Rev. Acad. Canar. Cienc., XII (Núms. 1-2), 175-188 (2000) - (Publicado en julio de 2001)

On A Class Of Generalized Elliptic-Type Integrals

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Abstract

In this paper, we introduce and study a new class of generalized elliptic-type integrals of the form

$$R_{\gamma}^{\omega}(x,l,\beta)=\int_{0}^{\pi}\frac{\cos^{\frac{2l}{\omega}-1}(\theta/2)\left(1-\cos^{\frac{2}{\omega}}(\theta/2)\right)^{\beta-l-1}\sin(\theta/2)}{\left(1-x^{2}\cos\theta\right)^{\gamma+\frac{1}{2}}}\,d\theta$$

where $0\leq x<1,\omega>0,\mathrm{Re}(\beta)>\mathrm{Re}(l)\geq 0,\mathrm{Re}(\gamma)>-\frac{1}{2}$. This new generalization might have some applications in radiation field problems .These integrals generalize the family of integrals studied earlier by Kalla et al. (Applicable analysis, vol. 22(1986) 273-287) .We obtain a series expansion of $R_{\gamma}^{\omega}(x,l,\beta)$ and establish its relationship with Wright's hypergeometric functions ${}_2R_1^{\omega}(a,b;c;x)$. Moreover, we obtain some recurrence relations, representations and approximations. Some differential properties and an integral involving the generalized elliptic-type integral is evaluated.Several known results follow as special cases of the results drived here.

Keywords : Elliptic-type integrals, Wright's hypergeometric functions , Asymptotic formulas .

A.M.S. subject classification : 33C75 , 33C20 .

1. Introduction

Elliptic-type integrals are encountered in the application of a Legendre polynomial expansion method to certain physical problems dealing with the computation of the radiation field off axis from a certain uniform circular disc radiating according to an arbitrary angular distribution law [6] , give rise to the Epstein and Hubbell [5] family of elliptic-type integrals of the form

$$\Omega_\gamma(x)=\int_0^\pi \left(1-x^2\cos\theta\right)^{-\gamma-\frac{1}{2}}\,d\theta \qquad (1.1)$$

for $0\le x<1$, $\mathrm{Re}(\gamma)>-\frac{1}{2}$ .These integrals have been modified and generalized by a large number of authors [2 , 4 , 7-14]. For example , Kalla [7] has treated the family of integrals of the form

$$S_\gamma(x,\beta)=\int_0^\pi \sin^{2\beta}\theta\,\left(1-x^2\cos\theta\right)^{-\gamma-\frac{1}{2}}\,d\theta \qquad (1.2)$$

for $0\le x<1, \mathrm{Re}(\beta), \mathrm{Re}(\gamma)>-\frac{1}{2}$, which generalize (1.1) since $S_\gamma(x,0)=\Omega_\gamma(x)$. Furthermore , Kalla et al. [9] have generalized and studied the family of integrals of the form

$$R_\gamma(x,l,\beta)=\int_0^\pi \cos^{2l-1}(\theta/2)\,\sin^{2\beta-2l-1}(\theta/2)\,\left(1-x^2\cos\theta\right)^{-\gamma-\frac{1}{2}}\,d\theta \qquad (1.3)$$

for $0\le x<1, \mathrm{Re}(\beta)>\mathrm{Re}(l)\ge 0, \mathrm{Re}(\gamma)>-\frac{1}{2}$ .They observed that the of integrals of the form (1.1) and (1.2) are considered as a special cases of $R_\gamma(x,l,\beta)$, since $\Omega_\gamma(x)=R_\gamma\left(x,\frac{1}{2},1\right)$ and $S_\gamma(x,\beta)=2^{2\beta}\times R_\gamma\left(x,\beta+\frac{1}{2},2\beta+1\right)$. Moreover , Kalla and Tuan [11] considered a unified form of different elliptic-type integrals where they first expressed them in terms of the Lauricella's hypergeometric functions of three variables $F_D^{(3)}$ and then obtained their asymptotic expansion valid for $x \to 1$.

