PRODUCTO NULO DE MATRICES

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Sabemos que ($M_2,+,.$) es un anillo no íntegro,es decir,que existen matrices no nulas cuyo producto es la matriz nula. Tratemos de encontrar cuáles son las c.n. y s.para que dos matrices A y B verifiquen que $A.B=0$:

Sean

$$A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad y \quad B=\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

Si suponemos que $A.B=0$,entonces

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Longrightarrow$$

$$\Longrightarrow \left\{ \begin{array}{ll} a_{11}\, b_{11} + a_{12}\, b_{21} = 0 \quad (1) & a_{21}\, b_{11} + a_{22}\, b_{21} = 0 \quad (3) \\ a_{11}\, b_{12} + a_{12}\, b_{22} = 0 \quad (2) & a_{21}\, b_{12} + a_{22}\, b_{22} = 0 \quad (4) \end{array} \right.$$

Despejando ℓ_{11} en (1) y (3).

$$\left.\begin{array}{l} \ell_{11} = \dfrac{a_{12}\,\ell_{21}}{a_{11}} \\[2ex] \ell_{11} = -\dfrac{a_{22}\,\ell_{21}}{a_{21}} \end{array}\right\} \Rightarrow -\frac{a_{12}\,\ell_{21}}{a_{11}} = -\frac{a_{22}\,\ell_{21}}{a_{21}} \Rightarrow a_{12}a_{21}=a_{11}a_{22} \quad (1)$$

Y despejando ℓ_{12} en (2) y (4)

$$\left.\begin{array}{l} \ell_{12} = -\dfrac{a_{12}\,\ell_{22}}{a_{11}} \\[2ex] \ell_{12} = -\dfrac{a_{22}\,\ell_{22}}{a_{21}} \end{array}\right\} \Rightarrow -\frac{a_{12}\,\ell_{22}}{a_{11}} = -\frac{a_{22}\,\ell_{22}}{a_{21}} \Rightarrow (1)$$

Por lo tanto,tomando ℓ_{21} y ℓ_{22} arbitrariamente,las condiciones buscadas son

$$\boxed{\begin{array}{l} a_{12}\,a_{21} = a_{11}\,a_{22} \\[2ex] \ell_{11} = -\dfrac{a_{12}\,\ell_{21}}{a_{11}} \\[2ex] \ell_{12} = -\dfrac{a_{22}\,\ell_{22}}{a_{21}} \end{array}}$$

Veamos ahora que estas condiciones son suficientes :

$$A \,.\, B = \begin{pmatrix} a_{11} & \dfrac{a_{11}\,a_{22}}{a_{21}} \\[2ex] a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} -\dfrac{a_{12}\,\ell_{21}}{a_{11}} & -\dfrac{a_{22}\,\ell_{22}}{a_{21}} \\[2ex] \ell_{21} & \ell_{22} \end{pmatrix} =$$

$$= \begin{pmatrix} x & y \\ z & t \end{pmatrix}$$

y calculando por separado x,y,z,t,resulta

$$
\begin{cases}
x = -a_{11} \cdot \dfrac{a_{12}\, b_{21}}{a_{11}} + \dfrac{a_{22}\, a_{11}}{a_{21}} \cdot b_{21} = \\[2ex]
\quad = -a_{12}\, b_{21} + \dfrac{a_{12}\, a_{21}}{a_{21}} \cdot b_{21} = -a_{12}\, b_{21} + a_{12}\, b_{21} = 0 \\[2ex]
y = -a_{11} \cdot \dfrac{a_{22}\, b_{22}}{a_{21}} + \dfrac{a_{22}\, a_{11}}{a_{21}}\, b_{22} = 0 \\[2ex]
z = -a_{21} \cdot \dfrac{a_{12}\, b_{21}}{a_{11}} + a_{22}\, b_{21} = \\[2ex]
\quad = -a_{11} \cdot \dfrac{a_{22}\, b_{21}}{a_{11}} + a_{22}\, b_{21} = 0 \\[2ex]
t = -a_{21} \cdot \dfrac{a_{22}\, b_{22}}{a_{21}} + a_{22}\, b_{22} = 0
\end{cases}
$$

Ejemplo.- El producto $\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} -3 & 12 \\ 1 & -4 \end{pmatrix}$ es nulo,ya que:

$$
\begin{cases}
a_{11}\, a_{22} = 1.6 = 6 \\
a_{12}\, a_{21} = 3.2 = 6
\end{cases}
$$

$$
\begin{cases}
b_{11} = -3 \\
-\dfrac{a_{12}\, b_{21}}{a_{11}} = -\dfrac{3\ .\ 1}{1} = -3
\end{cases}
$$

$$
\begin{cases}
b_{12} = 12 \\
-\dfrac{a_{22}\, b_{22}}{a_{21}} = -\dfrac{6\ .(-4)}{2} = 12
\end{cases}
$$

En efecto, $\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} -3 & 12 \\ 1 & -4 \end{pmatrix} = \begin{pmatrix} 1(-3)+3.1 & 1.12+3(-4) \\ 2(-3)+6.1 & 2.12+6(-4) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$