Rev.Acad.Canar.Cienc., XI (Núms. 1-2), 71-81 (1999)

FURTHER RESULTS ON THE FAMILY OF GENERALIZED RADIATION INTEGRALS $I_{m,n}^{(p,q,u,\mathrm{v})}$

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Abstract

In this paper we continue the investigation of the family of generalized radiation integrals defined by

$$I_{m,n}^{(p,q,u,\mathrm{v})}\left[\begin{array}{l} a,b,c,d,\lambda \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_u),(\gamma_\mathrm{v}) \end{array}\right] = \frac{\sigma\ a^{1-m\rho}}{4\pi}\frac{\prod_{j=1}^{q}\Gamma(\beta_j)}{\Gamma(\alpha)\prod_{j=1}^{p}\Gamma(\alpha_j)} \times$$

$$\int_0^b x^\lambda (x^m+c)^\delta G_{p+1,q+1}^{n,p+1}\left(\frac{a^m}{x^m+c}\middle| \begin{array}{l} 1+\rho-(\alpha_p),1+\delta \\ \rho,1+\rho-(\beta_q) \end{array}\right) {}_uF_\mathrm{v}\left(\begin{array}{l}(\eta_u); \\ (\gamma_\mathrm{v});\end{array} \frac{x^m}{d^m}\right)dx$$

where $a,b,c,d>0; -1<\mathrm{Re}(\lambda)<m\min\mathrm{Re}(\rho,1+\rho-(\beta_j))-m\,[\max\mathrm{Re}(1-(\eta_k))-1]-m\,\mathrm{Re}(\delta)-1,\ j=1,...,n-1,\ k=1,...,u;\ m,n,p,q\in\mathbb{N};\ \gamma_i\neq 0,-1,-2,...,$ $i=1,2,...,\mathrm{v}$.

For this family of generalized radiation integrals we present several cases of reducing of the order, various recurrence relations and some relations of special type.

Key words: Radiation integrals, recurrence relations, reducing of the order.

Introduction

The response $I(a,b)$ of an omnidirectional radiation detector at a height h directly over a corner of a plane isotropic rectangular (plaque) source of length l, width w and uniform strength σ can be expressed as [3]:

$$I(a,b)=\frac{\sigma}{4\pi}\int_0^b \arctan\left(\frac{a}{\sqrt{x^2+1}}\right)\frac{dx}{\sqrt{x^2+1}}, \tag{1}$$

where $a=w/h>0,\ b=l/h>0$.

This integral was originally investigated by Hubbell et al. [3] in civil defense studies, in order to predict radiation fields due to radioactive fallout, from nuclear accidents or weapons, deposited on rectangular surfaces such as roof sections; the integral (1) has also subsequently found important applications in irradiation technology.

Various generalizations of (1) have been given by several authors, so we have: Kalla et al. [4] defined and studied the following integral involving the Gauss hypergeometric function ${}_2F_1(\alpha,\beta;\gamma;x)$:

$$H\left[\begin{array}{c} a,b,p,\lambda \\ \alpha,\beta,\gamma \end{array}\right]=\frac{\sigma a}{4\pi}\int_0^b x^{\lambda}(x^2+p)^{-\alpha}\ {}_2F_1\left(\alpha,\beta;\gamma;-\frac{a^2}{x^2+p}\right)dx \tag{2}$$

$$\mathrm{Re}(\gamma)>\mathrm{Re}(\beta)>0,\ -1<\lambda<2\alpha-1; p,a,b>0.$$

Galué [1] generalized the integral (2) by means of

$$I\left[\begin{array}{c} a,b,p,\lambda,\mu \\ \alpha,\beta,\gamma \end{array}\right] = \frac{\sigma a}{4\pi}\int_0^b x^{\lambda}(x^2+p)^{-\alpha}\left(1-\frac{x^2}{b^2}\right)^{\mu} \times \ {}_2F_1\left(\alpha,\beta;\gamma;-\frac{a^2}{x^2+p}\right)dx \tag{3}$$

where $\mathrm{Re}(\gamma)>\mathrm{Re}(\beta)>0;\ -1<\lambda<2\alpha-2\mu-1;\ \mu>-1;\ p,a,b>0;\ 0<a\leq b<\infty$.

Other generalization of (2) has been given by Saigo and Srivastava [6] in the form

$$S_m^{(p,q)}\left[\begin{array}{c} a,b,c,\lambda \\ \alpha,(\alpha_p),(\beta_q) \end{array}\right]=\frac{\sigma a}{4\pi}\int_0^b x^{\lambda}(x^m+c)^{-\alpha} \times \ {}_{p+1}F_q\left[\begin{array}{cc} \alpha,(\alpha_p); & -\dfrac{a^m}{x^m+c} \\ (\beta_q); & \end{array}\right]dx \tag{4}$$

where $\min\{a,b,c\}>0,\ \lambda\in(-1,1)$ and ${}_pF_q((\alpha_p);(\beta_q);z)$ is the generalized hypergeometric function.