We consider special cases of $R_\gamma^\omega(x,l,\beta)$ in section 2 ,whereas in section 3 , we obtain its series expansion in terms of Wright's hypergeometric functions (the generalized form of the hypergeometric functions)

$$ {}_2R_1^\omega(a,b;c;x) = \frac{\Gamma(c)}{\omega \times \Gamma(b)\Gamma(c-b)} \int_0^1 t^{\frac{b}{\omega}-1} \left(1 - t^{\frac{1}{\omega}}\right)^{c-b-1} (1-xt)^{-a}\, dt \,, \qquad (1.4) $$

for $0 \le x < 1, \omega > 0, \mathrm{Re}(\beta) > \mathrm{Re}(l) > 0$, which have been investigated and studied recently by Kalla et al. [13, Eq.7] and Virchenko [14 , p.234 , Eq.(2.3)].Moreover, we obtain an asymptotic expansion of $R_\gamma^\omega(x,l,\beta)$ as $x \to 1$. In section 4, we establish some recurrence relation .Some differentiation formulae and an integral involving $R_\gamma^\omega(x,l,\beta)$ are considered in section 5 . It is interesting to observe that several families of elliptic-type integrals such as Epstein and Hubbell [5] and Kalla et al. [8,9] ... etc. follow as a particular cases of our formulae for $R_\gamma^\omega(x,l,\beta)$ established here .

2. Definition and Explicit Representation

We begin by defining the new class of generalized elliptic-type of integrals of the form

$$R_\gamma^\omega(x,l,\beta)=\int_0^\pi \frac{\cos^{\frac{2l}{\omega}-1}(\theta/2)\left(1-\cos^{\frac{2}{\omega}}(\theta/2)\right)^{\beta-l-1}\sin(\theta/2)}{\left(1-x^2\cos\theta\right)^{\gamma+\frac{1}{2}}}\,d\theta \qquad (2.1)$$

for $0\le x<1,\omega>0,\operatorname{Re}(\beta)>\operatorname{Re}(l)\ge 0,\operatorname{Re}(\gamma)>-\frac{1}{2}$. Next we mention some important special cases of (2.1) which are generalization of some elliptic-type integrals that have been treated before by several authors . For example ,we define $S_\gamma^\omega(x,\beta)$ to be the family of integrals of the form

$$S_\gamma^\omega(x,\beta)=\int_0^\pi \frac{\cos^{\frac{2\beta+1}{\omega}-1}(\theta/2)\times\left(1-\cos^{\frac{2}{\omega}}(\theta/2)\right)^{\beta-\frac{1}{2}}\sin(\theta/2)}{\left(1-x^2\cos\theta\right)^{\gamma+\frac{1}{2}}}\,d\theta \qquad (2.2)$$

for $\omega>0$, $0\le x<1,\operatorname{Re}(\beta),\operatorname{Re}(\gamma)>-\frac{1}{2}$,which are considered as a special case of (2.1) , since $S_\gamma^\omega(x,\beta)=2^{2\beta}\times R_\gamma^\omega\left(x,\beta+\frac{1}{2},2\beta+1\right)$, and if we let $\omega=1$ we get the family of integrals (1.2) . Furthermore, we define $\Omega_\gamma^\omega(x)$ to be the family of integrals of the form ,

$$\Omega_\gamma^\omega(x)=\int_0^\pi \frac{\cos^{\frac{1}{\omega}-1}(\theta/2)\times\left(1-\cos^{\frac{2}{\omega}}(\theta/2)\right)^{-\frac{1}{2}}\sin(\theta/2)}{\left(1-x^2\cos\theta\right)^{\gamma+\frac{1}{2}}}\,d\theta \qquad (2.3)$$

for $\omega>0$, $0\le x<1$,$\operatorname{Re}(\gamma)>-\frac{1}{2}$, which are considered as a special case of (2.1), since $\Omega_\gamma^\omega(x)=R_\gamma^\omega\left(x,\frac{1}{2},1\right)$ and a general case of (1.1)because $\Omega_\gamma(x)=\Omega_\gamma^1(x)$.

