1} E_k(\tilde{T}_n) = \frac{n}{2^{n-1}}.$$

Let $p \in \tilde{P}_n$ be. If $k<n$, and p_k denotes the best approximating polynomial on P_k, since $p(x)-p_k(x) \in \tilde{P}_n$, results
$$E_k(p) = \|p-p_k\|_\infty \geq \|\tilde{T}_n\|_\infty = \frac{1}{2^{n-1}}$$

As a consequence,
$$S(p) = \sum_{k=0}^{n-1} E_k(p) \geq \frac{n}{2^{n-1}} \qquad \blacksquare$$

2.2 Extremal property of the Chebychev polynomial of the second kind

The nth Chebychev polynomial of second kind is given by

$$U_n(x)=\frac{\operatorname{sen}(n+1)\vartheta}{\operatorname{sen}\vartheta}, \quad \cos\vartheta = x,\ x \in [-1,1],$$

The relation

$$U_n(x) = \frac{T'_{n+1}(x)}{(n+1)},$$

is well known.

On the other hand, $U_n(x) = 2^n x^n + \ldots$ and $\tilde{U}_n(x) = 2^{-n} U_n(x)$ is monic. We can stablish the next result.

Theorem 2

(a) $T(\tilde{U}_n) = \frac{1}{2^n}$

(b) $T(p) \geq \frac{1}{2^n}, \quad \forall\, p \in \tilde{P}_n$

Proof

(a)

$$T(U_n) = S\left(\int_{x_0}^{x} U_n(t)dt\right) = S\left(\frac{T_{n+1}(x)}{n+1} - \frac{T_{n+1}(0)}{n+1}\right) =$$

$$= S\left(\frac{T_{n+1}(x)}{n+1}\right) = \frac{1}{n+1} S(T_{n+1}) = 1 .$$

Then

$$T(\tilde{U}_n) = T(2^{-n} U_n) = 2^{-n} T(U_n) = 2^{-n}.$$

(b) Let $p(x) = x^n + \ldots \in \tilde{P}_n$ be, then

$$q(x) = \int_{x_0}^{x} p(t)dt = \frac{x^{n+1}}{n+1} + \ldots$$

and $T(p) = S(q) = S(x^{n+1} + \ldots)\frac{1}{n+1} \geq \left(\frac{n+1}{2^n}\right)\frac{1}{n+1} = 2^{-n}$ ■

2.3. Functions with equioscilations around a constant

We say that the function f presents m+1 equioscilations on [a,b] around a constant c if there is a set of m+2 points $A = \{x_0, x_1, \ldots, x_{m+1}\} \subset [a,b]$ such that

$$|f(x)-c| = \|f-c\|_\infty, \quad x \in A$$

$$f(x_k) - c = -(f(x_{k+1})-c) \quad (k=0,1,\ldots,m).$$

Proposition 1

If f present m+1 equioscilations around a contant on [a,b] and $f \in C[a,b]$, then

$$S(f+p) \geq S(f),\ \forall\ p \in P_m$$

Proof

From the Equioscilation Chebychev Theorem which characterizes the best approximating polynomial of a function on P_n in terms of equioscilations, (see for example [2]) we can assure that c is the best approximation of f on $P_m, P_{m-1}, \ldots, P_0$. Then

$$E_0(f) = E_1(f) = \ldots = E_{m-1}(f) = E_m(f)\ .$$

Let $p \in P_m$ be, then

$$E_k(f+p) = E_k(f) \quad \text{if } k \geq m \text{ and } E_k(f+p) \geq E_m(f)\ (k<m)\ .$$

Hence

$$S(f+p) = \sum_{k=0}^{m-1} E_k(f+p) + \sum_{k=m}^{\infty} E_k(f) \geq mE_m(f) + \sum_{k=m}^{\infty} E_k(f) =$$

$$= E_0(f)+E_1(f)+\ldots+E_{m-1}(f)+\sum_{k=m}^{\infty} E_k(f) = S(f) \quad \blacksquare$$

Remark. If $f=\tilde{T}_n$, then $\tilde{T}_n$ presents n equioscilations and we can assure by the above proposition that

$$S(\tilde{T}_n+p) \geq S(\tilde{T}_n), \quad \forall\ p \in P_{n-1}\ ,$$

note that,

$$\{\tilde{T}_n+p/\ p \in P_{n-1}\} = \tilde{P}_n\ .$$

Then we obtain

$$S(p) \geq S(\tilde{T}_n)\ ,\quad \forall\ p \in \tilde{P}_n$$

In this sense, Proposition 1, is a generalization of Theorem 1.

