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ELEMENTS OF A THEORY OF ORTHOGONAL RATIONAL FUNCTIONS

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Abstract

We give a brief survey to some basic elements of the theory of orthogonal rational functions. Two main cases are treated separately: I. All the poles are outside the closed unit disk, and the orthogonality measures have support in the unit circle. II. All the poles are on the extended real line, and the orthogonality measures have support in the real line. These situations generalize the theory of orthogonal polynomials on the unit circle (Szegő polynomials) and the theory of orthognal polynomials on the real line.

En este trabajo, exponemos brevemente, los elementos básicos de la teoría de funciones racionales ortogonales. Nos centraremos en dos casos fundamentales: I. Todos los polos se encuentran fuera del disco unidad cerrado, estando las medidas de ortogonalidad soportadas sobre la circunferencia unidad. II. Todos los polos se hallan sobre la recta real extendida, y las medidas de ortogonalidad con soporte en el eje real. Tales situaciones generalizan la teoría de polinomios ortogonales sobre la circunferencia unidad (Polinomios de Szegő) y la teoría de polinomios ortogonales sobre la recta real.

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1 INTRODUCTION.

A sequence $\{\varphi_n\}_{n=0}^{\infty}$ of polynomials is said to be an orthogonal polynomial sequence if φ_n is a polynomial of degree n and it is orthogonal to all polynomials of lower degree. The orthogonality may be with respect to a linear functional M or a measure μ (with support in $\mathbb{C}$). In particular, the most widely studied cases of such general orthogonal polynomials arise when the support μ is contained in the real line or the complex unit circle. For general treatises on orthogonal polynomials we refer to [1], [33], [43], [44, 45], [47], [51], [59], [74], [75], [76].

Polynomials may be viewed as rational functions whose poles are all fixed at infinity. By fixing a sequence of poles $\{\gamma_n\}$ in the extended complex plane (i.e., on the Riemann sphere), we obtain a theory of orthogonal rational functions. The poles can in principle be taken anywhere in the extended plane. Some of the γ_k can be repeated, possibly an infinite number of times. The sequence is fixed once and for all, and the order in which the γ_k occur (possible repetitions included) is also given.

We consider generalizations of the two special cases indicated above: orthogonality on the unit circle and on the real line. Polynomials orthogonal on the unit circle are generalized to orthogonal rational functions with poles outside the closed unit disk. Polynomials orthogonal on the real line are generalized to orthogonal rational functions with poles on the extended real line. There is a substantial difference between the two cases, since in the former case the poles lie outside the "support curve" of the orthogonality measures, while in the latter case the poles belong to the "support curve". (We do not by this mean that the support of the orthogonality measure need consist of the whole unit circle on the whole real line.) The orthogonal rational functions will behave differently. The main reason for this is that in the former case reflection of the poles with respect to the unit circle produce different points, while in the latter case reflection of the poles, with respect to the real line produce the same points, doubling the poles in a sense.

The cause of the difference between the two cases is then not any difference between the unit circle and the (extended) real line, but between the ways the poles are placed in relation to the "support curve", whether circle on line. By the Cayley transform $z \to w = \frac{i-z}{i+z}$ the extended real line is mapped to the unit circle and the extended upper half plane to the unit disk. This transformation maps all rational functions to all rational functions. Thus we may consider rational functions orthogonal on the real line with poles in the lower half plane as analogs to rational functions orthogonal on the unit circle with poles outside the unit disk. Similarly we may consider rational functions orthogonal on the unit circle with poles on the unit circle as analogs to rational functions orthogonal on the real line with poles on the extended real line. Orthogonal polynomials on the unit circle correspond to orthogonal rational functions on the real line with their only pole (infinitely repeated) at the point $-i$. Similarly orthogonal polynomials on the real line correspond to orthogonal rational functions on the unit circle with their only pole (infinitely repeated) at the point -1.

The case of the real line and the unit circle which are linked by the Cayley transform are essentially the same, and can be treated within a common framework. A unified and rather extensive treatment is given in the monograph "Orthogonal Rational Functions" [34] by the present authors. In this paper we give a very brief introduction to some basic elements of this theory. We here treat the situation with poles outside the "support curve" specified to the unit circle case, and the situation with poles in the "support curve" specified to the real line case. The generalizations of the classical polynomial situations are thus clearly seen.

From a purely mathematical point of view the theory of orthogonal rational functions was as far as we know initiated by Djrbashian about 1960. See [37, 38, 39, 40, 41, 42]. Independently, partly from an applied point of view, the same constructions were used by Bultheel, Bultheel and Dewilde, Dewilde and Dym about 1980. See [2, 3, 4], [36]. A general theory has been worked out in a series of papers by the present authors. See [6]-[35], and the monograph [34]. A number of contributions have also been given by Li, Pan and by Li and Pan. See [52]-[53], [68]-[73]. The periodic case, where the γ_k consist of a finite number of points cyclically repeated, was studied by González-Vera, Hendriksen and Njåstad. See [46], [48]-[49], [60]-[65]. When the poles of the orthogonal rational functions consist of the origin and the point at infinity, infinitely repeated, the rational functions are Laurent polynomials. For a presentation of the basic theory of orthogonal Laurent polynomials and related topics we refer to the survey article [50] by Jones and Njåstad, and the references given there.

The whole theory of orthogonal rational functions is related to the theory of polynomials orthogonal with respect to varying measures, first extensively studied by Lopez. We refer to [54]-[58].

We shall make use of the following notation. $\mathbb{C}$ denotes the complex plane, $\mathbb{D}$ the open unit disk, $\mathbb{U}$ the open upper half plane, $\mathbb{T}$ the unit circle, $\mathbb{R}$ the real line. Furthermore $\hat{\mathbb{C}}$ denotes the extended complex plane (the point at infinity added), $\hat{\mathbb{R}}$ the closure of $\mathbb{R}$ in $\hat{\mathbb{C}}$, $\hat{\mathbb{U}}$ the closure of $\mathbb{U}$ in $\hat{\mathbb{C}}$. We also write $\mathbb{E} = \mathbb{C} \setminus \{\mathbb{D} \cup \mathbb{T}\}$ and $\mathbb{V} = \mathbb{C} \setminus \{\mathbb{U} \cup \mathbb{R}\}$.

The substar transform f_* of a function f is defined as follows.
In the unit circle situation:

$$f_*(z) = \overline{f(1/\bar{z})}.$$

In the real line situation:

$$f_*(z) = \overline{f(\bar{z})}.$$

The *Riesz-Herglotz-Nevanlinna transform* $\Omega(\cdot, \mu)$ of a finite measure μ is defined as follows.
In the unit circle situation:

$$\Omega(z, \mu) = \int_{\mathbb{T}} \frac{t+z}{t-z} d\mu(t). \tag{1.1}$$

In the real line situation:

$$\Omega(z, \mu) = -i \int_{\mathbb{R}} \frac{1+tz}{t-z} d\mu(t). \tag{1.2}$$

The function (1.1) maps $\mathbb{D}$ into the right half plane, the function (1.2) maps $\mathbb{U}$ into the right half plane.

2 FUNCTION SPACES I

Let $\{\alpha_n\}_{n=1}^{\infty}$ be an arbitrary sequence of points in $\mathbb{D}$. We introduce the Blaschke factors ζ_k defined by

$$\zeta_k(z) = z_k \frac{z - \alpha_k}{1 - \overline{\alpha_k} z}, \quad z_k = -\frac{\overline{\alpha_k}}{|\alpha_k|}, \quad k = 1, 2, \ldots.$$

We set by convention $\frac{\overline{\alpha_k}}{|\alpha_k|} = -1$ when $\alpha_k = 0$, so that $\zeta_k(z) = z$ when $\alpha_k = 0$. The Blaschke products B_n are defined by

$$B_0 = 1, \quad B_n(z) = \prod_{k=1}^{n} \zeta_k(z), \quad n = 1, 2, \ldots.$$

We note that $B_n(z) = z^n$ for all n if $\alpha_k = 0$ for all k.

We shall use the notation π_n for the denominator polynomial in the rational function B_n, i.e.,

$$\pi_0 = 1, \quad \pi_n(z) = \prod_{k=1}^{n} (1 - \overline{\alpha_k} z), \quad n = 1, 2, \ldots.$$

We shall also use the notation

$$\omega_0 = 1, \quad \omega_n(z) = \prod_{k=1}^{n}(z - \alpha_k), \quad n = 1, 2, \ldots.$$

We observe that we may write

$$\omega_n(z) = \pi_n^{\#}(z),$$

where $\pi_n^{\#}$ denotes the *superstar transform* of the *polynomial* π_n (see e.g. [45],[47]). Hence

$$B_n(z) = \eta_n \frac{\omega_n(z)}{\pi_n(z)} = \eta_n \frac{\pi_n^{\#}(z)}{\pi_n(z)}, \quad \eta_n = \prod_{k=1}^{n} z_k.$$

We shall study spaces $\mathcal{L}_n$ and $\mathcal{L}$ of rational functions. The space $\mathcal{L}_n$ is given by

$$\mathcal{L}_n = \text{Span}\{B_0, B_1, \ldots, B_n\}, \quad n = 0, 1, 2, \ldots$$

and we set

$$\mathcal{L} = \bigcup_{n=0}^{\infty} \mathcal{L}_n.$$

(In [34] is used $\mathcal{L}_\infty$ for this union, and $\mathcal{L}$ for the closure of this space in an L_2-space. We shall here not have occasion to consider this closure, and use for convenience $\mathcal{L}$ for the space of rational functions itself.) A function f belongs to $\mathcal{L}_n$ if and only if it is of the form

$$f(z) = \frac{p(z)}{\pi_n(z)},$$

where p is a polynomial of degree at most n.