3. Asymptotic Expansion and Single Term Approximation

First we obtain a relationship between $R_\gamma^\omega(x,l,\beta)$ and Wright's hype eometric functions ${}_2R_1^\omega(a,b;c;x)$, given in (1.4). From the integral representation of (2.1), by changing the variable of integration, $\cos^2(\theta/2)=t$, and by using the integral representation of ${}_2R_1^\omega$, we find directly

$$R_\gamma^\omega(x,l,\beta)=(1+x^2)^{-\gamma-\frac{1}{2}}\times\int_0^1 t^{\frac{l}{\omega}-1}\left(1-t^{\frac{1}{\omega}}\right)^{\beta-l-1}\left(1-\frac{2x^2}{1+x^2}t\right)^{-\gamma-\frac{1}{2}}dt$$

$$=\frac{\omega\times B(l,\beta-l)}{(1+x^2)^{\gamma+\frac{1}{2}}}\times{}_2R_1^\omega\left(\gamma+\frac{1}{2},l;\beta;\frac{2x^2}{x^2+1}\right) \quad (3.1)$$

Next we establish an asymptotic expansion of $R_\gamma^\omega(x,l,\beta)$ as $x\to 1$. We observe that

$$R_\gamma^\omega(x,l,\beta)=(1-x^2)^{-\gamma-\frac{1}{2}}\int_0^1(1-t)^{\frac{l}{\omega}-1}\left(1-(1-t)^{\frac{1}{\omega}}\right)^{\beta-l-1}\left(1+\frac{2x^2}{1-x^2}t\right)^{-\gamma-\frac{1}{2}}dt. \quad (3.2)$$

Let $y=\eta t$, where $\eta=\dfrac{2x^2}{1-x^2}$, then

$$R_\gamma^\omega(x,l,\beta)=\frac{(1-x^2)^{-\gamma-\frac{1}{2}}}{\eta}\times\int_0^\eta(1+y)^{-\gamma-\frac{1}{2}}\times\left(1-\frac{y}{\eta}\right)^{\frac{l}{\omega}-1}\left(1-\left(1-\frac{y}{\eta}\right)^{\frac{1}{\omega}}\right)^{\beta-l-1}dy. \quad (3.3)$$

Binomial expansion of the factor $\left(1-\dfrac{y}{\eta}\right)^{\frac{l}{\omega}-1}$ leads to,

$$\left(1-\frac{y}{\eta}\right)^{\frac{l}{\omega}-1}\times\left(1-\left(1-\frac{y}{\eta}\right)^{\frac{1}{\omega}}\right)^{\beta-l-1}=\mathrm{R}_n+(\omega\eta)^{1-\beta+l}\sum_{k=0}^{n}\begin{pmatrix}\frac{l}{\omega}-1\\ k\end{pmatrix}\times\frac{y^{k+\beta-l-1}}{\eta^k}$$

and therefore we have

$$R_\gamma^\omega(x,l,\beta)=\mathrm{R}_n+\frac{\omega^{1-\beta+l}\eta^{l-\beta}}{\left(1-x^2\right)^{\gamma+\frac{1}{2}}}\sum_{k=0}^{n}\frac{1}{\eta^k}\times\begin{pmatrix}\frac{l}{\omega}-1\\ k\end{pmatrix}\times\left[\int_0^\infty\frac{y^{k+\beta-l-1}}{(1+y)^{\gamma+\frac{1}{2}}}dy-\mathrm{I}_\eta\right],$$

where $\mathrm{I}_\eta=\int_\eta^\infty\frac{y^{k+\beta-l-1}}{(1+y)^{\gamma+\frac{1}{2}}}dy$ .Now, since the error introduced by the addition of the

integrals I_η can be neglected as $\eta\to\infty$, then using beta integral

$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}=\int_0^\infty\frac{t^{a-1}}{(1+t)^{a+b}}dt$, $\mathrm{Re}(a+b)>\mathrm{Re}(a)>0$, we get

