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p-CLOSEDNESS IN TOPOLOGICAL SPACES

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ABSTRACT The present paper deals with a kind of covering property for topological spaces, called p-closedness, first initiated by Abu-Khadra [4], followed by a recent study of the same by Dontchev *et al.* [3]. We derive here a number of characterizations and certain relevant properties of such a concept mainly via certain newly introduced notions like p(θ)-continuity, p(θ)-subclosedness and strong p(θ)-closedness of graphs of functions.

KEYWORDS Preopen set, preclosure, p-closed space, p(θ)-continuity, p(θ)-subclosed graph, strongly p(θ)-closed graph.

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§ 1. INTRODUCTION

Compactness and its different allied forms, specially quasi H-closedness, have so far been studied in detail by many topologists. It is seen from literature that certain open-like sets, e.g. semiopen sets [5], regular open sets, have been employed for the above investgations. The notion of preopen sets was initiated by Mashhour *et al.* [7] and such sets along with some other relevant concepts have been investigated by many. Recently, Dontchev *et al.* [3] have taken up an investigation of a sort of covering property, called p-closedness for topological spaces with the help of the notions of preopen sets and some associated ones, the originator of the notion of p-closedness being Abo-Khadra [4] in 1989. We propose to undertake, in this paper, a further study of the same.

In [3], certain characterizations of p-closedness for topological spaces and their subsets have been found mainly in terms of filterbases. We shall add a few more to this list of formulations, in section 2.

In section 3, we define a type of functions called p(θ)-continuous ones, and introduce the notion of p(θ)-subclosedness of graphs of functions. These ideas are exploited to ultimately achieve a few characterizations of p-closed topological spaces.

In section 4, we try to obtain formulations of p-closed spaces by means of strong p(θ)-closedness of graphs of functions, a concept suitably defined in the same section.

In what follows, by spaces X and Y we mean topological spaces. For any subset A of a space X, we shall use the notations clA and intA to denote the closure and interior of A in X respectively. The word 'neighbourhood' will be abbreviated as 'nbd'.

§ 2. p-CLOSEDNESS, IN GENERAL

Let us, at the very outset, clarify certain key words and notations that are often taken resort to throughout the paper. These known terminologies and other pertinent details can be found in literature (e. g. see [2]).

DEFINITION 2.1 : A subset A of a topological space X is called ***preopen*** if $A \subseteq \text{intcl}A$; the complements of such sets in X are known as ***preclosed sets*** i.e., for a preclosed set A, $\text{clint}A \subseteq A$. The collection of all preopen sets in a space X will be denoted by **PO(X)**.

DEFINITION 2.2 : For any space X and any $A \subseteq X$, the union (intersection) of all preopen (preclosed) sets in X, each contained in (containing) A is called the ***preinterior*** (resp. ***preclosure***) of A in X, to be denoted by **pintA** (resp. **pclA**).

Since arbitrary union (intersection) of preopen (preclosed) sets in X is known to be preopen (preclosed), the preinterior (preclosure) of a set A in a space X can equivalently be defined as the largest (smallest) preopen (preclosed) set contained in A (containing A). The following results are also well-known.

THEOREM 2.3 : Let A be a subset of a space X. Then,

(a) pclA consists precisely of those points x of X such that $U \cap A \neq \phi$, for every preopen set U containing x ;

(b) $\text{pcl}(X \setminus A) = X \setminus \text{pint}A$.

We now append the definition of p-closedness, as introduced originally in [4].

DEFINITION 2.4 : A non-void subset A of a topological space X is said to be ***p-closed relative to X*** if for every cover $\{U_\alpha : \alpha \in \Lambda\}$ (here and hereafter Λ denotes an indexing set) of

A by preopen sets of X, there exists a finite subset Λ_0 of Λ such that $A \subseteq \cup\{\text{pcl}U_\alpha : \alpha \in \Lambda_0\}$. If, in addition, A = X, then X is called a ***p-closed space***.

THEOREM 2.5 : Suppose A and Y are subsets of a topological space X such that $A \subseteq Y \subseteq X$ and Y is open in X. Then A is p-closed relative to X if and only if A is p-closed relative to the subspace Y.

PROOF : First suppose that A is p-closed relative to Y, and let $\{V_\alpha : \alpha \in \Lambda\}$ be a cover of A by preopen sets of X. As $V_\alpha \in PO(X)$ and Y is open in X, one can check that $V_\alpha \cap Y \in PO(Y)$. Hence, $\{ V_\alpha \cap Y : \alpha \in \Lambda\}$ is a cover of A by preopen sets of Y. By p-closedness of A relative to Y, there exists some finite subset Λ_0 of Λ such that $A \subseteq \cup\{\text{pcl}_Y (V_\alpha \cap Y): \alpha \in \Lambda_0\}$ (here and afterwards also, the usual notation $\text{pcl}_Y B$ stands for the preclosure of a subset B of Y in the subspace Y of X). It is then easily seen that $A \subseteq \cup\{\text{pcl}V_\alpha : \alpha \in \Lambda_0\}$, proving that A is p-closed relative to X.