Recently Galué and Prieto [2] have considered generalized radiation integrals in the following form:

$$I_{m,n}^{(p,q)}\left[\begin{array}{l} a,b,c,\lambda \\ \alpha,(\alpha_p),(\beta_q) \end{array}\right] = \frac{\sigma\, a^{1-m\rho}}{4\pi} \frac{\prod_{j=1}^{q}\Gamma(\beta_j)}{\prod_{j=1}^{p}\Gamma(\alpha_j)} \times$$

$$\int_0^b x^\lambda (x^m+c)^\delta\, G_{p+1,q+1}^{n,p+1}\left(\frac{a^m}{x^m+c}\middle|\begin{array}{l} 1+\rho-(\alpha_p),1+\delta \\ \rho,1+\rho-(\beta_q) \end{array}\right) dx \qquad (5)$$

$a,b,c>0;\ -1<\mathrm{Re}(\lambda)<m\ \min\mathrm{Re}(\rho,1+\rho-\beta_j)-m\,\mathrm{Re}(\delta)-1,\ j=1,...,n-1;$ $m,n,p,q\in\mathbb{N}$ y $G_{p,q}^{m,n}\left(z\middle|\begin{array}{l}\alpha_p\\ \beta_q\end{array}\right)$ is the Meijer G-function,

and

$$I_{m,n}^{(p,q,u,\mathrm{v})}\left[\begin{array}{l} a,b,c,d,\lambda \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_u),(\gamma_\mathrm{v}) \end{array}\right] = \frac{\sigma\, a^{1-m\rho}}{4\pi} \frac{\prod_{j=1}^{q}\Gamma(\beta_j)}{\Gamma(\alpha)\prod_{j=1}^{p}\Gamma(\alpha_j)} \times$$

$$\int_0^b x^\lambda (x^m+c)^\delta\, G_{p+1,q+1}^{n,p+1}\left(\frac{a^m}{x^m+c}\middle|\begin{array}{l} 1+\rho-(\alpha_p),1+\delta \\ \rho,1+\rho-(\beta_q) \end{array}\right)\ {}_uF_\mathrm{v}\left(\begin{array}{l}(\eta_u);\\ (\gamma_\mathrm{v});\end{array}\ \frac{x^m}{d^m}\right) dx \qquad (6)$$

$a,b,c,d>0;\ -1<\mathrm{Re}(\lambda)<m\min\mathrm{Re}(\rho,1+\rho-(\beta_j))-m[\max\mathrm{Re}(1-(\eta_k))-1]-m\,\mathrm{Re}(\delta)-1,\ j=1,...,n-1,\ k=1,...,u;\ m,n,p,q\in\mathbb{N};\ \gamma_i\neq 0,-1,-2,...,$ $i=1,2,...,\mathrm{v}$.

In this paper we continue the investigation of the family of generalized radiation integrals defined in (6), and we present several cases of reducing of the order, various recurrence relations and some relations of special type.

Cases of Reducing of the Order

In this section, using the known formulas (7.2.3.14) through (7.2.3.21) in [5] we establish many interesting results with positive integers h and i for reducing the order of the integral (6). These are listed below:

$$I_{m,n}^{(p,q,u,\mathrm{v})}\left[\begin{array}{l} a,b,c,d,\lambda \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_{u-1}),\gamma_\mathrm{v}+1,(\gamma_\mathrm{v}) \end{array}\right] = I_{m,n}^{(p,q,u-1,\mathrm{v}-1)}\left[\begin{array}{l} a,b,c,d,\lambda \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_{u-1}),(\gamma_{\mathrm{v}-1}) \end{array}\right] +$$

$$\frac{\prod_{j=1}^{u-1}\eta_j}{\prod_{j=1}^{\mathrm{v}}\gamma_j}\frac{1}{d^m} I_{m,n}^{(p,q,u-1,\mathrm{v}-1)}\left[\begin{array}{l} a,b,c,d,\lambda+m \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_{u-1})+1,(\gamma_{\mathrm{v}-1})+1 \end{array}\right] \qquad (7)$$

$$I_{m,n}^{(p,q,u,\mathrm{v})}\left[\begin{array}{l} a,b,c,d,\lambda \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_{u-1}),\gamma_{\mathrm{v}}+h,(\gamma_{\mathrm{v}}) \end{array}\right]=\sum_{r=0}^{h}\binom{h}{r}\frac{\prod_{j=1}^{u-1}(\eta_j)_r}{\prod_{j=1}^{\mathrm{v}}(\gamma_j)_r}\times$$