$$R_\gamma^\omega(x,l,\beta)=\frac{\omega^{1-\beta+l}\eta^{l-\beta}\left(1-x^2\right)^{-\left(\gamma+\frac{1}{2}\right)}}{\Gamma\left(\gamma+\frac{1}{2}\right)\Gamma\left(1-\frac{l}{\omega}\right)}\sum_{k=0}^{n}\frac{\Gamma(\beta-l+k)\times\Gamma\left(1-\frac{l}{\omega}+k\right)\times\Gamma\left(\gamma+\frac{1}{2}-\beta+l-k\right)}{\eta^k\times k!}+R_n$$

$$\to(3.4)$$

For $\omega=1$, we obtain a result of [10 , p.273 , Eq.6] .

Single Term Approximation

A superior approach to obtain a single term approximation is to set

$$\left(1-\frac{y}{\eta}\right)^{\frac{l}{\omega}-1}\cong(1+y)^{\frac{1-\frac{l}{\omega}}{\eta}}\text{ and }\left(1-\left(1-\frac{y}{\eta}\right)^{\frac{1}{\omega}}\right)^{\beta-l-1}\cong\left(1-\left(1-\frac{y}{\omega\eta}\right)\right)^{\beta-l-1}=\left(\frac{y}{\omega\eta}\right)^{\beta-l-1}$$

$$R_\gamma^\omega(x,l,\beta) \cong \frac{\omega^{1-\beta+l}\eta^{l-\beta}}{(1-x^2)^{\gamma+\frac{1}{2}}}\left[\int_0^\infty \frac{y^{\beta-l-1}}{(1+y)^{\gamma+\frac{1}{2}+\frac{1}{\eta}\left(\frac{l}{\omega}-1\right)}}\,dy\right]$$

or equivalently ,

$$R_\gamma^\omega(x,l,\beta) \cong \frac{\Gamma(\beta-l)\times\Gamma\left(\gamma+\frac{1}{2}-\beta+l+\frac{\frac{l}{\omega}-1}{\eta}\right)}{\omega^{\beta-l-1}\eta^{\beta-l}\times(1-x^2)^{\gamma+\frac{1}{2}}\times\Gamma\left(\gamma+\frac{1}{2}+\frac{\frac{l}{\omega}-1}{\eta}\right)} \qquad \blacksquare$$

For $\omega=1$, we obtain a single term approximation for $R_\gamma(x,l,\beta)$, given earlier by [10 , p.274].

4. Recurrence Relations

In the previous section we established the relationship between $R_\gamma^\omega(x,l,\beta)$ and ${}_2R_1^\omega(a,b;c;x)$, that is

$$R_\gamma^\omega(x,l,\beta) = \frac{\omega\times B(l,\beta-l)}{(1+x^2)^{\gamma+\frac{1}{2}}} \times {}_2R_1^\omega\left(\gamma+\frac{1}{2},l;\beta;\frac{2x^2}{x^2+1}\right)$$

and since we have from [13,14]that

$${}_2R_1^\omega(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt^\omega)^{-a}\,dt$$

therefore we can define $R_\gamma^\omega(x,l,\beta)$ as the following

$$R_\gamma^\omega(x,l,\beta) = \int_0^\pi \frac{\cos^{2l-1}(\theta/2)\sin^{2\beta-2l-1}(\theta/2)}{\left[1+x^2-2x^2\cos^{2\omega}(\theta/2)\right]^{\gamma+\frac{1}{2}}}\quad d\theta\ ,$$

$$= \int_0^{\pi} \frac{\cos^{2l-2\omega}(\theta/2)\sin^{2\beta-2l-2}(\theta/2)\times\cos^{2\omega-1}(\theta/2)\sin(\theta/2)}{\left[1+x^2-2x^2\cos^{2\omega}(\theta/2)\right]^{\gamma+\frac{1}{2}}}\, d\theta . \quad (4.1)$$

From this definition of $R_\gamma^\omega(x,l,\beta)$ we can obtain a number of recurrence relations .