Conversely, let A be p-closed relative to X and $\{ V_\alpha : \alpha \in \Lambda\}$ be a cover of A by preopen sets of Y. As Y is open, and $V_\alpha \in PO(Y)$ for each $\alpha \in \Lambda$, we have $V_\alpha \in PO(X)$. Thus, $\{V_\alpha : \alpha \in \Lambda\}$ is a cover of A by preopen sets of X. By hypothesis, there is a finite subset Λ_0 of Λ such that $A \subseteq \cup\{\text{pcl}V_\alpha : \alpha \in \Lambda_0\}$, i.e., $A \cap Y = A \subseteq \cup\{(\text{pcl}V_\alpha) \cap Y: \alpha \in \Lambda_0\}$. It can be verified that $\text{pcl}V_\alpha \cap Y \subseteq \text{pcl}_Y V_\alpha$. Thus, $A \subseteq \cup\{\text{pcl}_Y (V_\alpha) : \alpha \in \Lambda_0\}$, and A becomes p-closed relative to Y.

COROLLARY 2.6 : An open subset A of a space X is p-closed iff it is p-closed relative to X.

THEOREM 2.7 : (a) Union of finite number of sets in a space X, each of which is p-closed relative to X, is p-closed relative to X.

(b) A subset A in a p-closed space X is a p-closed relative to X if A is preopen as well as preclosed.

PROOF : The straightforward proofs are omitted.

It is well-known that barring paracompactness the best known weaker form of compactness is quasi H-closedness, a few other widely studied compact-like covering properties being S-closedness [9], s-closedness [6] and near compactness [8]. A topological space X is called quasi H-closed [1] if every open cover $\mathcal{U}$ of X has a finite subfamily $\mathcal{U}_0$, the union of the closures of whose members is X. As to the relation of p-closedness of a space with the above standard covering properties, it has been shown by Dontchev *et al.* [3] that p-closedness of a space X is independent of each of compactness, near compactness, s-closedness and S-closedness of X. Now, since every open set is preopen and for any open set A in X, pclA = clA, it immediately follows that

THEOREM 2.8 : Every p-closed space is quasi H-closed.

The above result and also the fact that the converse of it is false, have also been observed in [3]. We give here another and a rather simple example of a compact space which is not p-closed.

EXAMPLE 2.9 : Let X denote the set of real numbers endowed with the cofinite topology. The space is clearly compact. Now, for any subset A of X, there are three possibilities as follows : (i) A is finite, (ii) A is infinite with finite complement, (iii) A is infinite with $X \setminus A$ infinite.

For case (i), intclA = intA = ϕ, so that A is not preopen. In case (ii), A is open and hence preopen. For case (iii), we have $A \subseteq X$ = intclA and in this case A is preopen. Let $U_i = (X \setminus N) \cup \{i\}$, for any $i \in N$ (the set of natural numbers). Thus $\mathcal{U} = \{U_i : i \in N\} \cup \{(X \setminus N)\}$ is a cover of X by preopen sets of X. We observe that $X \setminus N$ and $X \setminus U_i$ ($i \in N$) are also preopen (being sets of type (iii)). Hence there cannot exist any finite subfamily of $\mathcal{U}$, the preclosure of whose members may cover X. Thus, X is not p-closed.

Similar to the definition of θ-adherence and θ-convergence of filterbases, the ideas of pre-θ-adherence and pre-θ-convergence of filterbases were introduced in [2]. We shall recall them below, and include with them the corresponding definitions for nets.

DEFINITION 2.10 : Let X be a topological space, $A \subseteq X$ and $x \in X$.

(a) [3] A ***filterbase*** $\mathcal{F}$ on A is said to

(i) ***pre-θ-adhere*** at x (to be written as $x \in p(\theta)\text{-ad}\mathcal{F}$), if for each preopen set U containing x and each $F \in \mathcal{F}$, $F \cap \text{pcl}U \neq \phi$,

(ii) ***pre-θ-converge*** to x (to be denoted by $\mathcal{F} \xrightarrow{p(\theta)} x$) if for each preopen set U containing x, there exists $F \in \mathcal{F}$ such that $F \subseteq \text{pcl}U$.