We shall write

$$\mathcal{L}_{n*} = \{f_* : f \in \mathcal{L}_n\}, \quad \mathcal{L}_* = \{f_* : f \in \mathcal{L}\}.$$

We then have

$$\mathcal{L}_{n*} = \text{Span}\{B_{0*}, B_{1*}, \ldots, B_{n*}\}, \quad \mathcal{L}_* = \bigcup_{n=1}^{\infty} \mathcal{L}_{n*}$$

and we observe that

$$B_{k*}(z) = [B_k(z)]^{-1}.$$

We shall in this paper work with the standard basis $\{B_0, B_1, \ldots, B_n, \ldots\}$ for $\mathcal{L}$. Several other basis $\{C_0, C_1, \ldots, C_n, \ldots\}$ for $\mathcal{L}$ with the property $\mathcal{L}_n = \text{Span}\{C_0, C_1, \ldots, C_n\}$ for every n have been studied and may be useful. One such basis is

$$\{1, \frac{1}{1 - \overline{\alpha_1} z}, \frac{1}{1 - \overline{\alpha_2} z} B_1(z), \ldots, \frac{1}{1 - \overline{\alpha_n} z} B_{n-1}(z), \ldots\}.$$

When all the points α_k are distinct,

$$\{1, \frac{1}{1 - \overline{\alpha_1}, z}, \ldots, \frac{1}{1 - \overline{\alpha_n} z}, \ldots\}$$

is such a basis, and when $\alpha_k \neq 0$ for all k,

$$\{1, \frac{1}{\pi_1(z)}, \ldots, \frac{1}{\pi_n(z)}, \ldots\}$$

is such a basis.

The *superstar transform* f^* of a function in $\mathcal{L}_n \setminus \mathcal{L}_{n-1}$ is defined as

$$f^*(z) = B_n(z) f_*(z).$$

We note that f^* is a function in $\mathcal{L}_n$. We find that

$$\left[\sum_{k=0}^{n} a_k B_k(z)\right]^* = B_n(z) \sum_{k=0}^{n} \overline{a_k} \, [B_k(z)]^{-1}$$

(when $a_n \neq 0$). In particular $B_n^*(z) = 1$.

We call a_n the *leading coefficient* of the function $f(z) = \sum_{k=0}^{n} a_k B_k(z)$ (with respect to the basis $\{B_0, B_1, \ldots, B_n, \ldots\}$). We observe that $a_n = \overline{f^*(\alpha_n)}$. If the leading coefficient is 1, the function f is said to be *monic*.

General reference: Sections 2.1-2.2 of [34].

3 ORTHOGONAL FUNCTIONS I

Let M be a linear functional defined on the linear space $\mathcal{L} + \mathcal{L}_*$. Since all the points α_k are contained in $\mathbb{D}$, all the factors $(z - \alpha_k)$ are different from all the factors $(z - (\overline{\alpha_k})^{-1})$. Thus we find by partial fraction decomposition that $\mathcal{L} + \mathcal{L}_*$ is the same as the product space $\mathcal{L} \cdot \mathcal{L}_*$.

We shall assume that

$$M[f_*] = \overline{M[f]} \quad \text{for} \quad f \in \mathcal{L} + \mathcal{L}_* \tag{3.1}$$

and

$$M[f \cdot f_*] > 0 \quad \text{for} \quad f \in \mathcal{L}, \quad f \not\equiv 0. \tag{3.2}$$

(We recall that here $f_*(z) = \overline{f(1/\bar{z})}$.) For convenience we normalize M such that $M[1] = 1$.

Typical examples are functionals represented by positive measures as follows: Let μ be a finite positive measure on $\mathbb{T}$, and define

$$M[f] = \int_{\mathbb{T}} f(t) d\mu(t) \quad \text{for} \quad f \in \mathcal{L} + \mathcal{L}_*.$$

We easily verify that M satisfies (3.1)-(3.2).

The functional M gives rise to an inner product $\langle \cdot, \cdot \rangle$ on the space $\mathcal{L}$ through the formula

$$\langle f, g \rangle = M[f \cdot g_*], \quad f, g \in \mathcal{L}.$$

We denote by $\{\varphi_n\}_{n=0}^{\infty}$ the orthonormal basis for $\mathcal{L}$ associated with the sequence $\{\mathcal{L}_n\}$, with leading coefficient $\varphi_n^*(\alpha_n)$ real and positive. Thus we have

$$\mathcal{L}_n = \operatorname{Span}\{\varphi_0, \varphi_1, \ldots, \varphi_n\}$$

and

$$\langle \varphi_j, \varphi_k \rangle = \delta_{j,k}.$$

When $\alpha_k = 0$ for all k, the functions φ_n are simply the (normalized) Szegő polynomials determined by M.

We may write φ_n in the form

$$\varphi_n(z) = \frac{p_n(z)}{\pi_n(z)}, \tag{3.3}$$

where p_n is a polynomial of exact degree n. We then have

$$\varphi_{n*}(z) = \frac{z^n p_{n*}(z)}{\omega_n(z)}, \quad \varphi_n^*(z) = \eta_n \frac{z^n p_{n*}(z)}{\pi_n(z)}. \tag{3.4}$$

Since φ_n is orthogonal to all the functions $z^m[\pi_{n-1}(z)]^{-1}$ for $m = 0, 1, \ldots, n-1$, we may write

$$M_n[p_n(z) \cdot (z^m)_*] = 0, \quad m = 0, 1, \ldots, n-1,$$

where

$$M_n[f] = M\left[\frac{f(z)}{(1-\overline{\alpha_n}z)\prod_{k=1}^{n-1}(1-\overline{\alpha_k}z)(1-\overline{\alpha_k}z)_*}\right].$$

Thus the sequence $\{p_n\}_{n=0}^{\infty}$ of polynomials is orthogonal with respect to the sequence of varying complex inner products $\langle\cdot,\cdot\rangle_n$ on $\mathbb{T}$ given by

$$\langle p, q\rangle_n = M_n[p \cdot q_*].$$

In particular, if $M[f]$ is given by $M[f] = \int_{\mathbb{T}} f(t)d\mu(t)$, then the sequence $\{p_n\}_{n=0}^{\infty}$ is orthogonal with respect to the sequence of varying complex measures μ_n given by

$$d\mu_n(t) = \frac{d\mu(t)}{(1-\overline{\alpha_n}t)\prod_{k=1}^{n-1}|1-\overline{\alpha_k}t|^2}.$$

The *functions of the second kind* ψ_n associated with $\{\varphi_n\}$ are defined as follows:

$$\psi_0 = 1$$

$$\psi_n(z) = M\left[\frac{t+z}{t-z}\{\varphi_n(t) - \varphi_n(z)\}\right], \quad n = 1, 2, \ldots .$$

(Here M operates on its argument as a function of t.) We may write

$$\psi_n(z) = \frac{q_n(z)}{\pi_n(z)}, \tag{3.5}$$

where q_n is a polynomial of degree at most n, and thus $\psi_n \in \mathcal{L}_n$.

We may also write

$$\psi_n(z) = M\left[\frac{t+z}{t-z}\left\{\frac{f(t)}{f(z)}\varphi_n(t) - \varphi_n(z)\right\}\right], \quad n = 1, 2, \ldots,$$

where f is any function in $\mathcal{L}_{(n-1)*}$. We find that the superstar transform ψ_n^* is given by

$$\psi_n^*(z) = M\left[\frac{t+z}{t-z}\left\{\frac{g(t)}{g(z)}\varphi_n^*(t) - \varphi_n^*(z)\right\}\right], \quad n = 1, 2, \ldots,$$

where g is any function in $\mathcal{L}_{n*}$ satisfying $g(1/\overline{\alpha_n}) = 0$.

General reference: Sections 2.2 and 4.2 of [34].

4 RECURSION I

The orthonormal function φ_n satisfy a recurrence relation which has the same structure as the Szegő recursion for polynomials, and which reduces to this Szegő recursion in the polynomial case.