For example , to obtain the recurrence relation

$$\omega\, x^2(1-2\gamma)\, R_\gamma^\omega(x,l,\beta) = (l-\omega)R_{\gamma-1}^\omega(x,l-\omega,\beta-\omega) - (\beta-l-1)R_{\gamma-1}^\omega(x,l+1-\omega,\beta-\omega)$$

we use integration by parts, beside the definition of $R_\gamma^\omega(x,l,\beta)$, that is given by (4.1).

The following recurrence relations for $R_\gamma^\omega(x,l,\beta)$ can be easily derived from the definition (4.1) by using elementary trigonometric identities :

$$R_\gamma^\omega(x,l,\beta) = R_\gamma^\omega(x,l,\beta-1) - R_\gamma^\omega(x,l+1,\beta)$$

$$R_\gamma^\omega(x,l,\beta) = R_\gamma^\omega(x,l-1,\beta-1) - R_\gamma^\omega(x,l-1,\beta)$$

$$R_\gamma^\omega(x,l,\beta) = (1+x^2)R_{\gamma+1}^\omega(x,l,\beta) - 2x^2 R_{\gamma+1}^\omega(x,l+\omega,\beta+\omega)$$

For $\omega = 1$, we get the recurrence relations established in [9 , p.281,282] .

5. Differentiation Formulae

We begin this section by proving the following lemma ,

***Lemma* 5.1.**

$$\frac{d}{dx}\left[(a-x^2)^{\rho-\frac{1}{2}} \times R_\gamma^\omega\left(\frac{x}{\sqrt{a-x^2}},l,\beta\right)\right] = \frac{a\times(2\gamma+1)\times(a-x^2)^{\rho-\frac{3}{2}}}{x}\times$$

$$\left\{R_{\gamma+1}^\omega\left(\frac{x}{\sqrt{a-x^2}},l,\beta\right) - \left(1+\frac{x^2\times(2\rho-1)}{a\times(2\gamma+1)}\right)\times R_\gamma^\omega\left(\frac{x}{\sqrt{a-x^2}},l,\beta\right)\right\}, \quad (5.1)$$

$$\frac{d}{dx}\left[x^{\lambda}\times\left(a-x^{2}\right)^{\rho-\frac{1}{2}}\times R_{\gamma}^{\omega}\left(\frac{x}{\sqrt{a-x^{2}}},l,\beta\right)\right]=\frac{a\times(2\gamma+1)\times\left(a-x^{2}\right)^{\rho-\frac{3}{2}}}{x^{1-\lambda}}\times$$

$$\left\{R_{\gamma+1}^{\omega}\left(\frac{x}{\sqrt{a-x^{2}}},l,\beta\right)-\left(1+\frac{x^{2}\times(2\rho-1)}{a\times(2\gamma+1)}-\frac{\lambda\left(a-x^{2}\right)}{a\times(2\gamma+1)}\right)\times R_{\gamma}^{\omega}\left(\frac{x}{\sqrt{a-x^{2}}},l,\beta\right)\right\}$$

$$\rightarrow(5.2)$$

Proof : Since the result (5.2) can be derived easily from (5.1) , therefore we shall prove only the first one . From (3.1) we observe that ,

$$R_{\gamma}^{\omega}\left(\frac{x}{\sqrt{a-x^{2}}},l,\beta\right)=B(l,\beta-l)\times\left(\frac{a-x^{2}}{a}\right)^{\gamma+\frac{1}{2}}\times R^{\omega}\left(\gamma+\frac{1}{2},l;\beta;\frac{2x^{2}}{a}\right). \quad (5.3)$$

Using this and the following formulae [13 , Eq.12 and Eq.17]

$$\frac{d}{dx}R^{\omega}(a,b;c;x)=\frac{a\,\Gamma(c)\Gamma(b+\omega)}{\Gamma(b)\Gamma(c+\omega)}\,R^{\omega}(a+1,b+\omega;c+\omega;x)$$

$$R^{\omega}(a+1,b+\omega;c+\omega;x)=\frac{a\,\Gamma(c+\omega)\Gamma(b)}{x\Gamma(c)\Gamma(b+\omega)}\left[R^{\omega}(a+1,b;c;x)-R^{\omega}(a,b;c;x)\right]$$

we get the desired result(5.1) . ■

For $\omega=1$, we obtain a result of [8 , p.510 , Eq.9] .