$$I_{m,n}^{(p,q,u-1,\mathrm{v}-1)}\left[\begin{array}{l} a,b,c,d,\lambda \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_{u-1})+r,(\gamma_{\mathrm{v}-1})+r \end{array}\right]. \quad (8)$$

$$I_{m,n}^{(p,q,u,\mathrm{v})}\left[\begin{array}{l} a,b,c,d,\lambda \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_{u-2}),\gamma_{\mathrm{v}-1}+h,\gamma_{\mathrm{v}}+i,(\gamma_{\mathrm{v}}) \end{array}\right]=$$

$$\sum_{r=0}^{h}\sum_{s=0}^{i}\binom{h}{r}\binom{i}{s}(\gamma_{\mathrm{v}}+i)_r\,(\gamma_{\mathrm{v}-1}+r)_s\frac{\prod_{j=1}^{u-2}(\eta_j)_{r+s}}{\prod_{j=1}^{\mathrm{v}}(\gamma_j)_{r+s}}\times$$

$$I_{m,n}^{(p,q,u-2,\mathrm{v}-2)}\left[\begin{array}{l} a,b,c,d,\lambda \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_{u-2})+r+s,(\gamma_{\mathrm{v}-2})+r+s \end{array}\right] \quad (9)$$

where we use the result [7, pg. 17, No. (12) and No. (9)]

$$(-n)_k=\begin{cases} \frac{(-1)^k n!}{(n-k)!} & 0\le k\le n \\ 0 & k>n \end{cases} \quad (10)$$

$$(\lambda)_{m+n}=(\lambda)_m(\lambda+m)_n. \quad (11)$$

$$I_{m,n}^{(p,q,u,\mathrm{v})}\left[\begin{array}{l} a,b,c,d,\lambda \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_{u-1}),1,(\gamma_{\mathrm{v}-1}),2 \end{array}\right]=\frac{\prod_{j=1}^{\mathrm{v}-1}(\gamma_j-1)\ d^m}{\prod_{j=1}^{u-1}(\eta_j-1)}\times$$

$$\left\{I_{m,n}^{(p,q,u-1,\mathrm{v}-1)}\left[\begin{array}{l} a,b,c,d,\lambda-m \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_{u-1})-1,(\gamma_{\mathrm{v}-1})-1 \end{array}\right]-I_{m,n}^{(p,q)}\left[\begin{array}{l} a,b,c,\lambda-m \\ \alpha,(\alpha_p),(\beta_q) \end{array}\right]\right\}, \quad (12)$$

with $\eta_1,...,\eta_{u-1}\neq 1$ and $I_{m,n}^{(p,q)}\left[\begin{array}{l} a,b,c,\lambda \\ \alpha,(\alpha_p),(\beta_q) \end{array}\right]$ as defined in (5).

$$I_{m,n}^{(p,q,u,\mathrm{v})}\left[\begin{array}{l} a,b,c,d,\lambda \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_{u-1}),1,(\gamma_{\mathrm{v}-1}),3 \end{array}\right] = 2\,\frac{\prod_{j=1}^{\mathrm{v}-1}(\gamma_j-2)(\gamma_j-1)\; d^{2m}}{\prod_{j=1}^{u-1}(\eta_j-2)\;(\eta_j-1)} \quad\times$$

$$\left\{ I_{m,n}^{(p,q,u-1,\mathrm{v}-1)}\left[\begin{array}{l} a,b,c,d,\lambda-2m \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_{u-1})-2,(\gamma_{\mathrm{v}-1})-2 \end{array}\right] - \frac{\prod_{j=1}^{u-1}(\eta_j-2)}{\prod_{j=1}^{\mathrm{v}-1}(\gamma_j-2)}\frac{1}{d^m} \quad\times\right.$$

$$\left. I_{m,n}^{(p,q)}\left[\begin{array}{l} a,b,c,\lambda+m \\ \alpha,(\alpha_p),(\beta_q) \end{array}\right] - I_{m,n}^{(p,q)}\left[\begin{array}{l} a,b,c,\lambda \\ \alpha,(\alpha_p),(\beta_q) \end{array}\right]\right\}. \tag{13}$$

$$\eta_1,....,\eta_{u-1}\neq 1,2;\;\; \gamma_1,....,\gamma_{\mathrm{v}-1}\neq 2$$

$$I_{m,n}^{(p,q,u,\mathrm{v})}\left[\begin{array}{l} a,b,c,d,\lambda \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_{u-1}),1,(\gamma_{\mathrm{v}-1}),h+1 \end{array}\right] = (-1)^{u-\mathrm{v}}\frac{\prod_{j=1}^{\mathrm{v}-1}(1-\gamma_j)_h}{\prod_{j=1}^{u-1}(1-\eta_j)_h} \quad\times$$