Theorem 4.1 *The functions φ_n, φ_n^* satisfy a recursion of the following form:*

$$\begin{bmatrix} \varphi_n(z) \\ \varphi_n^*(z) \end{bmatrix} = e_n \frac{1-\overline{\alpha_{n-1}}z}{1-\overline{\alpha_n}z} \begin{bmatrix} u_n & 0 \\ 0 & v_n \end{bmatrix} \begin{bmatrix} 1 & \overline{\lambda_n} \\ \lambda_n & 1 \end{bmatrix} \begin{bmatrix} \zeta_{n-1}(z) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varphi_{n-1}(z) \\ \varphi_{n-1}^*(z) \end{bmatrix}, n = 1, 2, \ldots, \tag{4.1}$$

with initial conditions $\varphi_0 = \varphi_0^* = 1$. Here e_n is a positive constant, the constant u_n, where $|u_n| = 1$, is chosen such that $\varphi_n^*(\alpha_n) > 0$, and $v_n = \overline{u_n z_{n-1}} z_n$. Finally λ_n is given by

$$\lambda_n = \frac{1-\alpha_n\overline{\alpha_{n-1}}}{1-\overline{\alpha_n}\alpha_{n-1}} \overline{z_{n-1}} z_n \frac{\overline{\varphi_n(\alpha_{n-1})}}{\varphi_n^*(\alpha_{n-1})}. \tag{4.2}$$

The coefficient λ_n satisfies $|\lambda_n| < 1$.

The expression (4.2) for the coefficient λ_n is not very useful, since it uses function values of φ_n and φ_n^* to compute these functions. More practical expressions for λ_n are given by

$$\lambda_n = -\overline{z_{n-1}} \frac{\left\langle \varphi_k(z), \dfrac{z-\alpha_{n-1}}{1-\overline{\alpha_n}z} \varphi_{n-1}(z) \right\rangle}{\left\langle \varphi_k(z), \dfrac{1-\overline{\alpha_{n-1}}z}{1-\overline{\alpha_n}z} \varphi_{n-1}^*(z) \right\rangle}, \quad k = 0, 1, \ldots, n-1.$$

The constant e_n can be obtained as the square root of

$$e_n^2 = \frac{1-|\alpha_n|^2}{1-|\alpha_{n-1}|^2} \cdot \frac{1}{1-|\lambda_n|^2}. \tag{4.3}$$

In the polynomial situation, i.e. when $\alpha_n = 0$ for all n, the formula (4.1) takes the form

$$\begin{bmatrix} \varphi_n(z) \\ \varphi_n^*(z) \end{bmatrix} = e_n \begin{bmatrix} u_n & 0 \\ 0 & v_n \end{bmatrix} \begin{bmatrix} 1 & \overline{\lambda_n} \\ \lambda_n & 1 \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varphi_{n-1}(z) \\ \varphi_{n-1}^*(z) \end{bmatrix},$$

or

$$\begin{bmatrix} \varphi_n(z) \\ \varphi_n^*(z) \end{bmatrix} = \begin{bmatrix} e_n u_n z & e_n u_n \overline{\lambda_n} \\ \overline{e_n u_n} \lambda_n z & \overline{e_n u_n} \end{bmatrix} \begin{bmatrix} \varphi_{n-1}(z) \\ \varphi_{n-1}^*(z) \end{bmatrix},$$

which has the form of the Szegő recursion for Szegő polynomials.

The functions of the second kind satisfy a recurrence relation very similar to that satisfied by the orthogonal functions.

Theorem 4.2 *The functions ψ_n, ψ_n^* satisfy the recurrence relation*

$$\begin{bmatrix} \psi_n(z) \\ -\psi_n^*(z) \end{bmatrix} = e_n \frac{1-\overline{\alpha_{n-1}}z}{1-\overline{\alpha_n}z} \begin{bmatrix} u_n & 0 \\ 0 & v_n \end{bmatrix} \begin{bmatrix} 1 & \overline{\lambda_n} \\ \lambda_n & 1 \end{bmatrix} \begin{bmatrix} \zeta_{n-1}(z) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \psi_{n-1}(z) \\ -\psi_{n-1}^*(z) \end{bmatrix},$$

$n = 1, 2, \ldots$, where the recurrence coefficients are the same as those in Theorem 4.1. The initial conditions are $\psi_0 = \psi_0^* = 1$.

We have seen that orthonormal rational functions satisfy a recurrence relation as given in Theorem 4.1. A converse of this is also true. The following *Favard type theorem* follows immediately from results in Section 8.1 in [34] formulated in terms of measures, but can also be proved directly without recourse to representation theory for functionals.

Theorem 4.3 *Let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of complex numbers such that $|\lambda_n| < 1$, and let positive numbers e_n be determined through the formula (4.3). Define the functions φ_n recursively by $\varphi_0 = 1$,*

$$\varphi_n(z) = e_n u_n \frac{1-\overline{\alpha_{n-1}}z}{1-\overline{\alpha_n}z}[\zeta_{n-1}(z)\varphi_{n-1}(z) + \overline{\lambda_n}\varphi_{n-1}^*(z)],$$

$n = 1, 2, \ldots$, where u_n is chosen such that $|u_n| = 1, \varphi_n^*(\alpha_n) > 0$. Then the functions φ_n^* satisfy the recursion $\varphi_0^* = 1$,

$$\varphi_n^*(z) = e_n v_n \frac{1-\overline{\alpha_{n-1}}z}{1-\overline{\alpha_n}z}[\zeta_{n-1}(z)\lambda_n\varphi_{n-1}(z) + \varphi_{n-1}^*(z)],$$

$n = 1, 2, \ldots$, with $v_n = \overline{u_n z_{n-1}} z_n$. Furthermore there exists a linear functional M on $\mathcal{L} + \mathcal{L}_*$ such that $\{\varphi_n\}$ are the corresponding orthonormal functions.

General reference: Section 4.1-4.2 and 8.1 of [34].

5 QUADRATURE I

It can be shown that all the zeros of φ_n are contained in $\mathbb{D}$. They may be multiple zeros of any order. Thus these zeros are not suitable as nodes in a quadrature on $\mathbb{T}$. To obtain such quadrature formulas, we introduce *para-orthogonal functions* of order n. These are functions of the form

$$Q_n(z,\tau) = \varphi_n(z) + \tau\varphi_n^*(z), \quad \tau \in \mathbb{C}, \quad \tau \neq 0.$$

They satisfy

$$\langle Q_n(z,\tau), B_k(z)\rangle = 0, \quad k = 1, 2, \ldots, n-1, \tag{5.1}$$

$$\langle Q_n(z,\tau), 1\rangle \neq 0, \quad \langle Q_n(z,\tau), B_n(z)\rangle \neq 0. \tag{5.2}$$

It can be shown that a function that satisfies (5.1)-(5.2) is para-orthogonal as defined above.

With the para-orthogonal functions $Q_n(z,\tau)$ we associate function of the second kind $P_n(z,\tau)$ given by

$$P_n(z,\tau) = \psi_n(z) - \tau\psi_n^*(z).$$

These functions may also be produced by the formula

$$P_n(z,\tau) = M\left[\frac{t+z}{t-z}\left\{\frac{f(t)}{f(z)}Q_n(t,\tau) - Q_n(z,\tau)\right\}\right], \quad n = 2, 3, \ldots, \tag{5.3}$$

where f is any function in $\mathcal{L}_{(n-1)*}$ satisfying $f(1/\overline{\alpha_n}) = 0$.

In the following we assume that $|\tau| = 1$.

We may write

$$Q_n(z,\tau) = \frac{p_n(z) + \tau\eta_n z^n p_{n*}(z)}{\pi_n(z)}.$$

Since all the zeros of p_n are contained in $\mathbb{D}$ and $|\tau| = |\eta_n| = 1$, we find that $p_n(z) + \tau\eta_n z^n p_{n*}(z)$ is a polynomial of exact degree n for all τ.

Theorem 5.1 *The para-orthogonal function $Q_n(z,\tau)$ has exactly n zeros, all of which are simple and lie on $\mathbb{T}$.*

Note that since $\alpha_n \notin \mathbb{T}$ for all n, the zeros of $Q_n(z,\tau)$ are the same as the zeros of the polynomial $p_n(z) + \tau\eta_n z^n p_{n*}(z)$.

The zeros of a para-orthogonal rational functions are nodes in a rational Szegő quadrature formula.

Theorem 5.2 *Let $\zeta_{nk}(\tau)$, $k = 1, \ldots, n$, be the zeros of $Q_n(z,\tau)$, and set $\lambda_{nk}(\tau) = \dfrac{P_n(\zeta_{nk},\tau)}{2\zeta_{nk}Q_n'(\zeta_{nk},\tau)}$. Then the rational Szegő quadrature formula*

$$M[R] = \sum_{k=1}^{n} \lambda_{nk}(\tau) R(\zeta_{nk}(\tau)) \tag{5.4}$$

is exact for every $R \in \mathcal{L}_{n-1} + \mathcal{L}_{(n-1)*} = \mathcal{L}_{n-1} \cdot \mathcal{L}_{(n-1)*}$.