In a similar way one can show

$$\frac{d}{dx}\left[\left(1+x^{2}\right)^{\rho-\frac{1}{2}}\times R_{\gamma}^{\omega}(x,l,\beta)\right]=\frac{(2\gamma+1)\times\left(1+x^{2}\right)^{\rho-\frac{5}{2}}}{x}\times R_{\gamma+1}^{\omega}(x,l,\beta)+$$

$$\left(1+x^{2}\right)^{\rho-\frac{3}{2}}\times R_{\gamma}^{\omega}(x,l,\beta)\left[2x(\rho+\gamma)-\frac{2\gamma+1}{x}\right], \quad (5.4)$$

which leads to

$$\frac{d}{dx}R_\gamma^\omega(x,l,\beta)=\frac{2\gamma+1}{1+x^2}\times\left[\frac{R_{\gamma+1}^\omega(x,l,\beta)}{x(1+x^2)}+\left(x-\frac{1}{x}\right)\times R_\gamma^\omega(x,l,\beta)\right] \qquad (5.5)$$

6. An Integral Involving $R_\gamma^\omega(x,l,\beta)$

We shall establish an integral involving $R_\gamma^\omega(x,l,\beta)$, and mention a special case. We prove the following lemma :

***Lemma* 6.1.**

$$\int_0^1 \frac{x^{2\nu-1}(1-x^2)^{\mu-1}}{B(l,\beta-l)}\times\left(\frac{2}{2-\alpha^2x^2}\right)^{\gamma+\frac{1}{2}}\times R_\gamma^\omega\left(\frac{\alpha x}{\sqrt{2-\alpha^2x^2}},l,\beta\right)dx=\frac{B(\mu,\nu)}{2}\times$$

$$_3R_2^\omega\left(\gamma+\frac{1}{2},\nu:l;\nu+\mu:\beta;\alpha^2\right)$$

where $_3R_2^\omega(a,d:b;e:c;x)=\frac{\Gamma(c)}{\Gamma(b)}\sum_{k=0}^{\infty}\frac{(a)_k(d)_k\Gamma(b+\omega k)}{(e)_k\Gamma(c+\omega k)}\times\frac{x^k}{k!}$

and $B(\mu,\nu)=\frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\nu+\mu)}$

Proof: To prove this , we observe that

$$\frac{1}{B(l,\beta-l)}\times\left(\frac{2}{2-\alpha^2x^2}\right)^{\gamma+\frac{1}{2}}\times R_\gamma^\omega\left(\frac{\alpha x}{\sqrt{2-\alpha^2x^2}},l,\beta\right)={}_2R_1^\omega\left(\gamma+\frac{1}{2},l;\beta;\alpha^2x^2\right)$$

therefore

$$J_\gamma^\omega = \int_0^1 \frac{x^{2\nu-1}(1-x^2)^{\mu-1}}{B(l,\beta-l)} \times \left(\frac{2}{2-\alpha^2x^2}\right)^{\gamma+\frac{1}{2}} \times R_\gamma^\omega\left(\frac{\alpha x}{\sqrt{2-\alpha^2x^2}}, l, \beta\right) dx$$

$$= \int_0^1 x^{2\nu-1}\left(1-x^2\right)^{\mu-1} \times {}_2R_1^\omega\left(\gamma+\frac{1}{2}, l; \beta; \alpha^2x^2\right) dx$$

$$= \sum_{k=0}^{\infty} C_k \int_0^1 x^{2\nu+2k-1}(1-x^2)^{\mu-1}\, dx$$

where $C_k = \dfrac{\left(\gamma+\frac{1}{2}\right)_k \times \Gamma(\beta)\times\Gamma(l+\omega\, k)}{\Gamma(l)\Gamma(\beta+\omega\, k)} \times \dfrac{\alpha^{2k}}{k!}$.