$$I_{m,n}^{(p,q,u-1,\mathrm{v}-1)}\left[\begin{array}{l} a,b,c,d,\lambda \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_{u-1})-h,(\gamma_{\mathrm{v}-1})-h \end{array}\right] -$$

$$\sum_{s=1}^{h}\frac{\prod_{j=1}^{\mathrm{v}-1}(1-\gamma_j)_s}{\prod_{j=1}^{u-1}(1-\eta_j)_s}\,\frac{(-1)^{(u-\mathrm{v})(1+h-s)}}{(h-s)!\;d^{m(h-s)}}\,I_{m,n}^{(p,q)}\left[\begin{array}{l} a,b,c,\lambda+m(h-s) \\ \alpha,(\alpha_p),(\beta_q) \end{array}\right], \tag{14}$$

where we have used the result [7, pg. 17, No. (10)]

$$(\lambda)_{n-k} = \frac{(-1)^k(\lambda)_n}{(1-\lambda-n)_k} \qquad 0\le k\le n. \tag{15}$$

$$I_{m,n}^{(p,q,u,\mathrm{v})}\left[\begin{array}{l} a,b,c,d,\lambda \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_u),(\gamma_{\mathrm{v}-2}),\eta_{u-1}+1,\eta_u+1 \end{array}\right] = \frac{1}{(\eta_u-\eta_{u-1})} \quad\times$$

$$\left\{\eta_u \, I_{m,n}^{(p,q,u-1,\mathrm{v}-1)} \left[\begin{array}{l} a, b, c, d, \lambda \\ \alpha, (\alpha_p), (\beta_q) \\ (\eta_{u-1}), (\gamma_{\mathrm{v}-2}), \eta_{u-1} + 1 \end{array} \right] - \right.$$

$$\left. \eta_{u-1} \, I_{m,n}^{(p,q,u-1,\mathrm{v}-1)} \left[\begin{array}{l} a, b, c, d, \lambda \\ \alpha, (\alpha_p), (\beta_q) \\ (\eta_{u-2}), \eta_u, (\gamma_{\mathrm{v}-2}), \eta_u + 1 \end{array} \right] \right\}, \eta_{u-1} \neq \eta_u. \qquad (16)$$

$$I_{m,n}^{(p,q,u,\mathrm{v})} \left[\begin{array}{l} a, b, c, d, \lambda \\ \alpha, (\alpha_p), (\beta_q) \\ (\eta_u), (\gamma_{\mathrm{v}-2}), \eta_{u-1} + h, \eta_u + 1 \end{array} \right] = \frac{(\eta_{u-1})_h}{(\eta_{u-1} - \eta_u)_h} \times$$

$$I_{m,n}^{(p,q,u-1,\mathrm{v}-1)} \left[\begin{array}{l} a, b, c, d, \lambda \\ \alpha, (\alpha_p), (\beta_q) \\ (\eta_{u-2}), \eta_u, (\gamma_{\mathrm{v}-2}), \eta_u + 1 \end{array} \right] - \frac{(\eta_{u-1})_h \eta_u}{(\eta_{u-1} - \eta_u)_h} \times$$

$$\sum_{i=1}^{h} \frac{(\eta_{u-1} - \eta_u - 1)_i}{(\eta_{u-1})_i} \, I_{m,n}^{(p,q,u-1,\mathrm{v}-1)} \left[\begin{array}{l} a, b, c, d, \lambda \\ \alpha, (\alpha_p), (\beta_q) \\ (\eta_{u-1}), (\gamma_{\mathrm{v}-2}), \eta_{u-1} + i \end{array} \right] \qquad (17)$$

$$\eta_{u-1} \neq \eta_u.$$

Recurrence Relations

In this section we establish various recurrence relations for $I_{m,n}^{(p,q,u,\mathrm{v})}[\cdots]$, where the upper parameters are different from the corresponding lower parameters by an integer. The results presented are based on the definition (6) and formulas (7.2.3.25) through (7.2.3.36) in [5]. We list the following interesting results without going into details.