An alternative expression for the weight $\lambda_{nk}(\tau)$ is

$$\lambda_{nk}(\tau) = \left[\sum_{j=0}^{n-1} |\varphi_j(\zeta_{nk}(\tau)|^2\right]^{-1}.$$

We point out that the degree of exactness of this formula is one less than maximal in the sense that the formula has $2n$ parameters $\zeta_{n1}(\tau), \ldots, \zeta_{nn}(\tau), \lambda_{n1}(\tau), \ldots, \lambda_{nn}(\tau)$, while the dimension of the space $\mathcal{L}_{n-1} + \mathcal{L}_{(n-1)}*$ is only $2n-1$. The formula is thus not analogous to a Gaussian quadrature formula, but it is a direct generalization of Szegő quadrature formulas in the polynomial case. In the polynomial situation the space $\mathcal{L}_{n-1} + \mathcal{L}_{(n-1)*}$ reduces to the space $\Lambda_{-(n-1),n-1}$ of Laurent polynomials of the form $L(z) = \sum_{k=-(n-1)}^{n-1} a_k z^k$. Note that there is one Szegő quadrature formula for every τ, while Gaussian quadrature formulas are unique.

The following result follows from Theorem 5.2 and the fact that the argument in formula (5.3) belongs to $\mathcal{L}_{n-1} + \mathcal{L}_{(n-1)*}$.

Theorem 5.3 *Let $\zeta_{nk}(\tau), \lambda_{nk}(\tau), k = 1, \ldots, n$, be as in Theorem 5.2. Then we have*

$$\frac{P_n(z,\tau)}{Q_n(z,\tau)} = -\sum_{k=1}^{n} \lambda_{nk}(\tau)\frac{\zeta_{nk}(\tau) + z}{\zeta_{nk}(\tau) - z} \tag{5.5}$$

for $n = 1, 2 \ldots$

General reference: Sections 5.1-5.4 of [34].

6 INTERPOLATION AND CONVERGENCE I

We shall in this section for convenience assume that the functional M is derived from a positive measure on $\mathbb{T}$ as described in Section 2. We recall the definition (1.1) of the Riesz-Herglotz-Nevanlinna transform $\Omega(z,\mu)$. We shall discuss how the rational functions $-\psi_n/\varphi_n$, ψ_n^*/φ_n^* and $-P_n(z,\tau)/Q_n(z,\tau)$ (for $|\tau| = 1$) interpolate and converge to $\Omega(z,\mu)$.

Theorem 6.1 *For $|\tau| = 1$ the quotients $-P_n(z,\tau)/Q_n(z,\tau)$ interpolate $\Omega(z,\mu)$ at the table $\{0, \infty, \alpha_1, 1/\overline{\alpha_1}, \ldots, \alpha_{n-1}, 1/\overline{\alpha_{n-1}}\}$ in the following sense:*

$$\Omega(z,\mu) + \frac{P_n(z,\tau)}{Q_n(z,\tau)} = g(z) z B_{n-1}(z) \tag{6.1}$$

$$\Omega(z,\mu) + \frac{P_n(z,\tau)}{Q_n(z,\tau)} = h(z) z^{-1} [B_{n-1}(z)]^{-1} \tag{6.2}$$

for $n = 1, 2, \ldots$, where g is holomorphic in $\mathbb{D}$ and h is holomorphic in $\mathbb{E}$.

Since $\alpha_k \in \mathbb{D}$ and $1/\overline{\alpha_k} \notin \mathbb{D} \cup \mathbb{T}$ for all k we may also write (6.1)-(6.2) in the form

$$\Omega(z,\mu) + \frac{P_n(z,\tau)}{Q_n(z,\tau)} = z G(z) \omega_{n-1}(z)$$

$$\Omega(z,\mu) + \frac{P_n(z,\tau)}{Q_n(z,\tau)} = z^{-1} H(z) \pi_{n-1}(z),$$

with G holomorphic in $\mathbb{D}$, H holomorphic in $\mathbb{E}$.

For the orthonormal functions and the functions of the second kind an extra interpolation condition is satisfied. On the other hand, at a part of the interpolation table only linearized interpolation is obtained.

Theorem 6.2 *The quotients $-\frac{\psi_n}{\varphi_n}$ and $\frac{\psi_n^*}{\varphi_n^*}$ interpolate $\Omega(z,\mu)$ at the tables $\{0, \infty, \alpha_1, 1/\overline{\alpha_1}, \ldots, \alpha_{n-1}, 1/\overline{\alpha_{n-1}}, 1/\overline{\alpha_n}\}$ and $\{0, \infty, \alpha_1, 1/\overline{\alpha_1}, \ldots, \alpha_{n-1}, 1/\overline{\alpha_{n-1}}, \alpha_n\}$, respectively, in the following sense:*

$$\Omega(z,\mu)\varphi_n(z) + \psi_n(z) = z G(z) B_{n-1}(z) \tag{6.3}$$

$$\Omega(z,\mu) + \frac{\psi_n(z)}{\varphi_n(z)} = z^{-1} h(z) [B_n(z)]^{-1} \tag{6.4}$$

$$\Omega(z,\mu) - \frac{\psi_n^*(z)}{\varphi_n^*(z)} = z g(z) B_n(z) \tag{6.5}$$

$$\Omega(z,\mu)\varphi_{n*}(z) - \psi_{n*}(z) = z^{-1} H(z) [B_{n-1}(z)]^{-1}, \tag{6.6}$$

where g and G are holomorphic in $\mathbb{D}$, h and H are holomorphic in $\mathbb{E}$.

Recall the formulas (3.3)-(3.4) and (3.5). All the zeros of φ_n lie in $\mathbb{D}$, all the zeros of φ_{n*} and φ_n^* lie in $\mathbb{E}$. We may therefore write (6.3)-(6.6) in the following form:

$$\Omega(z,\mu) p_n(z) + q_n(z) = z \Gamma(z) \omega_{n-1}(z)$$

$$\Omega(z,\mu) + \frac{q_n(z)}{p_n(z)} = z^{-1} \delta(z) \pi_n(z)$$

$$\Omega(z,\mu) - \frac{q_{n*}(z)}{p_{n*}(z)} = z \gamma(z) \omega_n(z)$$

$$\Omega(z,\mu) p_{n*}(z) - q_{n*}(z) = z^{-1} \Delta(z) \pi_{n-1}(z),$$

where γ and Γ are holomorphic in $\mathbb{D}$, δ and Δ are holomorphic in $\mathbb{E}$.

In the polynomial situation ($\alpha_n = 0$ for all n) the expressions $zB_{n-1}(z), zB_n(z), [zB_{n-1}(z)]^{-1}$, $[zB_n(z)]^{-1}$ reduce to $z^n, z^{n+1}, z^{-n}, z^{-(n+1)}$. Cf. formulas in [51].

Since ψ_n/φ_n and ψ_n^*/φ_n^* are rational functions of type $[n/n]$, the content of Theorem 6.2 may be expressed as follows: $-\psi_n/\varphi_n$ is the $[n/n]$ multipoint Padé approximant to $\Omega(z,\mu)$ at the table $\{0, \infty, \alpha_1, 1/\overline{\alpha_1}, \ldots, \alpha_{n-1}, 1/\overline{\alpha_{n-1}}, 1/\overline{\alpha_n}\}$, and ψ_n^*/φ_n^* is the $[n/n]$ multipoint Padé approximant to $\Omega(z,\mu)$ at the table $\{0, \infty, \alpha_1, 1/\overline{\alpha_1}, \ldots, \alpha_{n-1}, 1/\overline{\alpha_{n-1}}, \alpha_n\}$.

We close this section with a theorem concerning convergence of the interpolating functions.

Theorem 6.3 *Assume that the condition $\sum_{n=1}^{\infty}(1-|\alpha_n|)=\infty$ is satisfied. Then the following hold:*
A. $\{\psi_n^*(z)/\varphi_n^*(z)\}$ *converges to* $\Omega(z,\mu)$ *locally uniformly in* $\mathbb{D}$.
B. $\{-\psi_n(z)/\varphi_n(z)\}$ *converges to* $\Omega(z,\mu)$ *locally uniformly in* $\mathbb{E}$.
C. $\{-P_n(z,\tau)/Q_n(z,\tau)\}$ *converges to* $\Omega(z,\mu)$ *locally uniformly in* $\mathbb{D}\cup\mathbb{E}$.

General reference: Section 6.1-6.2 and 9.2 in [34].