Now since, $\displaystyle\int_0^1 x^{2\nu+2k-1}(1-x^2)^{\mu-1}\, dx = \frac{B(\mu,\nu+k)}{2} = \frac{B(\mu,\nu)}{2} \times \frac{(\nu)_k}{(\nu+\mu)_k}$,

we get,

$$J_\gamma^\omega = \frac{B(\mu,\nu)}{2} \times \frac{\Gamma(\beta)}{\Gamma(l)} \sum_{k=0}^{\infty} \frac{(\nu)_k \times \left(\gamma+\frac{1}{2}\right)_k \times \Gamma(l+\omega\, k)}{(\nu+\mu)_k \times \Gamma(\beta+\omega\, k)} \times \frac{\alpha^{2k}}{k!} = \frac{B(\mu,\nu)}{2} \times$$

$${}_3R_2^\omega\left(\gamma+\frac{1}{2}, \nu : l ; \nu+\mu : \beta ; \alpha^2\right). \quad \blacksquare$$

For $\omega = 1$, $l = \dfrac{1}{2}$ and $\beta = 1$, we obtain a result of [8, p.511, Eq.18].

7. Results Involving $B_\gamma(x, l, \beta)$

As we mentioned in section 1, Epstein and Hubbell elliptic-type integral (1.1) has been generalized by several authors [2 , 4 , 7-14] , due to their applications in radiation field problems . Al-Saqabi [2] has studied the following family of elliptic-type integrals ;

$$B_\gamma(x,l,\beta)=\int_0^\pi \cos^{2l}\theta\ \sin^{2\beta}\theta\ \left(1-x^2\cos\theta\right)^{\gamma+\frac{1}{2}}\ d\theta \qquad (7.1)$$

for $0\le x<1, \mathrm{Re}(\beta), \mathrm{Re}(\gamma)>-\dfrac{1}{2}$, $l=0,1,2,\ldots$

The representation of elliptic-type integrals in terms of hypergeometric series of argument x^4 has some advantages for numerical computations . For example [12]

$$\Omega_\gamma(x)=\frac{\pi}{2}\times\ {}_2F_1\left(\frac{\gamma}{2}+\frac{1}{4}\ ,\ \frac{\gamma}{2}+\frac{3}{4}\ ;\ 1\ ;\ x^4\right).$$

Similarly , we can establish that

$$B_\gamma(x,l,\beta)=B\left(l+\frac{1}{2},\beta+\frac{1}{2}\right)\ {}_3F_2\left(\frac{\gamma}{2}+\frac{1}{4}\ ,\ \frac{\gamma}{2}+\frac{3}{4}\ ,\ l+\frac{1}{2}\ ;\ \frac{1}{2}\ ,\ l+\beta+1\ ;\ x^4\right).$$

$$\rightarrow(7.2)$$

Corresponding results for $S_\gamma(x,\beta)$ and $\Omega_\gamma(x)$ can be derived from (7.2).

Further , we record here an integral involving $B_\gamma(x,l,\beta)$

$$\int_0^1 x^{2\nu-1}\left(1-x^2\right)^{\mu-1}\times\ B_\gamma(\alpha x,l,\beta)\ dx=$$

$$=\frac{B(\nu,\mu)\times B\left(l+\frac{1}{2},\beta+\frac{1}{2}\right)}{2}\times\ {}_5F_4\left(\begin{matrix}\frac{\gamma}{2}+\frac{1}{4}\ ,\ \frac{\gamma}{2}+\frac{3}{4}\ ,\ \frac{\nu}{2}\ ,\ \frac{\nu+1}{2}\ ,\ l+\frac{1}{2} \\ ;\alpha^4 \\ \frac{1}{2}\ ,\ l+\beta+1\ ,\ \frac{\nu+\mu}{2}\ ,\ \frac{\nu+\mu+1}{2}\end{matrix}\right)$$