$$I_{m,n}^{(p,q,u,\mathrm{v})} \left[\begin{array}{l} a, b, c, d, \lambda \\ \alpha, (\alpha_p), (\beta_q) \\ (\eta_u), (\gamma_\mathrm{v}) \end{array} \right] = \frac{\eta_u}{(\eta_u - \eta_{u-1})} \, I_{m,n}^{(p,q,u,\mathrm{v})} \left[\begin{array}{l} a, b, c, d, \lambda \\ \alpha, (\alpha_p), (\beta_q) \\ (\eta_{u-1}), \eta_u + 1, (\gamma_\mathrm{v}) \end{array} \right] -$$

$$\frac{\eta_{u-1}}{(\eta_u - \eta_{u-1})} \, I_{m,n}^{(p,q,u,\mathrm{v})} \left[\begin{array}{l} a, b, c, d, \lambda \\ \alpha, (\alpha_p), (\beta_q) \\ (\eta_{u-2}), \eta_{u-1} + 1, \eta_u, (\gamma_\mathrm{v}) \end{array} \right] \qquad (18)$$

$$I_{m,n}^{(p,q,u,\mathrm{v})} \left[\begin{array}{l} a, b, c, d, \lambda \\ \alpha, (\alpha_p), (\beta_q) \\ (\eta_u), (\gamma_\mathrm{v}) \end{array} \right] = \frac{\gamma_\mathrm{v} - 1}{(\gamma_\mathrm{v} - \eta_u - 1)} \, I_{m,n}^{(p,q,u,\mathrm{v})} \left[\begin{array}{l} a, b, c, d, \lambda \\ \alpha, (\alpha_p), (\beta_q) \\ (\eta_u), (\gamma_{\mathrm{v}-1}), \gamma_\mathrm{v} - 1 \end{array} \right] -$$

$$\frac{\eta_u}{(\gamma_v - \eta_u - 1)} \; I_{m,n}^{(p,q,u,v)} \left[\begin{array}{l} a, b, c, d, \lambda \\ \alpha, (\alpha_p), (\beta_q) \\ (\eta_{u-1}), \eta_u + 1, (\gamma_v) \end{array} \right]. \tag{19}$$

$$I_{m,n}^{(p,q,u,v)} \left[\begin{array}{l} a, b, c, d, \lambda \\ \alpha, (\alpha_p), (\beta_q) \\ (\eta_u), (\gamma_v) \end{array} \right] = \frac{\gamma_v - 1}{(\gamma_v - \gamma_{v-1})} \; I_{m,n}^{(p,q,u,v)} \left[\begin{array}{l} a, b, c, d, \lambda \\ \alpha, (\alpha_p), (\beta_q) \\ (\eta_u), (\gamma_{v-1}), \gamma_v - 1 \end{array} \right] -$$

$$\frac{\gamma_{v-1} - 1}{(\gamma_v - \gamma_{v-1})} \; I_{m,n}^{(p,q,u,v)} \left[\begin{array}{l} a, b, c, d, \lambda \\ \alpha, (\alpha_p), (\beta_q) \\ (\eta_u), (\gamma_{v-2}), \gamma_{v-1} - 1, \gamma_v \end{array} \right]. \tag{20}$$

$$I_{m,n}^{(p,q,u,v)} \left[\begin{array}{l} a, b, c, d, \lambda \\ \alpha, (\alpha_p), (\beta_q) \\ (\eta_u), (\gamma_v) \end{array} \right] = \frac{\eta_{u-1}(\gamma_v - \eta_u)}{\gamma_v(\eta_{u-1} - \eta_u)} \quad \times$$

$$I_{m,n}^{(p,q,u,v)} \left[\begin{array}{l} a, b, c, d, \lambda \\ \alpha, (\alpha_p), (\beta_q) \\ (\eta_{u-2}), \eta_{u-1} + 1, \eta_u, (\gamma_{v-1}), \gamma_v + 1 \end{array} \right] -$$

$$\frac{\eta_u(\gamma_v - \eta_{u-1})}{\gamma_v(\eta_{u-1} - \eta_u)} \; I_{m,n}^{(p,q,u,v)} \left[\begin{array}{l} a, b, c, d, \lambda \\ \alpha, (\alpha_p), (\beta_q) \\ (\eta_{u-1}), \eta_u + 1, (\gamma_{v-1}), \gamma_v + 1 \end{array} \right]. \tag{21}$$

$$I_{m,n}^{(p,q,u,v)} \left[\begin{array}{l} a, b, c, d, \lambda \\ \alpha, (\alpha_p), (\beta_q) \\ (\eta_u), (\gamma_v) \end{array} \right] = \frac{(\gamma_v - 1)(\gamma_{v-1} - \eta_u)}{\gamma_{v-1}(\gamma_v - \eta_u - 1)} \quad \times$$

$$I_{m,n}^{(p,q,u,v)} \left[\begin{array}{l} a, b, c, d, \lambda \\ \alpha, (\alpha_p), (\beta_q) \\ (\eta_u), (\gamma_{v-2}), \gamma_{v-1} + 1, \gamma_v - 1 \end{array} \right] -$$