7 LINEAR FRACTIONAL TRANSFORMATIONS I

We shall in this section discuss a system of nested disks associated with the functional M. The results can be proved by applying a Liouville-Ostrogradskii type formula

$$\varphi_n^*(z)\psi_n(z)+\varphi_n(z)\psi_n^*(z)=2\frac{(1-|\alpha_n|^2)zB_n(z)}{(z-\alpha_n)(1-\overline{\alpha_n}z)}$$

and Christoffel-Darboux type formulas

$$\frac{\varphi_n^*(z)\overline{\varphi_n^*(w)}-\varphi_n(z)\overline{\varphi_n(w)}}{1-\zeta_n(z)\overline{\zeta_n(w)}}=\sum_{k=0}^{n-1}\varphi_k(z)\overline{\varphi_k(w)},$$

$$\frac{\varphi_n^*(z)\overline{\psi_n^*(w)}+\varphi_n(z)\overline{\psi_n(w)}}{1-\zeta_n(z)\overline{\zeta_n(w)}}+\frac{2}{1-z\bar{w}}=-\sum_{k=0}^{n-1}\varphi_k(z)\overline{\psi_k(w)},$$

$$\frac{\psi_n^*(z)\overline{\psi_n^*(w)}-\psi_n(z)\overline{\psi_n(w)}}{1-\zeta_n(z)\overline{\zeta_n(w)}}=\sum_{k=0}^{n-1}\psi_k(z)\overline{\psi_k(w)}.$$

We set

$$\mathbb{D}_0=\{z\in\mathbb{D}:z\neq\alpha_k\quad\text{for}\quad k=1,2,\ldots\}$$
$$\mathbb{E}_0=\{z\in\mathbb{E}:z\neq 1/\overline{\alpha_k}\quad\text{for}\quad k=1,2,\ldots\}.$$

For a fixed point $z\in\mathbb{D}_0\cup\mathbb{E}_0$ the values of $s=-P_n(z,\tau)/Q_n(z,\tau)$ describe a circle $K_n(z)$ when τ take all values in $\mathbb{T}$. The closed disk $\Delta_n(z)$ bounded by $K_n(z)$ is described by

$$s\in\Delta_n(z)\Leftrightarrow\sum_{k=0}^{n-1}|\psi_k(z)-s\varphi_k(z)|^2\le\frac{2(s+\bar{s})}{1-|z|^2},$$

and its radius $r_n(z)$ is given by

$$r_n(z)=\frac{2|z|}{1-|z|^2}\left[|B_{n-1}(z)|\sum_{k=0}^{n-1}|\varphi_k(z)|^2\right]^{-1}\tag{7.1}$$

The system of disks $\{\Delta_n(z)\}$ is nested, i.e. $\Delta_{n+1}(z)\subset\Delta_n(z)$. The intersection

$$\Delta_\infty(z)=\bigcap_{n=1}^{\infty}\Delta_n(z)$$

is therefore either a proper closed disk or a single point. It follows from (7.1) that $\Delta_\infty(z)$ is a single point if and only if the sequence $\{[|B_n(z)|\sum_{k=0}^{n-1}|\varphi_k(z)|^2]^{-1}\}$ tends to zero. When $\sum_{n=1}^{\infty}(1-|\alpha_n|)<\infty$ this condition is equivalent to divergence of the series $\sum_{k=0}^{\infty}|\varphi_k(z)|^2$.

Theorem 7.1 *Let $z_0 \in \mathbb{D}_0 \cup \mathbb{E}_0$ and assume that $\Delta_\infty(z)$ is a proper disk. Then $\Delta_\infty(z)$ is a proper disk for every $z \in \mathbb{D}_0 \cup \mathbb{E}_0$, and the series $\sum_{k=0}^{\infty} |\varphi_k(z)|^2$ and $\sum_{k=0}^{\infty} |\psi_k(z)|^2$ converge locally uniformly in $\mathbb{D}_0 \cup \mathbb{E}_0 \cup \mathbb{T}$.*

(In [34] only locally uniform convergence in $\mathbb{D}_0 \cup \mathbb{E}_0$ is stated, but the proof gives the stronger result.)

As a consequence of this theorem we get a dichotomy:
Either $\Delta_\infty(z)$ is a proper disk for every $z \in \mathbb{D}_0 \cup \mathbb{E}_0$, the *limit circle case*, or $\Delta_\infty(z)$ is a single point for every $z \in \mathbb{D}_0 \cup \mathbb{E}_0$, the *limit point case.*

Finally we mention that if $\sum_{n=1}^{\infty}(1-|\alpha_n|) = \infty$, then $\Delta_\infty(z)$ reduces to a point for every $z \in \mathbb{D}_0 \cup \mathbb{E}_0$ and we have the limit point case.

General reference: Section 10.2 of [34].

8 MOMENT PROBLEMS I

As before we suppose that we are given a linear functional M on $\mathcal{L} + \mathcal{L}_*$, satisfying (3.1)-(3.2). By the moment problem for M we mean: Find measures μ on $\mathbb{T}$ with infinite support such that

$$M[f] = \int_{\mathbb{T}} f(t) d\mu(t) \tag{8.1}$$

for all $f \in \mathcal{L}$. A measure with this property is called a solution of the moment problem. The problem is *determinate* if there is exactly one solution, *indeterminate* if there is more than one solution. Note that because of (3.1) we also have $M[g] = \int_{\mathbb{T}} g(t) d\mu(t)$ for all $g \in \mathcal{L}_*$ when μ is a solution.

For μ to be a solution, it is of course sufficient that (8.1) is satisfied for every element of some basis for $\mathcal{L}$. For example, (8.1) is equivalent to

$$M[B_n] = \int_{\mathbb{T}} B_n(t) d\mu(t), \quad n = 0, 1, 2, \ldots.$$

The constants $M[B_n]$, or $M[C_n]$ for any basis $\{C_n\}$, may be considered as moments of M, and this motivates the expression *moment problem* for (8.1).

We recall the quadrature formulas (5.4) of Section 5. We define the measures $\mu_n(\cdot, \tau)$ as the discrete measure with support $\{\zeta_{n1}(\tau), \ldots, \zeta_{nn}(\tau)\}$ and mass $\lambda_{nk}(\tau)$ at $\zeta_{nk}(\tau), k = 1, \ldots, n$. From the quadrature formulas (5.4) we find that

$$M[B_m] = \int_{\mathbb{T}} B_m(t) d\mu_n(t, \tau)$$

when $m < n$. Similarly we may write (5.5) as

$$\frac{P_n(z,\tau)}{Q_n(z,\tau)} = -\int_{\mathbb{T}} \frac{t+z}{t-z} d\mu_n(t,\tau) = -\Omega(z, \mu_n(\cdot, \tau)). \tag{8.2}$$

It follows from (8.2) that $\Omega(z, \mu_n(, \tau)) \in K_n(z)$ when $z \in \mathbb{D}_0 \cup \mathbb{E}_0$.

From Helly's selection and convergence theorems it can be deduced that for every z on the boundary $K_\infty(z)$ of $\Delta_\infty(z)$ there is a subsequence of a sequence $\{\mu_n(\cdot, \tau_n)\}$ which converges to a solution ν and such that the corresponding subsequence of $\{\Omega(z, \mu_n(\cdot, \tau_n))\}$ converges to $\Omega(z, \nu)$. Thus for every boundary point s of $\Delta_\infty(z)$ there is a solution ν of the moment problem such that $\Omega(z, \nu) = s$. The set of solutions of the moment problem is easily seen to be convex, from which it follows that for every $s \in \Delta_\infty(z)$ there is a solution μ such that $\Omega(z, \mu) = s$. On the other hand, by the aid of Bessel's inequality it can be shown that $\Omega(z,\mu) \subset \Delta_\infty(z)$ for any solution μ of the moment problem.

Summing up this discussion, we get:

Theorem 8.1 *Let $z \in \mathbb{D}_0 \cup \mathbb{E}_0$. Then $\Delta_\infty(z)$ consists of exactly all values $\Omega(z,\mu)$, where μ is a solution of the moment problem.*

Since a measure is uniquely determined by its transform, this implies:

Corollary 8.2 *The moment problem is indeterminate in the limit circle case, determinate in the limit point case. In particular the problem is always determinate when $\sum_{n=1}^\infty (1-|\alpha_n|) = \infty$.*

We note that if the sequence $\{\alpha_n\}$ consists of a finite number of points repeated in some way, then $\sum_{n=1}^\infty (1-|\alpha_n|) = \infty$ and hence the moment problem is determinate. In particular this is the case in the polynomial situation, when $\alpha_n = 0$ for all n.

General reference: Section 10.1, 10.3 of [34].

9 FUNCTION SPACES II

Let $\{\alpha_n\}_{n=1}^\infty$ be a sequence of points in the extended real line $\hat{\mathbb{R}}$. For technical reasons we assume there is a point in $\hat{\mathbb{R}}$ which is different from all the α_n. There is no restriction in assuming this point to be at the origin, i.e., that $\alpha_n \neq 0$ for all n.

We shall make use of the factors Z_k defined by

$$Z_k(z) = \frac{z}{1-\alpha_k^{-1}z}, \quad k = 1,2,\ldots$$

and the products b_n defined by

$$b_0 = 1, \quad b_n(z) = \prod_{k=1}^n Z_k(z), \quad n = 1,2,\ldots.$$

We shall here use the notation ω_n as follows:

$$\omega_0 = 1, \quad \omega_n(z) = \prod_{k=1}^n (1-\alpha_k^{-1}z), \quad n = 1,2,\ldots.$$

Thus we may write

$$b_n(z) = \frac{z^n}{\omega_n(z)}, \quad n = 0,1,2,\ldots.$$

In particular, $b_n(z) = z^n$ for all n when $\alpha_k = \infty$ for all k. We note that $Z_{k*}(z) = Z_k(z)$, $b_{n*}(z) = b_n(z)$, $\omega_{n*}(z) = \omega_n(z)$. (Recall that here $f_*(z) = \overline{f(\bar{z})}$.)