2.4 Other extremal property of the Chebychev polynomial of the second kind

If we consider the L_1 nom in C[a,b],

$$\|f\|_1=\int_a^b |f(t)|\,dt$$

and p_n denote a best approximation of f on P_n relative to this norm, we can consider the series

$$S_1(f) = \sum_{k=0}^{\infty} \|f-p_k\|_1 \quad ,$$

$\tilde{U}_n$ are extremal with S_1:

3. Extremal properties of Orthogonal Polynomials

3.1 Other extremal property of the Chebychev polynomial of the second kind

Proposition 2

(a) $S_1(\tilde{U}_n) = \dfrac{n}{2^{n-1}}$

(b) $S_1(p) \geq S_1(\tilde{U}_n)$, $\forall\ p \in \tilde{P}_n$

Proof

(a) The best approximation of $\tilde{U}_n$ on $P_0, P_1, \ldots, P_{n-1}$ related to the 1 norm is 0. This result follows from Theorem 2 in [1] pp. 22. Then

$$S_1(\tilde{U}_n) = n\int_{-1}^{1} |\tilde{U}_n(t)|\,dt = \frac{n}{2^{n-1}}$$

(see for example [3] pp.32-33).

(b) Let $p \in \tilde{P}_n$ be. If $k<n$, and p_k denote the best approximating polynomial on P_k in the 1 norm, $p(x)-p_k(x) \in \tilde{P}_n$, hence

$$\|p-p_k\|_1 \geq \|\tilde{U}_n\|_1 = \frac{1}{2^{n-1}} ,$$

and for $k\geq n$, the best approximating polynomial of p on P_k in the L_1 norm is p, Then

$$S_1(p) = \sum_{k=0}^{n-1} \|p-p_k\|_1 \geq \frac{n}{2^{n-1}} \quad \blacksquare$$

Analogously in the L_p norm

$$\|f\|_p = \left(\int_a^b |f(t)|^p \right)^{1/p} \quad ,$$

if p_n denotes a best approximation of f on P_n in the L_p norm, we can consider the series

$$S_p(f) = \sum_{k=0}^{\infty} \|f-p_k\|_p \ .$$

A special t. eatment require the case of the normed weighted

$$\|f\|_{2w}=\left(\int_a^b |f|^2 w\right)^{1/2} ,$$

w(x) being a function weight.

We introducing the series

$$S_{2w}(f) = \sum_{k=0}^{\infty} \|f-\tilde{p}_k\|_{2w},$$

where $\tilde{p}_n$ is the best approximation of f on P_n in the 2 w-norm. We have the next extremal properties of orthogonal polinomials.

3.2 Extremal maximal property

Let w(x) be a weigh function on [a,b]. (p_n) the sequence of orthonormal polynomials respect to w:

$$\langle p_m,p_n\rangle_{2w}=\int_a^b p_m(x)P_n(x)w(x)dx = \begin{cases} 0 & (m\neq n) \\ 1 & (m=n) \end{cases}$$

The nth orthonormal polynomial has the next extremal property.

Theorem 3

(a) $S_{2w}(p_n) = n$

(b) $S_{2w}(p) \leq S_{2w}(p_n) = n$, $\forall$ $p \in P_n$ such that $\|p\|_{2w} = 1$

Proof

(a) Let $f \in C[a,b]$ be, then we know that the best approximating polynomial on P_m is

$$f_m = \sum_{k=0}^{m} \langle f,p_k\rangle p_k ,$$

for $f=p_n$, then the best approximating polynomial on P_m is $q_m=p_n$ for $m\geq n$ and

$$q_m = \sum_{k=0}^{m} \langle p_n,p_k\rangle p_k = 0 \text{ for } m<n,$$

then $$S_{2w}(p_n) = \sum_{m=0}^{\infty} \|p_n-q_m\|_{2w} = \sum_{m=0}^{n-1} \|p_n\|_{2w} = n .$$

(b) Let $p \in P_n$ be,

$$p = \sum_{k=0}^{n} A_k p_k , A_k=\langle p,p_k\rangle ,$$

$$\|p\|^2_{2w} = A_0^2 + A_1^2 + \ldots A_n^2 = 1 \ ,$$

then

$$S_{2w}(p) = \sum_{r=0}^{n-1} \|p - q_r\|_{2w} \ ,$$

where

$$q_r = \sum_{k=0}^{r} A_k p_k \ ,$$

then

$$S_{2w}(p) = \sum_{r=0}^{n-1} \| \sum_{k=r+1} A_k \ p_k \|_{2w} =$$

$$= (A_1^2 + A_2^2 + \ldots + A_n^2)^{1/2} +$$

$$+ (A_2^2 + A_3^2 + \ldots + A_n^2)^{1/2} +$$

$$+ (A_3^2 + A_4^2 + \ldots + A_n^2)^{1/2} +$$

$$\vdots$$

$$+ (A_n^2)^{1/2} =$$

$$= (1 - A_0^2)^{1/2} + (1 - A_0^2 - A_1^2)^{1/2} + \ldots + (1 - A_0^2 - A_1^2 - \ldots - A_{n-1}^2)^{1/2} \le 1 + \ldots + 1 = n \ \blacksquare$$