$$\frac{\eta_u(\gamma_{v-1} - \gamma_v + 1)}{\gamma_{v-1}(\gamma_v - \eta_u - 1)} \; I_{m,n}^{(p,q,u,v)} \left[\begin{array}{l} a, b, c, d, \lambda \\ \alpha, (\alpha_p), (\beta_q) \\ (\eta_{u-1}), \eta_u + 1, (\gamma_{v-2}), \gamma_{v-1} + 1, \gamma_v \end{array} \right]. \tag{22}$$

$$I_{m,n}^{(p,q,u,v)} \left[\begin{array}{l} a, b, c, d, \lambda \\ \alpha, (\alpha_p), (\beta_q) \\ (\eta_u), (\gamma_v) \end{array} \right] = I_{m,n}^{(p,q,u,v)} \left[\begin{array}{l} a, b, c, d, \lambda \\ \alpha, (\alpha_p), (\beta_q) \\ (\eta_{u-1}), \eta_u + 1, (\gamma_v) \end{array} \right] -$$

$$\frac{\prod_{j=1}^{u-1} (\eta_j)}{\prod_{j=1}^{v} (\gamma_j)} \frac{1}{d^m} I_{m,n}^{(p,q,u,v)} \left[\begin{array}{l} a, b, c, d, \lambda + m \\ \alpha, (\alpha_p), (\beta_q) \\ (\eta_u) + 1, (\gamma_v) + 1 \end{array} \right]. \tag{23}$$

$$I_{m,n}^{(p,q,u,\mathrm{v})}\left[\begin{array}{l} a,b,c,d,\lambda \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_u),(\gamma_\mathrm{v}) \end{array}\right] = I_{m,n}^{(p,q,u,\mathrm{v})}\left[\begin{array}{l} a,b,c,d,\lambda \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_u),(\gamma_{\mathrm{v}-1}),\gamma_\mathrm{v}+1 \end{array}\right] +$$

$$\frac{\prod_{j=1}^{u}(\eta_j)}{\prod_{j=1}^{\mathrm{v}-1}(\gamma_j)} \frac{1}{\gamma_\mathrm{v}(\gamma_\mathrm{v}+1)d^m} I_{m,n}^{(p,q,u,\mathrm{v})}\left[\begin{array}{l} a,b,c,d,\lambda+m \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_u)+1,(\gamma_{\mathrm{v}-1})+1,\gamma_\mathrm{v}+2 \end{array}\right]. \qquad (24)$$

$$I_{m,n}^{(p,q,u,\mathrm{v})}\left[\begin{array}{l} a,b,c,d,\lambda \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_u),(\gamma_\mathrm{v}) \end{array}\right] = I_{m,n}^{(p,q,u,\mathrm{v})}\left[\begin{array}{l} a,b,c,d,\lambda \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_{u-2}),\eta_{u-1}+1,\eta_u-1,(\gamma_\mathrm{v}) \end{array}\right] -$$

$$\frac{(\eta_u-\eta_{u-1}-1)}{d^m} \frac{\prod_{j=1}^{u-2}(\eta_j)}{\prod_{j=1}^{\mathrm{v}}(\gamma_j)} I_{m,n}^{(p,q,u,\mathrm{v})}\left[\begin{array}{l} a,b,c,d,\lambda+m \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_{u-1})+1,\eta_u,(\gamma_\mathrm{v})+1 \end{array}\right]. \qquad (25)$$

$$I_{m,n}^{(p,q,u,\mathrm{v})}\left[\begin{array}{l} a,b,c,d,\lambda \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_u),(\gamma_\mathrm{v}) \end{array}\right] = I_{m,n}^{(p,q,u,\mathrm{v})}\left[\begin{array}{l} a,b,c,d,\lambda \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_{u-1}),\eta_u+1,(\gamma_{\mathrm{v}-1}),\gamma_\mathrm{v}+1 \end{array}\right] -$$

$$\frac{\prod_{j=1}^{u}(\eta_j)}{\prod_{j=1}^{\mathrm{v}-1}(\gamma_j)} \frac{(\gamma_\mathrm{v}-\eta_u)}{\gamma_\mathrm{v}(\gamma_\mathrm{v}+1)d^m} I_{m,n}^{(p,q,u,\mathrm{v})}\left[\begin{array}{l} a,b,c,d,\lambda+m \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_u)+1,(\gamma_{\mathrm{v}-1})+1,\gamma_\mathrm{v}+2 \end{array}\right]. \qquad (26)$$

$$I_{m,n}^{(p,q,u,\mathrm{v})}\left[\begin{array}{l} a,b,c,d,\lambda \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_u),(\gamma_\mathrm{v}) \end{array}\right] = \frac{\eta_{u-1}}{\gamma_\mathrm{v}} I_{m,n}^{(p,q,u,\mathrm{v})}\left[\begin{array}{l} a,b,c,d,\lambda \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_{u-2}),\eta_{u-1}+1,\eta_u+1,(\gamma_{\mathrm{v}-1}),\gamma_\mathrm{v}+1 \end{array}\right] +$$

