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ON MIXED TRILATERAL GENERATING FUNCTIONS

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Abstract :

In this note the bilateral generating relations involving modified extended Jacobi polynomial $F_n(\alpha-n,\beta;x)$ derived by the present author have been generalized into mixed trilateral generating relations from Lie group view point. Some applications and several special cases of interest of our result are also discussed.

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1. INTRODUCTION

In a recent paper [1], the present author has proved the following bilateral generating relation :

$$(1.1) \qquad (1-\lambda t)^{\alpha}\left(1-\lambda\frac{x-a}{b-a}t\right)^{-\alpha-\beta-1} G\left(\frac{x-\lambda b\frac{x-a}{b-a}t}{1-\lambda\frac{x-a}{b-a}t}, \frac{zt}{1-\lambda t}\right)$$

$$= \sum_{n=0}^{\infty} F_n(\alpha-n,\beta;x)\,\sigma_n(z)\,t^n$$

where,

$$G(x,w) = \sum_{n=0}^{\infty} a_n F_n(\alpha-n, \beta; x)\, w^n ,$$

$$\sigma_n(z) = \sum_{k=0}^{n} a_k \binom{n}{k} z^k ,$$

and $F_n(\alpha,\beta;x)$ — the extended Jacobi polynomial is defined by [2]:

$$(1.2) \qquad F_n(\alpha,\beta;x) = \frac{(-1)^n}{n!}(x-a)^{-\alpha}(b-x)^{-\beta}\left(\frac{\lambda}{b-a}\right)^n$$

$$\times D^n\left[(x-a)^{n+\alpha}(b-x)^{n+\beta}\right], \quad D = \frac{d}{dx} .$$

The aim at writing this note is to generalize the above bilateral generating relation into mixed trilateral generating relation by group-theoretic method. In fact, the main result of this note is as follows :

<u>Theorem</u> : If

$$(1.3) \qquad G(x,u,w) = \sum_{n=0}^{\infty} a_n F_n(\alpha-n,\beta;x)\, g_n(u)\, w^n ,$$

where $g_n(u)$ an arbitrary polynomial of degree n, then

$$(1.4)\qquad (1-\lambda t)^{\alpha}\left(1-\lambda\frac{x-a}{b-a}t\right)^{-1-\alpha-\beta} G\left(\frac{x-\lambda b\frac{x-a}{b-a}t}{1-\lambda\frac{x-a}{b-a}t}, u, \frac{zt}{1-t\lambda}\right)$$

$$= \sum_{n=0}^{\infty} F_n(\alpha-n,\beta;x)\,\sigma_n(z,u)\,t^n$$

where

$$\sigma_n(z,u) = \sum_{k=0}^{n} a_k \binom{n}{k} g_k(u)\, z^k .$$

The importance of the above theorem lies in the fact that one can get a large number of generating relations from (1.4) by attributing suitable values to a_n in (1.3).

2. PROOF OF THE THEOREM

From [3], we notice that

$$(2.1)\qquad e^{wR} f(x,y) = (1+\lambda wy)^{\alpha}\left\{1+w\lambda\frac{x-a}{b-a}y\right\}^{-\alpha-\beta-1}$$

$$\times f\left(\frac{x+\lambda bw\frac{x-a}{b-a}y}{1+\lambda w\frac{x-a}{b-a}y}, \frac{y}{1+\lambda wy}\right)$$

where

$$R = \frac{\lambda}{b-a}(x-a)(b-x)\,y\frac{\partial}{\partial x} - \lambda y^2\frac{\partial}{\partial y} + \left\{(b-x)\alpha - (\beta+1)(x-a)\right\}\frac{\lambda y}{b-a},$$

such that

$$(2.2)\qquad R\left(F_n(\alpha-n,\beta;x)\,y^n\right) = -(n+1)\,F_{n+1}(\alpha-n-1,\beta;x)\,y^{n+1} .$$

Let us consider

$$(2.3) \qquad G(x,u,w) = \sum_{n=o}^{\infty} a_n F_n(\alpha-n,\beta;x)\, g_n(u)\, w^n .$$

Now replacing w by wyz and then operating (exp(wR)) on both sides, we get

$$(2.4) \qquad e^{wR}\, G(x,u,wyz) = e^{wR} \sum_{n=o}^{\infty} a_n (wz)^n \, (F_n(\alpha-n,\beta;x)\, y^n)\, g_n(u).$$

The left member of (2.4) is

$$(2.5) \qquad (1+\lambda wy)^{\alpha} \left(1+\lambda wy\, \frac{x-a}{b-a}\right)^{-\alpha-\beta-1}$$

$$\times\; G\left(\frac{x+\lambda bwy\, \frac{x-a}{b-a}}{1+\lambda wy\, \frac{x-a}{b-a}},\; u,\; \frac{wyz}{1+\lambda wy} \right)$$