From this result we can derive the following integrals for $S_\gamma(x,\beta)$ and $\Omega_\gamma(x)$

$$\int_0^1 x^{2\nu-1}\left(1-x^2\right)^{\mu-1}\times S_\gamma(\alpha x,\beta)\,dx$$

$$=\frac{B(\nu,\mu)\times B\left(\frac{1}{2},\beta+\frac{1}{2}\right)}{2}\times {}_4F_3\left(\begin{matrix}\frac{\gamma}{2}+\frac{1}{4},\ \frac{\gamma}{2}+\frac{3}{4},\ \frac{\nu}{2},\ \frac{\nu+1}{2}\\ \qquad\qquad\qquad\qquad ;\alpha^4\\ \beta+1,\ \frac{\nu+\mu}{2},\ \frac{\nu+\mu+1}{2}\end{matrix}\right).$$

$$\int_0^1 x^{2\nu-1}\left(1-x^2\right)^{\mu-1}\ \Omega_\gamma(\alpha x)\,dx=\frac{\pi}{2}\times B(\nu,\mu)\times {}_4F_3\left(\begin{matrix}\frac{\gamma}{2}+\frac{1}{4},\ \frac{\gamma}{2}+\frac{3}{4},\ \frac{\nu}{2},\ \frac{\nu+1}{2}\\ \qquad\qquad\qquad\qquad ;\alpha^4\\ 1,\ \frac{\nu+\mu}{2},\ \frac{\nu+\mu+1}{2}\end{matrix}\right)$$

References

[1] Abramowitz, A. and Stegun , I. *Handbook of Mathematical Functions*.Dover Publications , New York , (1972) .

[2] Al-Saqabi, B. A generalization of elliptic-type integrals. *Hadronic Jour*, **10**,331-336 , (1976) .

[3] Al-Zamel, A. and Kalla, S.L . Epstein-Hubbell elliptic-type integrals and its generalizations formulas. *Applied Mathematics and Computation* 77,9-32, (1996).

[4] Al-Zamel , A.Tuan ,V.K. and Kalla ,S.L .Generalized elliptic-type integrals and asymptotic formulas . *Appl. Math. Comput.* **114**,13-25, (2000).

[5] Epstein, L.F.and Hubbell, J.H .Evaluation of a generalized elliptic-type integrals. *J.Res. vol.* **67**, 1-17 , (1963) .

[6] Hubbell, J.H .Bach, R.L. and Herbold, R.J. Radiation field from a circular disk source . *J. Res.N.B.S.*,65-249 , (1961) .

[7] Kalla, S.L. Results on generalized elliptic-type integrals . *Mathematical Structure Computational Mathematics-Mathematical Modeling* (Edited by Bl. Sendov) Bulg. Acad. Sci. (special vol.), 216-219, (1984).

[8] Kalla, S.L and Al-Saqabi, B. Some results for a class of generalized elliptic-type integrals . *Results in Maths*. **20**, 507-516, (1991).

[9] Kalla, S.L , Conde, S. and Hubbell, J.H . Some results on generalized elliptic-type integrals . *Applicable analysis , vol.* **22**, 273-287 , (1986) .

[10] Kalla, S.L. Leubner, C. and Hubbell, J.H . Further results on generalized elliptic-type integrals. *Applicable analysis, vol.* **25**, 269-274, (1987).

[11] Kalla, S.L and Tuan, V.K. Asymptotic formulas for generalized elliptic-type integrals. *Computers Math. Applic.* **32**, 49-55, (1996).

[12] Todorov, P.G. and Hubbell, J.H .On the Epstein-Hubbell generalized elliptic-type integrals elliptic-type integrals. *Appl. Math. Comput.* **61**,157-167,(1993) .

[13] Virchenko, N. Kalla, S.L and Al-Zamel, A . Some results on a generalized hypergeometric function. *Integral Transform.Spec.funct.* (to appear) .

[14] Virchenko, N. On some generalizations of the functions of hypergeometric type. *Fractional calculus and applied analysis , vol.* **2**, no.3, 233-244 , (1999) .