$$\frac{(\gamma_\mathrm{v}-\eta_{u-1})}{\gamma_\mathrm{v}} I_{m,n}^{(p,q,u,\mathrm{v})}\left[\begin{array}{l} a,b,c,d,\lambda \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_{u-1}),\eta_u+1,(\gamma_{\mathrm{v}-1}),\gamma_\mathrm{v}+1 \end{array}\right] -$$

$$\frac{\prod_{j=1}^{u-2}(\eta_j)}{\prod_{j=1}^{\mathrm{v}}(\gamma_j)} \frac{\eta_{u-1}}{d^m} I_{m,n}^{(p,q,u,\mathrm{v})}\left[\begin{array}{l} a,b,c,d,\lambda+m \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_u)+1,(\gamma_\mathrm{v})+1 \end{array}\right]. \qquad (27)$$

$$I_{m,n}^{(p,q,u,\text{v})}\left[\begin{array}{l} a,b,c,d,\lambda \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_u),(\gamma_\text{v}) \end{array}\right] = \frac{(\gamma_{\text{v}-2}-1)(\gamma_{\text{v}-1}-1)(\eta_{u-1}-\gamma_\text{v})(\eta_u-\gamma_\text{v})}{(\eta_{u-1}-1)(\eta_u-1)(\gamma_{\text{v}-2}-\gamma_\text{v})(\gamma_{\text{v}-1}-\gamma_\text{v})} \times$$

$$I_{m,n}^{(p,q,u,\text{v})}\left[\begin{array}{l} a,b,c,d,\lambda \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_{u-2}),\eta_{u-1}-1,\eta_u-1,(\gamma_{\text{v}-3}),\gamma_{\text{v}-2}-1,\gamma_{\text{v}-1}-1,\gamma_\text{v} \end{array}\right] +$$

$$\frac{(\gamma_{\text{v}-2}-1)(\gamma_\text{v}-1)(\eta_{u-1}-\gamma_{\text{v}-1})(\eta_u-\gamma_{\text{v}-1})}{(\eta_{u-1}-1)(\eta_u-1)(\gamma_{\text{v}-2}-\gamma_{\text{v}-1})(\gamma_\text{v}-\gamma_{\text{v}-1})} \times$$

$$I_{m,n}^{(p,q,u,\text{v})}\left[\begin{array}{l} a,b,c,d,\lambda \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_{u-2}),\eta_{u-1}-1,\eta_u-1,(\gamma_{\text{v}-3}),\gamma_{\text{v}-2}-1,\gamma_{\text{v}-1},\gamma_\text{v}-1 \end{array}\right] +$$

$$\frac{(\gamma_{\text{v}-1}-1)(\gamma_\text{v}-1)(\eta_{u-1}-\gamma_{\text{v}-2})(\eta_u-\gamma_{\text{v}-2})}{(\eta_{u-1}-1)(\eta_u-1)(\gamma_{\text{v}-1}-\gamma_{\text{v}-2})(\gamma_\text{v}-\gamma_{\text{v}-2})} \times$$

$$I_{m,n}^{(p,q,u,\text{v})}\left[\begin{array}{l} a,b,c,d,\lambda \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_{u-2}),\eta_{u-1}-1,\eta_u-1,(\gamma_{\text{v}-2}),\gamma_{\text{v}-1}-1,\gamma_\text{v}-1 \end{array}\right]. \tag{28}$$

$$I_{m,n}^{(p,q,u,\text{v})}\left[\begin{array}{l} a,b,c,d,\lambda \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_u),(\gamma_\text{v}) \end{array}\right] = \frac{\eta_{u-2}\ \eta_{u-1}(\gamma_{\text{v}-1}-\eta_u)(\gamma_\text{v}-\eta_u)}{\gamma_{\text{v}-1}\gamma_\text{v}(\eta_{u-2}-\eta_u)(\eta_{u-1}-\eta_u)} \times$$

$$I_{m,n}^{(p,q,u,\text{v})}\left[\begin{array}{l} a,b,c,d,\lambda \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_{u-3}),\eta_{u-2}+1,\eta_{u-1}+1,\eta_u,(\gamma_{\text{v}-2}),\gamma_{\text{v}-1}+1,\gamma_\text{v}+1 \end{array}\right] +$$

$$\frac{\eta_{u-2}\ \eta_u(\gamma_{\text{v}-1}-\eta_{u-1})(\gamma_\text{v}-\eta_{u-1})}{\gamma_{\text{v}-1}\gamma_\text{v}(\eta_{u-2}-\eta_{u-1})(\eta_u-\eta_{u-1})} \times$$