The right member of (2.4) is

$$\sum_{n=o}^{\infty} \sum_{k=o}^{\infty} a_n (wz)^n \frac{w^k}{k!} R^k (F_n(\alpha-n,\beta;x)\, y^n)\, g_n(u)$$

$$= \sum_{n=o}^{\infty} \sum_{k=o}^{\infty} a_n (wz)^n \frac{w^k}{k!} (-1)^k (n+1)_k F_{n+k}(\alpha-n-k,\beta;x)\, y^{n+k}\, g_n(u).$$

$$= \sum_{n=o}^{\infty} \sum_{k=o}^{\infty} a_n (-z)^n \frac{(-wy)^{n+k}}{k!} (n+1)_k F_{n+k}(\alpha-n-k,\beta;x)\, g_n(u)$$

$$= \sum_{n=o}^{\infty} \sum_{k=o}^{n} a_{n-k} (-z)^{n-k} \frac{(-wy)^{n}}{k!} (n-k+1)_k F_{n}(\alpha-n,\beta;x)\, g_{n-k}(u)$$

$$(2.6) \qquad \sum_{n=o}^{\infty} (-wy)^n F_n(\alpha-n,\beta;x) \sum_{k=o}^{n} a_{n-k} \frac{(n-k+1)_k}{k!} g_{n-k}(u)\, (-z)^{n-k}$$

Now equating the above two members (2.5) and (2.6), we get

$$(2.7)\qquad (1+\lambda wy)^{\alpha}\left(1+\lambda wy\,\frac{x-a}{b-a}\right)^{-\alpha-\beta-1}$$

$$\times\; G\left(\frac{x+\lambda bwy\,\frac{x-a}{b-a}}{1+\lambda wy\,\frac{x-a}{b-a}},\; u\,,\; \frac{wyz}{1+\lambda wy}\right)$$

$$=\sum_{n=o}^{\infty}(-wy)^{n}\,F_{n}(\alpha-n,\beta;x)\sum_{k=o}^{n}a_{n-k}\,\frac{(n-k+1)_{k}}{k\,!}\,g_{n-k}(u)\,(-z)^{n-k}\,.$$

Replacing wy by "-t" and z by "-z" in (2.7), we get

$$(2.8)\qquad (1-\lambda t)^{\alpha}\left(1-\lambda t\,\frac{x-a}{b-a}\right)^{-\alpha-\beta-1}$$

$$\times\; G\left(\frac{x-\lambda bt\,\frac{x-a}{b-a}}{1-\lambda t\,\frac{x-a}{b-a}},\; u\,,\; \frac{tz}{1-\lambda t}\right)$$

$$=\sum_{n=o}^{\infty}t^{n}\,F_{n}(\alpha-n,\beta;x)\,\sigma_{n}(u,z)$$

where

$$\sigma_{n}(u,z)=\sum_{k=o}^{n}a_{k}\,\frac{(n+1)_{n-k}}{(n-k)\,!}\,g_{k}(u)\,z^{k}$$

$$=\sum_{k=o}^{n}a_{k}\binom{n}{k}\,g_{k}(u)\,z^{k}\;,$$

this completes the proof of the theorem.

3. APPLICATIONS

We now give some applications of our result.

(i) As application, we first consider the following generating relation [4] :

$$(3.1)\qquad \sum_{n=0}^{\infty} \frac{n!}{(1+\delta)_n} F_n(\alpha-n,\beta;x)\, F_n(\nu-n,\, \delta-n;u)\, w^n$$

$$= \left(1+\frac{\lambda^2}{b-a}(u-b)w\right)^{\nu}\left(1+\frac{\lambda^2}{b-a}(u-a)w\right)^{\delta} F_n\Bigg(1+\alpha+\beta;\, -\delta,\, -\nu,\, 1+\beta;$$

$$\frac{-\lambda^2(x-b)(u-a)w}{(b-a)^2+\lambda^2(b-a)(u-a)w},\ \frac{-\lambda^2(x-b)\,(u-b)\,w}{(b-a)^2+\lambda^2(b-a)(u-b)w}\Bigg).$$