(b) A net $\{x_\alpha : \alpha \in (D, \geq)\}$(where $(D,\geq)$ is directed set) in A is said to

(i) ***pre-θ-adhere*** at x (written as $x \in p(\theta)\text{-ad}(x_\alpha)$), if for each preopen set U containing x and each $\alpha \in D$, there exists a $\beta \in D$ with $\beta \geq \alpha$ such that $x_\beta \in$ pclU;

(ii) ***pre-θ-converge*** to x (in notation, $x_\alpha \xrightarrow{p(\theta)} x$) if the net is eventually in pclU, for each preopen set U containing x. If a net $\{x_\alpha\}$ pre-θ-adheres at some $x \in X$ (***pre-θ-converges*** to some $x \in X$), we shall sometimes write that x is a ***pre-θ-adherent point of*** the net (resp. the net is ***pre-θ-convergent to x***). Similar terminologies apply to filterbases also.

DEFINITION 2.11 : For a subset A of a space X and a point x of X, we say that x is in the ***pre-θ-closure*** of A and write $x \in p(\theta)\text{-cl}A$, if for every preopen set U containing x, pclU $\cap A \neq \phi$.

The rest of this section is forwarded to the characterizations of p-closedness in terms of the concepts detailed so far. The first theorem in this endeavour follows next, giving a long list of formulations of p-closedness of subsets relative to a space, the first three being already known from Dontchev *et al.* [3].

THEOREM 2.12 : For any non-void subset A of a space X, the following statements are equivalent :

(a) A is p-closed relative to X.

(b) Every maximal filterbase on X which meets A, pre-θ-converges to some points of A.

(c) Every filterbase on X which meets A, pre-θ-adheres at some point of A.

(d) For every family $\{U_\alpha : \alpha \in \Lambda\}$ of non-void preclosed sets with $\left(\bigcap_{\alpha\in\Lambda} U_\alpha\right) \cap A = \phi$, there is a finite subset Λ_0 of Λ such that $\left(\bigcap_{\alpha\in\Lambda_0} \text{pint}\, U_\alpha\right) \cap A = \phi$.

(e) Every maximal filterbase on A is pre-θ-convergent to some point of A.

(f) Every filterbase on A is pre-θ-adheres to some point of A.

(g) For every family $\{B_\alpha : \alpha \in \Lambda\}$ of non-empty sets in X with $\left[\bigcap_{\alpha\in\Lambda} (p(\theta) - \text{cl}B_\alpha)\right] \cap A = \phi$, there exists a finite subset Λ_0 of Λ such that $\left(\bigcap_{\alpha\in\Lambda_0} B_\alpha\right) \cap A = \phi$.

(h) Every net in A pre-θ-adheres at some point of A

(i) Every ultranet in A pre-θ-converges to some point of A.

(j) Every net in A has a pre-θ-convergent subnet.

PROOF : The equivalence of (a), (b), (c), (d) has been established in [3], and '(c) ⇒ (f)' is obvious. We prove the other implications as follows.

(b) ⇔ (e) : If $\mathscr{F}$ is a maximal filterbase on X such that $\mathscr{F}$ meets A, then $\mathscr{F}^* = \{F \cap A : F \in \mathscr{F}\}$ is a maximal filterbase on A. Hence '(b) ⇒ (e)' is clear. The converse is obvious.

(f) ⇒ (g) : Let $\mathscr{B} = \{B_\alpha : \alpha \in \Lambda\}$ be a family of non-void sets in X such that for every finite subset Λ_0 of Λ, $\left(\bigcap_{\alpha \in \Lambda_0} B_\alpha\right) \cap A \neq \phi$. Then $\mathscr{F} = \{\left(\bigcap_{\alpha \in \Lambda_0} B_\alpha\right) \cap A : \Lambda_0$ is a finite subset of $\Lambda\}$ is a filterbase on A. By (f), let $a \in A \cap (p(\theta)\text{-ad}\mathscr{F})$. Then for each $\alpha \in \Lambda$ and each preopen set U containing a, $A \cap B_\alpha \cap (pclU) \neq \phi$, i.e., $B_\alpha \cap pclU \neq \phi$. Hence $a \in p(\theta)\text{-cl}B_\alpha$ for each $\alpha \in \Lambda$ and consequently, $\left(\bigcap_{\alpha \in \Lambda} p(\theta) - clB_\alpha\right) \cap A \neq \phi$.

(g) ⇒ (a) : Let $\{U_\alpha : \alpha \in \Lambda\}$ be a cover of A by preopen sets of X. Then $A \cap \left[\bigcap_{\alpha \in \Lambda} (X \setminus U_\alpha)\right] = \phi$.