We shall again study spaces $\mathcal{L}_n$ and $\mathcal{L}$ of rational functions. These are given by

$$\mathcal{L}_n = \text{Span}\{b_0, b_1, \ldots, b_n\}, \quad n = 0,1,2,\ldots$$

and

$$\mathcal{L} = \bigcup_{n=0}^\infty \mathcal{L}_n.$$

A function f belongs to $\mathcal{L}_n$ if and only if it can be written in the form

$$f(z) = \frac{p(z)}{\omega_n(z)},$$

where p is a polynomial of degree at most n. In particular $\mathcal{L}_n$ equals the space of polynomials of degree at most n if $\alpha_k = \infty$ for all k.

As in Section 2 we could work with other simple bases $\{c_0, c_1, \ldots, c_n, \ldots\}$ for $\mathcal{L}$ such that $\mathcal{L}_n = \mathrm{Span}\{c_0, c_1, \ldots, c_n\}$. For example, if all the points α_k are finite and distinct,

$$\{1, \frac{1}{1-\alpha_1^{-1}z}, \ldots, \frac{1}{1-\alpha_n^{-1}z}, \ldots\}$$

is such a base, and when all α_k are finite,

$$\{1, \frac{1}{\omega_1(z)}, \ldots, \frac{1}{\omega_n(z)}, \ldots\}$$

is such a base.

A function f in $\mathcal{L}_n$ has the representation $f(z) = \sum_{k=0}^{n} a_k b_k(z)$. We call a_n the *leading coefficient* of f (with respect to the basis $\{b_n\}$). When $a_n = 1$, the function is said to be *monic*.

General reference: Section 11.1 of [34].

10 ORTHOGONAL FUNCTIONS II

Let M be a linear functional defined on the linear space $\mathcal{L} \cdot \mathcal{L}$, satisfying

$$M[f_*] = \overline{M[f]}, f \in \mathcal{L} \cdot \mathcal{L} \tag{10.1}$$

and

$$M[f \cdot f_*] > 0, f \in \mathcal{L}, f \not\equiv 0. \tag{10.2}$$

Without loss of generality we assume that $M[1] = 1$.

Note that in the present situation we have $\mathcal{L}_* = \mathcal{L}$. In the previous situation we had $\mathcal{L} \cdot \mathcal{L}_* = \mathcal{L} + \mathcal{L}_*$, and it was sufficient for M to be defined on $\mathcal{L} + \mathcal{L}_*$. In the present situation we have in general $\mathcal{L} \cdot \mathcal{L} \neq \mathcal{L} + \mathcal{L}_* = \mathcal{L}$, and we need to require M to be defined on $\mathcal{L} \cdot \mathcal{L}$. The equality $\mathcal{L} \cdot \mathcal{L} = \mathcal{L}$ holds when the sequence $\{\alpha_k\}$ consists of points which are all repeated an infinite number of times in some order. In particular this is the case when $\alpha_k = \infty$ for all k (the polynomial case).

Examples of functionals M satisfying (10.1)-(10.2) can be obtained as follows:

Let μ be a positive measure on $\mathbb{R}$ with the property that all functions in $\mathcal{L} \cdot \mathcal{L}$ are integrable. Define M by

$$M[F] = \int_{\mathbb{R}} F(t) d\mu(t), \quad F \in \mathcal{L} \cdot \mathcal{L}. \tag{10.3}$$

Then clearly M satisfies (10.1)-(10.2).

The functional M gives rise to an inner product $\langle \cdot, \cdot \rangle$ on the space $\mathcal{L}$ through the formula

$$\langle f, g \rangle = M[f \cdot g_*], \quad f, g \in \mathcal{L}. \tag{10.4}$$

Let $\{\varphi_n\}_{n=0}^{\infty}$ be an orthonormal basis associated with the sequence $\{\mathcal{L}_n\}$. I.e.,

$$\mathcal{L}_n = \mathrm{Span}\{\varphi_0, \varphi_1, \ldots, \varphi_n\}$$

and

$$\langle \varphi_j, \varphi_k \rangle = \delta_{jk}.$$

The leading coefficient of φ_n shall be chosen to be real and positive.

The function φ_n may be represented in the form

$$\varphi_n(z) = \frac{p_n(z)}{\omega_n(z)}$$

where p_n is a polynomial of exact degree n. Since φ_n is orthogonal to all functions of the form $z^m/\omega_{n-1}(z)$, $m = 0, 1, \ldots, n-1$, we may write

$$M_n[p_n(z) \cdot z^m] = 0, \quad m = 0, 1, \ldots, n-1,$$

where

$$M_n[f] = M \left[\frac{f(z)}{(1 - \alpha_n^{-1} z)\omega_{n-1}(z)^2} \right].$$

This means that the sequence $\{p_n\}$ of polynomials is orthogonal with respect to the sequence of varying (not necessarily positive) inner products $\langle \cdot, \cdot \rangle_n$ given by

$$\langle p, q \rangle_n = M_n[p \cdot q_*].$$

Thus if $M[f]$ is given by (10.3), then the sequence $\{p_n\}$ is orthogonal with respect to the sequence of varying measures μ_n given by

$$d\mu_n(t) = \frac{d\mu(t)}{(1 - \alpha_n^{-1} t)\omega_{n-1}(t)^2}.$$

The functions of the second kind are defined as follows:

$$\psi_0(z) = iz$$

$$\psi_n(z) = M \left[-i \frac{1 + tz}{t - z} \{\varphi_n(t) - \varphi_n(z)\} \right], \quad n = 1, 2, \ldots.$$

These functions have the form

$$\psi_n(z) = \frac{q_n(z)}{\omega_n(z)}, n = 1, 2, \ldots,$$

where q_n is a polynomial of degree at most n. Thus $\psi_n \in \mathcal{L}_n$ for $n = 1, 2, \ldots$.

General reference: Sections 11.1-11.2 of [34].

11 RECURSION II

The function $\varphi_n = \dfrac{p_n}{\omega_n}$ is called *regular* if $p_n(\alpha_{n-1}) \neq 0$, *singular* otherwise. When the sequence $\{\varphi_n\}$ is regular, i.e., when all φ_n are regular, it satisfies a three-term recurrence relation which generalizes the recursion for orthonormal polynomials.

Theorem 11.1 *Assume that the orthonormal sequence $\{\varphi_n\}$ is regular. Then the functions $\{\varphi_n, \psi_n\}$ satisfy a recurrence relation of the form*

$$\begin{bmatrix} \psi_n(z) \\ \varphi_n(z) \end{bmatrix} = \left(A_n \frac{z}{1 - \alpha_n^{-1} z} + B_n \frac{1 - \alpha_{n-2}^{-1} z}{1 - \alpha_n^{-1} z} \right) \begin{bmatrix} \psi_{n-1}(z) \\ \varphi_{n-1}(z) \end{bmatrix} + C_n \frac{1 - \alpha_{n-2}^{-1} z}{1 - \alpha_n^{-1} z} \begin{bmatrix} \psi_{n-2}(z) \\ \varphi_{n-2}(z) \end{bmatrix}.$$

with $n = 2, 3, \ldots$ *and* $\alpha_0 = \infty$. *The constants* A_n, B_n, C_n *satisfy*

$$A_n + B_n \frac{1 - \alpha_{n-2}^{-1}\alpha_{n-1}}{\alpha_n} \neq 0 \tag{11.1}$$

$$C_n \neq 0. \tag{11.2}$$

In the polynomial case (i.e., $\alpha_n = \infty$ for all n) all the polynomials φ_n are regular. Hence we obtain

$$\begin{bmatrix} \psi_n(z) \\ \varphi_n(z) \end{bmatrix} = (A_n z + B_n) \begin{bmatrix} \psi_{n-1}(z) \\ \varphi_{-1}(z) \end{bmatrix} + C_n \begin{bmatrix} \psi_{n-2}(z) \\ \varphi_{n-2}(z) \end{bmatrix}, \quad n = 2, 3, \ldots$$

which has the form of the classical recursion for orthonormal functions.

As in the classical situation, a converse of Theorem 11.1 is true. The argument is, however, rather more complicated. This is so partly because from a given recursion for functions in $\mathcal{L}$ we need to define a functional M on $\mathcal{L} \cdot \mathcal{L}$ with respect to which the functions are orthogonal.

Theorem 11.2 *Let* $\{\varphi_n\}$ *be a sequence of functions such that* $\varphi_0 = 1$, $\varphi_1(z) = \frac{\kappa z + \lambda}{1 - \alpha_1^{-1} z}$ *with* $\kappa \neq 0$, $\varphi_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$ *for* $n = 1, 2, \ldots$. *Assume that there exist constants* A_n, B_n, C_n, $n = 2, 3, \ldots$ *satisfying (11.1)-(11.2) such that*

$$\varphi_n(z) = \left(A_n \frac{z}{1 - \alpha_n^{-1} z} + B_n \frac{1 - \alpha_{n-2}^{-1} z}{1 - \alpha_n^{-1} z} \right) \varphi_{n-1}(z) + C_n \frac{1 - \alpha_{n-2}^{-1} z}{1 - \alpha_n^{-1} z} \varphi_{n-2}(z)$$

for $n = 2, 3, \ldots$. *Then there exists a linear functional* M *on* $\mathcal{L} \cdot \mathcal{L}$ *such that* $\{\varphi_n\}$ *forms a regular orthonormal sequence with respect to the inner product (10.4).*

General reference: Sections 11.1, 11.9 of [34].