Example

For $w(x) = \frac{1}{\sqrt{1-x^2}}$ we obtain the maximal property of the orthonormal Chebychev polynomial

$$\frac{2}{\pi} T_n(x) = \overline{T}_n(x) \ ,$$

$$S_{2w}(\overline{T}_n) \ge S_{2w}(p), \ \forall \ p \in P_n \ / \ \|p\|^2_{2w} = \int_{-1}^{1} \frac{p^2(x)}{\sqrt{1-x^2}} \, dx = 1$$

3.3 Extremal (minimal) property

Let $(\tilde{p}_n)$ be the sequence of monic orthogonal polynomials respect to w. $\tilde{p}_n$ is extremal in the space (C_{2w}, S_{2w}).

Theorem 4

(a) $S_{2w}(\tilde{p}_n) = n\|\tilde{p}_n\|_{2w}$

(b) $S_{2w}(p) \geq n\|\tilde{p}_n\|_{2w}, \ \forall\ p \in \tilde{P}_n$

Proof

(a) $\dfrac{\tilde{p}_n}{\|\tilde{p}_n\|_{2w}} = p_n$ nth orthonormal polynomial. Then by Theorem 4 (a), we have

$$S_{2w}\left(\frac{\tilde{p}_n}{\|\tilde{p}_n\|_{2w}}\right) = \frac{1}{\|\tilde{p}_n\|_{2w}} S_{2w}(\tilde{p}_n) = n ,$$

hence

$$S_{2w}(\tilde{p}_n) = n\|\tilde{p}_n\|_{2w} .$$

(b) Let $p \in \tilde{P}_n$ be. Consider

$$\int_a^b (\tilde{p}_n - p)^2\ w = \int_a^b \tilde{p}_n^{\,2}\ w + \int_a^b p^2\ w - 2\int_a^b \tilde{p}_n p\ w \tag{3.1}$$

by the other hand

$$p(x) = \sum_{k=0}^{n} A_k p_k(x) \quad ,$$

$$A_k = \langle p,p_k\rangle,\ (k=0,1,2,\ldots,n-1) \qquad A_n = \frac{1}{\|\tilde{p}_n\|_{2w}} .$$

Then $A_n p_n = \tilde{p}_n$ and

$$2\int_a^b \left(\sum_{k=0}^{n} A_k p_k\right) \tilde{p}_n\ w = 2A_n \int_a^b p_n \tilde{p}_n\ w = \int_a^b \tilde{p}_n^{\,2}\ w \tag{3.2}$$

From (3.1) and (3.2) result

$$\int_a^b (\tilde{p}_n - p)^2\ w = \int_a^b p^2\ w - \int_a^b \tilde{p}_n^{\,2}\ w \geq 0 , \tag{3.3}$$

That is

$$\|\tilde{p}_n - p\|_{2w}^2 = \|p\|_{2w}^2 - \|\tilde{p}_n\|_{2w}^2 \geq 0 , \tag{3.4}$$

Thus

$$\|\tilde{p}_n\|_{2w}^2 \leq \|p\|_{2w}^2 . \tag{3.5}$$

$S_{2w}(p) = \sum_{k=0}^{n-1} \|p-q_k\|_{2w}$, where q_k is the best polynomial approximation to p on P_k in the norm $\|\ \|_{2w}$. Note that $r_k = p-q_k \in \tilde{P}_n$, hence by (3.5) we have

$$\|r_k\|_{2w} \geq \|\tilde{p}_n\|_{2w} \quad (k=1,2,\ldots,n-1) .$$

Thus

$$S_{2w}(p)) \geq n\|\tilde{p}_n\|_{2w} \quad \blacksquare$$

Corollary

$$n\|\tilde{p}_n\|_{2w} \leq S_{2w}(p) \leq n\|p\|_{2w} \quad , \forall\ p \in \tilde{P}_n$$

Proof

The first inequality holds by Theorem 4b. For the second inequality note that

$$q = \frac{p}{\|p\|_{2w}} \text{ is such that } \|q\|_{2w} = 1,$$

hence by Theorem 3b,

$$S_{2w}(q) = \frac{S_{2w}(p)}{\|p\|_{2w}} \leq n \quad \blacksquare$$

4. References

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