If we choose $a_n = \frac{n!}{(1+\delta)_n}$, $g_n(u) = F_n(\nu-n,\nu-n;u)$ then by the application of our theorem, we get the following generalization of (3.1),

$$(3.2)\qquad (1-\lambda t)^{\alpha}\left(1-\lambda\frac{x-a}{b-a}t\right)^{-1-\alpha-\beta}(1-t\lambda)^{-\nu-\delta}\left(1-t\lambda+\lambda^2 tz\,\frac{u-b}{b-a}\right)^{\nu}$$

$$\times\left(1-t\lambda+\lambda^2 tz\,\frac{u-a}{b-a}\right)^{\delta} F_n\Bigg(1+\alpha+\beta;\, -\delta,\, -\nu,\, 1+\beta;$$

$$\frac{-\lambda^2 tz\,\frac{u-a}{b-a}\cdot\frac{x-b}{b-a}}{\left(1-\lambda t\,\frac{x-a}{b-a}\right)\left(1-t\lambda+\lambda^2 tz\,\frac{u-a}{b-a}\right)},\ \frac{-\lambda^2 tz\,\frac{u-b}{b-a}\cdot\frac{x-b}{b-a}}{\left(1-\lambda t\,\frac{x-a}{b-a}\right)\left(1-t\lambda+\lambda^2 tz\,\frac{u-b}{b-a}\right)}\Bigg)$$

$$= \sum_{n=0}^{\infty} F_n(\alpha-n,\beta;x)\,\sigma_n(u,z)\,t^n$$

where

$$\sigma_n(u,z) = \sum_{k=0}^{n}\frac{k!}{(1+\delta)_k}\binom{n}{k} F_k(\nu-k,\, \delta-k;\, u)\, z^k .$$

(ii) We now consider the following generating relation [5] :

$$(3.3)\qquad \sum_{n=0}^{\infty} \frac{n!}{(1+b_1)_n} F_n(\alpha-n,\beta;x)\, F_n(a_1-n,\, b_1;\, u)\, w^n$$

$$= (1-\lambda^2 w)^{1+a_1+b_1+\alpha} \left(1-\lambda^2 w\, \frac{x-a}{b-a}\right)^{-(1+\alpha+\beta)} \left(1-\lambda^2 w\, \frac{u-a}{b-a}\right)^{-(1+a_1+b_1)}$$

$$\times\; F_1\Big(1+a_1+b_1,\; b_1-\beta,\; 1+\alpha+\beta,\; 1+b_1,$$

$$\frac{-\lambda^2 w(u-b)/(b-a)}{1-\lambda^2 w(u-a)/(b-a)},\; \frac{\lambda^2 w(x-b)(u-b)/(b-a)^2}{(1-\lambda^2 w(x-a)/(b-a))(1-\lambda^2 w(u-a)/(b-a))}\Big).$$

If we take $a_n = \dfrac{n!}{(1+b_1)_n}$ and $g_n(u) = F_n(a_1-n,\, b_1;\, u)$, then by the application of our theorem, we get the following generalization of the relation (3.3).

$$(3.4)\qquad (1-\lambda t-\lambda^2 z t)^{1+a_1+b_1+\alpha} \times \left\{1-\lambda t(1+\lambda z)\,\frac{x-a}{b-a}\right\}^{-1-\alpha-\beta}$$

$$\times\; \left(1-\lambda t-\lambda^2 z t\,\frac{u-a}{b-a}\right)^{-1-a_1-b_1}$$

$$\times\; F_1\Big(1+a_1+b_1,\; b_1-\beta,\; 1+\alpha+\beta,\; 1+b_1,\; \frac{-\lambda^2 tz\,(u-b)/(b-a)}{1-\lambda t-\lambda^2 tz(u-a)/(b-a)},$$

$$\frac{\lambda^2 z t\,\frac{x-b}{b-a}\cdot\frac{u-b}{b-a}}{\left\{1-\lambda t(1+\lambda z)\,\frac{x-a}{b-a}\right\}\left(1-\lambda t-\lambda^2 z t\,\frac{u-a}{b-a}\right)}\Big)$$

$$= \sum_{n=0}^{\infty} F_n(\alpha-n, \beta; x)\ \sigma_n(u,z)\ t^n$$

where

$$\sigma_n(u,z) = \sum_{k=0}^{n} \frac{k!}{(1+b_1)_k} \binom{n}{k} F_k(a_1-k, b_1, u)\ z^k .$$

4. SOME SPECIAL CASES

We now discuss some special cases of our result.