12 QUADRATURE II

A *quasi-orthogonal rational function* of order n is a function of the form

$$Q_n(z, \tau) = \varphi_n(z) + \tau \frac{1 - \alpha_{n-1}^{-1} z}{1 - \alpha_n^{-1} z} \varphi_{n-1}(z), \quad \tau \in \hat{\mathbb{C}}. \tag{12.1}$$

($Q_n(z, \infty)$ means $\varphi_{n-1}(z)$.) These functions satisfy

$$\langle Q_n(z, \tau), b_k(z) \rangle = 0, \quad k = 0, 1, \ldots, n-2,$$

and all functions satisfying this conditions is of form (12.1).

With the quasi-orthogonal functions we associate functions of the second kind given by

$$P_n(z, \tau) = \psi_n(z) + \tau \frac{1 - \alpha_{n-1}^{-1} z}{1 - \alpha_n^{-1} z} \psi_{n-1}(z).$$

These functions may also be described by the formula

$$P_n(z, \tau) = M \left[-i \frac{1 + tz}{t - z} \left\{ \frac{1 - \alpha_n^{-1} t}{1 - \alpha_{n-1}^{-1} t} Q_n(t, z) - Q_n(z, \tau) \right\} \right], \quad n = 2, 3, \ldots. \tag{12.2}$$

We may write

$$Q_n(z, \tau) = \frac{p_n(z, \tau)}{\omega_n(z)}, \quad P_n(z, \tau) = \frac{q_n(z, \tau)}{\omega_n(z)},$$

where p_n and q_n are polynomials of degree at most n.

In the following we shall assume that $\tau \in \hat{\mathbb{R}}$.

A value of τ for which none of the points $\{0, \alpha_1, \ldots, \alpha_n\}$ are zeros of $p_n(z, \tau)$ is called a *regular* value for φ_n. There can be at most n non-regular values.

Theorem 12.1 *Assume that φ_n is regular and that τ is a regular value for φ_n. Then $Q_n(z,\tau)$ has n simple zeros, all lying in $\tau \in \hat{\mathbb{R}} \setminus \{0, \alpha_1, \ldots, \alpha_n\}$.*

Note that the zeros are the same as the zeros of the numerator polynomial $p_n(z,\tau)$.

Theorem 12.2 *Assume that φ_n is regular and that τ is a regular value for φ_n. Let $\xi_{nk}(\tau), k = 1, \ldots, n$, be the zeros of $Q_n(z,\tau)$, and set*

$$\lambda_{nk}(\tau) = \frac{i}{1+\xi_{nk}(\tau)^2} \cdot \frac{P_n(\xi_{nk}(\tau),\tau)}{Q_n'(\xi_{nk}(\tau),\tau)}.$$

Then the quadrature formula

$$M[R] = \sum_{k=1}^{n} \lambda_{nk}(\tau) R(\xi_{nk}(\tau))$$

is exact for every $R \in \mathcal{L}_{n-1} \cdot \mathcal{L}_{n-1}$.

An alternative expression for the weight $\lambda_{nk}(\tau)$ is

$$\lambda_{nk}(\tau) = \left[\sum_{j=0}^{n-1} |\varphi_j(\xi_{nk}(\tau))|^2 \right]^{-1}.$$

The degree of exactness of this formula is one less than maximal in the sense that the formula has $2n$ parameters $\lambda_{n1}(\tau), \ldots, \lambda_{nn}(\tau), \xi_{n1}(\tau), \ldots, \xi_{nn}(\tau)$, while the dimension of the space $\mathcal{L}_{n-1} \cdot \mathcal{L}_{n-1}$ is only $2n-1$. When $\tau = 0$, the degree of exactness is increased by one, and it is therefore natural to call this formula a *rational Gaussian quadrature formula*.

Theorem 12.3 *Assume that φ_n is regular and that $\tau = 0$ is a regular value for φ_n. Then the quadrature formula*

$$M[R] = \sum_{k=0}^{n} \lambda_{nk}(0) R(\xi_{nk}(0))$$

is exact for every $R \in \mathcal{L}_n \cdot \mathcal{L}_{n-1}$.

The following result follows from Theorem 12.2 and the fact that the argument in formula (12.2) belongs to $\mathcal{L}_{n-1} \cdot \mathcal{L}_{n-1}$.

Theorem 12.4 *Let $\xi_{nk}(\tau), \lambda_{nk}(\tau), k = 1, \ldots, n$, be as in Theorem 12.2. Then we have*

$$\frac{P_n(z,\tau)}{Q_n(z,\tau)} = -i \sum_{k=1}^{n} \lambda_{nk}(\tau) \frac{1+\xi_{nk}(\tau) z}{\xi_{nk}(\tau) - z}.$$

General reference: Sections 11.5-11.6, 11.10 of [34].

13 INTERPOLATION AND CONVERGENCE II

As in Section 6 we shall here only consider the situation when the functional M is given as an integral. This means that there exists a positive measure μ on $\mathbb{R}$ such that all functions in $\mathcal{L} \cdot \mathcal{L}$ are integrable, with M given by formula (10.3). We recall the definition (1.2) of the Riesz-Herglotz-Nevanlinna transform $\Omega(z,\mu)$ of μ.

Theorem 13.1 *Let φ_n be regular and let τ be a regular value for φ_n. Then the quotient $-\frac{P_n(z,\tau)}{Q_n(z,\tau)}$ interpolates $\Omega(z,\mu)$ at the table $\{i,-i,\alpha_1,\alpha_1,\ldots,\alpha_{n-1},\alpha_{n-1}\}$ in the following sense:*

$$-\frac{P_n(i,\tau)}{Q_n(i,\tau)} = \Omega(i,\mu), \quad -\frac{P_n(-i,\tau)}{Q_n(-i,\tau)} = \Omega(-i,\mu)$$

$$\lim_{z\to\alpha_m} \left\{\frac{P_n(z,\tau)}{Q_n(z,\tau)}\right\}^{(k)} + \Omega^{(k)}(z,\mu) = 0$$

(subscript means differentiation) for $k = 0,1,\ldots,\alpha_m^\# - 1$, $m = 1,2,\ldots,n-1$, where $\alpha_m^\#$ denotes the multiplicity of α_m in the sequence $\{\alpha_1,\alpha_1,\ldots,\alpha_m,\alpha_m,\ldots,\alpha_{n-1},\alpha_{n-1}\}$. The limit is to be understood as angular limtt in arbitrary regions $\varepsilon < |\arg(z-\alpha_m)| < \pi-\varepsilon$, $\varepsilon > 0$, $k = 1,\ldots,n-1$.

For the pùre quotients ψ_n/φ_n we have a somewhat stronger result.

Theorem 13.2 *Assume that φ_n is regular and that $\tau = 0$ is a regular value for φ_n. Then the quotient $-\psi_n/\varphi_n$ interpolates $\Omega(z,\mu)$ at the table $\{i,-i,\alpha_1,\alpha_1,\ldots,\alpha_{n-1},\alpha_{n-1},\alpha_n\}$ in the following sense:*

$$\frac{\psi_n(z)}{\varphi_n(z)} + \Omega(z,\mu) = (z-i)(z+i)\Gamma_n(z)\omega_n(z)\omega_{n-1}(z),$$

where Γ_n is holomorphic in $\mathbb{C}\setminus\mathbb{R}$ and bounded in any region $\varepsilon < |\arg(z-\alpha_k)| < \pi-\varepsilon$, $\varepsilon > 0$, $k = 1,\ldots,n$.

Since $\frac{\psi_n}{\varphi_n}$ is a rational function of type $[n/n]$ the meaning of Theorem 13.2 is: The function $-\psi_n/\varphi_n$ is the $[n/n]$ multipoint Padé approximant to $\Omega(z,\mu)$ at the table $\{i,-i,\alpha_1,\alpha_1,\ldots,\alpha_{n-1},\alpha_{n-1},\alpha_n\}$.

Let $\mu_n(\cdot,\tau)$ denote the discrete measure with support $\{\xi_{n1}(\tau),\ldots,\xi_{nk}(\tau)\}$ and mass $\lambda_{nk}(\tau)$ at $\xi_{nk}(\tau)$, $k = 1,\ldots,n$.

Theorem 13.3 *Assume that the sequence $\{\varphi_n\}$ is regular, and for each n let τ_n be a regular value of φ_n. If the sequence $\{\mu_{n_k}(\cdot,\tau_{n_k})\}$ converges to μ, then*

$$\lim_{k\to\infty}\left[-\frac{P_{n_k}(z,\tau_{n_k})}{Q_{nk}(z,\tau_{n_k})}\right] = \Omega(z,\mu)$$

locally uniformly in $\mathbb{U}$.

For the sequence $\{\psi_n/\varphi_n\}$ the following result holds.