$$I_{m,n}^{(p,q,u,\text{v})}\left[\begin{array}{l} a,b,c,d,\lambda \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_{u-3}),\eta_{u-2}+1,\eta_{u-1},\eta_u+1,(\gamma_{\text{v}-2}),\gamma_{\text{v}-1}+1,\gamma_\text{v}+1 \end{array}\right] +$$

$$\frac{\eta_{u-1}\ \eta_u(\gamma_{\text{v}-1}-\eta_{u-2})(\gamma_\text{v}-\eta_{u-2})}{\gamma_{\text{v}-1}\gamma_\text{v}(\eta_{u-1}-\eta_{u-2})(\eta_u-\eta_{u-2})} \times$$

$$I_{m,n}^{(p,q,u,\text{v})}\left[\begin{array}{l} a,b,c,d,\lambda \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_{u-2}),\eta_{u-1}+1,\eta_u+1,(\gamma_{\text{v}-2}),\gamma_{\text{v}-1}+1,\gamma_\text{v}+1 \end{array}\right]. \tag{29}$$

Relations of Special Type

We list here four more recurrence relations for $I_{m,n}^{(p,q,u,\text{v})}[\cdots]$. From the definition (6) and formulas (7.2.3.38) to (7.2.3.41) in [5], it is easy to establish the following relations that are independient of any multipliers.

$$I_{m,n}^{(p,q,u,\text{v})}\begin{bmatrix} a,b,c,d,\lambda \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_u),(\gamma_{\text{v}-1}),1-\gamma_{\text{v}-1} \end{bmatrix} + I_{m,n}^{(p,q,u,\text{v})}\begin{bmatrix} a,b,c,d,\lambda \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_u),(\gamma_{\text{v}-2}),-\gamma_{\text{v}-1},\gamma_{\text{v}-1}+1 \end{bmatrix}$$

$$= 2\ I_{m,n}^{(p,q,u,\text{v})}\begin{bmatrix} a,b,c,d,\lambda \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_u),(\gamma_{\text{v}-2}),1-\gamma_{\text{v}-1},1+\gamma_{\text{v}-1} \end{bmatrix}. \tag{30}$$

$$I_{m,n}^{(p,q,u,\text{v})}\begin{bmatrix} a,b,c,d,\lambda \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_u),(\gamma_{\text{v}-2}),-\eta_u,1+\eta_u \end{bmatrix} =$$

$$2\ I_{m,n}^{(p,q,u,\text{v})}\begin{bmatrix} a,b,c,d,\lambda \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_u),(\gamma_{\text{v}-2}),1-\eta_u,1+\eta_u \end{bmatrix} -$$

$$I_{m,n}^{(p,q,u-1,\text{v}-1)}\begin{bmatrix} a,b,c,d,\lambda \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_{u-1}),(\gamma_{\text{v}-2}),1-\eta_u \end{bmatrix}. \tag{31}$$

$$I_{m,n}^{(p,q,u,\text{v})}\begin{bmatrix} a,b,c,d,\lambda \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_u),(\gamma_{\text{v}-1}),1+\eta_u \end{bmatrix} +$$

$$I_{m,n}^{(p,q,u,\text{v})}\begin{bmatrix} a,b,c,d,\lambda \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_{u-1}),-\eta_u,(\gamma_{\text{v}-1}),1-\eta_u \end{bmatrix} =$$

$$2\ I_{m,n}^{(p,q,u+1,\text{v}+1)}\begin{bmatrix} a,b,c,d,\lambda \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_{u-1}),-\eta_u,\eta_u,(\gamma_{\text{v}-1}),1-\eta_u,1+\eta_u \end{bmatrix}. \tag{32}$$

$$I_{m,n}^{(p,q,u,\text{v})}\begin{bmatrix} a,b,c,d,\lambda \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_{u-1}),1-\eta_{u-1},(\gamma_{\text{v}}) \end{bmatrix} + I_{m,n}^{(p,q,u,\text{v})}\begin{bmatrix} a,b,c,d,\lambda \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_{u-2}),-\eta_{u-1},1+\eta_{u-1},(\gamma_{\text{v}}) \end{bmatrix}$$

$$= 2\ I_{m,n}^{(p,q,u,\text{v})}\begin{bmatrix} a,b,c,d,\lambda \\ \alpha,(\alpha_p),(\beta_q) \\ (\eta_{u-2}),-\eta_{u-1},\eta_{u-1},(\gamma_{\text{v}}) \end{bmatrix}. \tag{33}$$

Acknowledgement

The author would like to thank CONDES - Universidad del Zulia for financial support.

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