Special Case 1 : Putting $-a = b = 1$, $\lambda=1$ in our theorem we get the following result on mixed trilateral generating functions involving Jacobi polynomials :

Result 1 : If

(4.1) $$G(x,u,w) = \sum_{n=0}^{\infty} a_n\ P_n^{(\beta,\alpha-n)}(x)\ g_n(u)\ w^n$$

then

(4.2) $$(1-t)^{\alpha} \left\{1 - \frac{t}{2}(x+1)\right\}^{-\alpha-\beta-1} G\left(\frac{x - \frac{t}{2}(x+1)}{1 - \frac{t}{2}(x+1)}, u, \frac{zt}{1-t}\right)$$

$$= \sum_{n=0}^{\infty} P_n^{(\beta,\alpha-n)}(x)\ \sigma_n(u,z)\ t^n$$

where

$$\sigma_n(u,z) = \sum_{k=0}^{n} a_k \binom{n}{k} g_k(u)\ z^k ,$$

which is found derived in [6] .

It may be pointed out that the result-1 with the help of the symmetry relation

$$P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x)$$

yields the following analogous result.

Result 2 : If

(4.3) $$G(x,u,w) = \sum_{n=0}^{\infty} a_n P_n^{(\alpha-n,\beta)}(x)\, g_n(u)\, w^n$$

then

(4.4) $$(1+t)^{\alpha}\left\{1+\frac{t}{2}(1-x)\right\}^{-1-\alpha-\beta} G\left(\frac{x-\frac{t}{2}(1-x)}{1+\frac{t}{2}(1-x)}, u, \frac{zt}{1+t}\right)$$

$$= \sum_{n=0}^{\infty} P_n^{(\alpha-n,\beta)}(x)\, \sigma_n(u,z)\, t^n$$

where

$$\sigma_n(u,z) = \sum_{k=0}^{n} a_k \binom{n}{k} g_k(u)\, z^k,$$

which is the correct version of the result derived by S. Das [7].

Special Case 2 : Putting $a=0$, $\beta=b$ and $\lambda=1$ and then taking limit as $b \to \infty$ we get the following result on mixed trilateral generating relation involving Laguerre polynomial.

Result 3 : If

(4.5) $$G(x,u,w) = \sum_{n=0}^{\infty} a_n L_n^{(\alpha-n)}(x)\, g_n(u)\, w^n$$

then

(4.6) $$(1+t)^{\alpha} \exp(-xt)\, G(x(1+t), u, \frac{zt}{1+t}) = \sum_{n=0}^{\infty} L_n^{(\alpha-n)}(x)\, \sigma_n(u,z)\, t^n$$

where

$$\sigma_n(u,z) = \sum_{k=o}^{n} a_k \binom{n}{k} g_k(u) z^k$$

which is found derived in [6].

Special Case 3 : Putting $\alpha=\beta$, $b = -a = \sqrt{\alpha}$ and recalling $\lambda = \frac{2}{\sqrt{\alpha}}$ and then taking limit as $\alpha \to \infty$ we get the following result on mixed trilateral generating relation involving Hermite polynomials :

Result 4 : If

$$(4.7) \qquad G(x,u,w) = \sum_{n=o}^{\infty} a_n \frac{H_n(x)}{n!} g_n(u) w^n$$

then

$$(4.8) \qquad \exp(2xt-t^2)\, G(x-t,u,zt) = \sum_{n=o}^{\infty} \frac{H_n(x)}{n!} \sigma_n(z,u)\, t^n$$

where

$$\sigma_n(u,z) = \sum_{k=o}^{n} a_k \binom{n}{k} g_k(u) z^k ,$$

which is found derived in [8].

Special Case 4 : Putting $b = -a = 1$, $\lambda=1$, $\alpha = \nu-\epsilon-1$, $\beta = \epsilon-1$ and replacing x by $(1 + \frac{2x\epsilon}{s})$ and t by (sw/ϵ) and then taking limit as $\epsilon \to \infty$ in our theorem and finally using the relation [2] :

$$(4.9) \qquad \underset{\epsilon \to \infty}{\mathrm{Lt}} \frac{\Gamma(n+1)}{\epsilon^n} F_n(\nu-\epsilon-1, \epsilon-1 ; 1 + \frac{2x\epsilon}{s}) = Y_n(x,\nu;s)$$

we get the following result on bilateral generating relation involving Bessel polynomials :

Result 5 : If

$$(4.10) \qquad G(x,u,w) = \sum_{n=o}^{\infty} a_n Y_n(x,\alpha-n;s)\, g_n(u) \frac{(sw)^n}{n!}$$

then

$$(4.11) \qquad \exp(sw)\,(1-xw)^{-v+1}\, G\left(\frac{x}{1-xw}, u, wz\right)$$

$$= \sum_{n=o}^{\infty} Y_n(x,v-n;s) \frac{(ws)^n}{n!} \sigma_n(u,z)$$

where

$$\sigma_n(u,z) = \sum_{k=o}^{n} a_k \binom{n}{k} g_k(u)\, z^k$$

which is noteworthy.

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