Theorem 13.4 *Assume that $\{\varphi_n\}$ is a regular sequence. Let $\{\varphi_{n_k}\}$ be a subsequence of $\{\varphi_n\}$ such that $\tau = 0$ is a regular value for each φ_{n_k} and such that $\{\mu_{n_k}(\cdot,0)\}$ converges to μ. Then*

$$\lim_{k\to\infty}\left[-\frac{\psi_{n_k}(z)}{\varphi_{n_k}(z)}\right] = \Omega(z,\mu)$$

locally uniformly in $\mathbb{U}$.

General reference: Section 11.10-11.11 of [34].

14 LINEAR FRACTIONAL TRANSFORMATIONS II

Again we shall make use of Liouville-Ostrogradskii type formulas and Christoffel-Darboux type formulas. These have the form

$$(1-\alpha_n^{-1}z)(1-\alpha_{n-1}^{-1}z)[\psi_n(z)\varphi_{n-1}(z)-\varphi_n(z)\psi_{n-1}(z)] = -iE_n(1+z^2)$$

and

$$\begin{aligned}\varphi_n(w)\varphi_{n-1}(z)(1-\alpha_n^{-1}w)(1-\alpha_{n-1}^{-1}z) &- \varphi_n(z)\varphi_{n-1}(w)(1-\alpha_n^{-1}z)(1-\alpha_{n-1}^{-1}w)\\ &= E_n(w-z)\sum_{k=0}^{n-1}\varphi_k(z)\varphi_k(w)\end{aligned}$$

$$\begin{aligned}\psi_n(w)\varphi_{n-1}(z)(1-\alpha_n^{-1}w)(1-\alpha_{n-1}^{-1}z) &- \varphi_n(z)\psi_{n-1}(w)(1-\alpha_n^{-1}z)(1-\alpha_{n-1}^{-1}w)\\ &= E_n(w-z)\left[\sum_{k=0}^{n-1}\varphi_k(z)\psi_k(w) - i\frac{1+zw}{z-w}\right]\end{aligned}$$

$$\begin{aligned}\psi_n(w)\psi_{n-1}(z)(1-\alpha_n^{-1}w)(1-\alpha_{n-1}^{-1}z) &- \psi_n(z)\psi_{n-1}(w)(1-\alpha_n^{-1}z)(1-\alpha_{n-1}^{-1}w)\\ &= E_n(w-z)\left[\sum_{k=0}^{n}\psi_k(z)\psi_k(w) - 1\right].\end{aligned}$$

Here E_n is a constant, $E_n \neq 0$ if φ_n is regular, $E_n = 0$ if φ_n is singular.

We shall for simplicity assume that the sequence $\{\varphi_n\}$ is regular. The concluding results of this and the next section are, however, true without this assumption.

The results of this section rely on the formulas above.

For a fixed $z \in \mathbb{C}\setminus\mathbb{R}$ the mapping $\tau \to -\frac{P_n(z,\tau)}{Q_n(z,\tau)}$ transforms $\hat{\mathbb{R}}$ to a circle $K_n(z)$. We denote by $\Delta_n(z)$ the closed disk bounded by $K_n(z)$. This disk is described by

$$s \in \Delta_n(z) \Leftrightarrow |1-s|^2 + \sum_{k=0}^{n-1}|\psi_k(z)+s\varphi_k(z)|^2 \leq (s+\bar{s})\frac{|z-i|^2}{z-\bar{z}},$$

and its radius $r_n(z)$ is given by

$$r_n(z) = \frac{|z-i|^2}{z-\bar{z}}\left[\sum_{k=0}^{n-1}|\varphi_k(z)|^2\right]^{-1}. \tag{14.1}$$

We have $\psi_n(i) = -\varphi_n(i)$, thus $\Delta_n(i)$ reduces to a point.

The sequence $\{\Delta_n(z)\}$ is nested i.e., $\Delta_{n+1}(z) \subset \Delta_n(z)$, and as in Section 7 the intersection

$$\Delta_\infty(z) = \bigcap_{n=1}^{\infty}\Delta_n(z)$$

is either a proper closed disk or a single point. It follows from (14.1) that for $z \neq i, \Delta_\infty(z)$ is a single point if and only if the series $\sum_{k=1}^{\infty}|\varphi_k(z)|^2$ diverges.

We set $\mathbb{C}_0 = \mathbb{C}\setminus\{\mathbb{R}\cup\{i\}\cup\{-i\}\}$. The following invariance result holds.

Theorem 14.1 *Let $z_0 \in \mathbb{C}_0$ and assume that $\Delta_\infty(z_0)$ is a proper disk. Then $\Delta_\infty(z)$ is a proper disk for every $z \in \mathbb{C}_0$ and the series $\sum_{k=0}^{\infty} |\varphi_k(z)|^2$ and $\sum_{k=0}^{\infty} |\varphi_k(z)|^2$ converge locally uniformly in $\mathbb{C}_0$.*

Thus again we get a dichotomy:

Either $\Delta_\infty(z)$ is a proper disk for every $z \in \mathbb{C}_0$, the *limit circle case*, or $\Delta_\infty(z)$ is single point for every $z \in \mathbb{C}_0$, the *limit point case.*

General reference: Sections 11.3-11.4, 11.7 of [34].

15 MOMENT PROBLEMS II

We assume that we are given a linear functional M on $\mathcal{L} \cdot \mathcal{L}$ satisfying (10.1)-(10.2). A positive measure μ on $\mathbb{R}$ with infinite support is said to solve *the moment problem on $\mathcal{L}$* if

$$M[f] = \int_{\mathbb{R}} f(t)d\mu(t), f \in \mathcal{L}, \tag{15.1}$$

and to solve *the moment problem on $\mathcal{L} \cdot \mathcal{L}$* if

$$M[F] = \int_{\mathbb{R}} F(t)d\mu(t), F \in \mathcal{L} \cdot \mathcal{L}, \tag{15.2}$$

A measure which solves the moment problem on $\mathcal{L} \cdot \mathcal{L}$ also solves the problem on $\mathcal{L}$, since $\mathcal{L} \subset \mathcal{L} \cdot \mathcal{L}$.

Clearly it is sufficient for μ to solve the moment problem on $\mathcal{L}$ or on $\mathcal{L} \cdot \mathcal{L}$ that (15.1) or (15.2) is satisfied for the functions in some generating system for $\mathcal{L}$ or $\mathcal{L} \cdot \mathcal{L}$. For example, (15.1) is equivalent to

$$M[b_m] = \int_{\mathbb{R}} b_m(t)d\mu(t), m = 0, 1, 2, \ldots,$$

and (15.2) is equivalent to

$$M[b_m \cdot b_n] = \int_{\mathbb{R}} b_m(t)b_n(t)d\mu(t), m, n = 0, 1, 2, \ldots.$$

The constants $M[b_m]$ or $M[b_m b_n]$ may be termed *moments*, from which the expression *moment problems* arise.

We shall also in this section assume that the sequence $\{\varphi_n\}$ is regular.

We recall the measures $\mu_n(\cdot, \tau)$ introduced in Section 13. Let $z \in \mathbb{C}_0$. As in Section 8 we find that for every s on the boundary of $\Delta_\infty(z)$ there is a subsequence of a sequence $\{\mu_n(\cdot, \tau_n)\}$ which converges to a solution ν of *the moment problem on $\mathcal{L}$* and such that $\Omega(z, \nu) = s$. It should be noted that in order to carry out the proof, we need to know that M *is defined on* $\mathcal{L} \cdot \mathcal{L}$. From the convexity of the set of solutions, it follows that for every $s \in \Delta_\infty(z)$ there is a solution μ of the moment problem on $\mathcal{L}$ such that $\Omega(z, \mu) = s$. By the aid of Bessel's inequality it can be shown that $\Omega(z, \mu) \in \Delta_\infty(z)$ for every *solution of the moment problem on $\mathcal{L} \cdot \mathcal{L}$*. Thus we have:

Theorem 15.1 *Let $z \in \mathbb{C}_0$. Then*

$$\{\Omega(z, \mu) : \mu \text{ solution on } \mathcal{L} \cdot \mathcal{L}\} \subset \Delta_\infty(z) \subset \{\Omega(z, \mu) : \mu \text{ solution on } \mathcal{L}\}.$$

Again since a measure is determined by its transform, we conclude:

Corollary 15.2 *The moment problem on $\mathcal{L} \cdot \mathcal{L}$ (if solvable) is determinate in the limit point case. The moment problem on $\mathcal{L}$ is indeterminate in the limit circle case.*

We note that if the sequence $\{\alpha_n\}$ consists of points which are all repeated an infinite number of times, i.e., if $\mathcal{L} = \mathcal{L} \cdot \mathcal{L}$, then the moment problem is determinate exactly in the limit point case.

General reference: Section 11.8 of [34].

16 CONCLUSION

We have presented some of the basic features of the theory of orthogonal rational functions. We refer to the reference list for detailed treatments both of the topics we have discussed here and of problem areas that we have not considered. There is much room left for studies of the case when the poles can be partly on the "support curve", partly outside it. Dewilde and Dym [36] considered a situation of this kind, and a few remarks can be found in [34]. Some applications of orthogonal rational functions in the area of signal processing and system theory are discussed in Chapter 12 of [34], in [28] and in [66]-[67].

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