If for some $\alpha \in \Lambda$, $X \setminus pclU_\alpha = \phi$, then we are through. If $(X \setminus pclU_\alpha)$ (= B_α, say) $\neq \phi$ for all $\alpha \in \Lambda$, then $\mathscr{B} = \{B_\alpha : \alpha \in \Lambda\}$ is a family of nonempty sets such that

$$\left[\bigcap_{\alpha \in \Lambda} (p(\theta) - clB_\alpha)\right] \cap A \subseteq A \cap \left[\bigcap_{\alpha \in \Lambda} (X \setminus U_\alpha)\right] = \phi \qquad (1)$$

In fact, let $x \in p(\theta)\text{-cl}B_\alpha = p(\theta)\text{-cl}(X \setminus pclU_\alpha)$. Then for every preopen set V_x containing x, $(X \setminus pclU_\alpha) \cap pclV_x \neq \phi$. Since $U_\alpha \in PO(X)$, if $x \in U_\alpha$ then $(X \setminus pclU_\alpha) \cap (pclU_\alpha) \neq \phi$ which is not possible. Thus $x \notin U_\alpha$ so that $x \in X \setminus U_\alpha$. Hence $p(\theta)\text{-cl}B_\alpha \subseteq X \setminus U_\alpha$ and (i) follows. By (g), there is a finite subset Λ_0 of Λ such that $\left(\bigcap_{\alpha \in \Lambda_0} B_\alpha\right) \cap A = \phi$ i.e., $A \subseteq X \setminus \bigcap_{\alpha \in \Lambda_0} (X \setminus pclU_\alpha) = \bigcup_{\alpha \in \Lambda_0} pclU_\alpha$ which proves (a).

(f) ⇔ (h) : Suppose (f) holds. Let $\{x_n : n \in (D, \geq)\}$ be a net in A. Consider the filterbase $\mathscr{F} = \{T_n : n \in D\}$ generated by the net, where $T_n = \{x_m : m \in D \text{ and } m \geq n\}$. By (f), there exists $a \in A \cap (p(\theta)\text{-ad}\mathscr{F})$. Then for each preopen set U containing a and each $F \in \mathscr{F}$, $pclU \cap F \neq \phi$ i.e., $pclU \cap T_n \neq \phi$, for all $n \in D$. Hence $a \in A \cap [p(\theta)\text{-ad}(x_n)]$, and (h) follows. The proof of '(h) ⇒ (f)' is quite similar.

(h) ⇒ (i) : Let $\{x_n : n \in (D, \geq)\}$ be an ultranet in A. By (h), there exists $a \in [p(\theta)\text{-ad}(x_n)] \cap A$. Let U be a preopen set containing a. Since the given net is an ultranet in A, it is eventually in either $(A \cap pclU)$ or $A \setminus (A \cap pclU)$. But since the net is frequently in $(A \cap pclU)$ (as $a \in p(\theta)\text{-ad}(x_n)$), we conclude that the net is eventually in pclU. Hence $x_n \xrightarrow{p(\theta)} a$.

(i) ⇒ (j) : We know that every net in A has a subnet which is an ultranet. Thus any given net in A has a subnet which pre-θ-converges to some point of A (by (i)), and (j) follows.

(j) ⇒ (h) : Let $T : E \to A$ be a pre-θ-convergent subnet of a given net $S : D \to A$, and suppose $T \xrightarrow{p(\theta)} a \in A$. Then $T = S \circ g$, where $g : E \to D$ is a function such that for each $n \in D$ there exists $p \in E$ with the property that $g(m) \geq n$ in D whenever $m \in E$ with $m \geq p$. Let U be a preopen set containing a and $n \in D$. There is $m_1 \in E$ such that $T(m) \in pclU$, for all $m \geq m_1 (m \in E)$. For the given $n \in D$, let $p \in E$ with the above stated property and let $m_2 \in E$ such that $m_2 \geq p, m_1$. Then $g(m_2) \geq n$ in D, and we have $T(m_2) = S \circ g(m_2) \in pclU$(since $m_2 \geq m_1$). Hence $a \in (p(\theta)\text{-ad}S) \cap A$. This completes the proof of the theorem.

Putting A = X in the above theorem, we obtain the following characterizations of p-closed spaces (we note here that direct proofs of the equivalence of the statements in (a) - (d) of the following theorem have been given by Dontchev *et al.* [3]).

THEOREM 2.13 : For a space X, the following are equivalent :

(a) X is a p-closed space.

(b) Every maximal filterbase on X pre-θ-converges.

(c) Every filterbase on A is pre-θ-adheres to some point of A.

(d) For every family $\{U_\alpha : \alpha \in \Lambda\}$ of non-void preclosed sets in X with $\bigcap_{\alpha\in\Lambda} U_\alpha = \phi$, there is a finite subset Λ_0 of Λ such that $\bigcap_{\alpha\in\Lambda_0} \text{pint}\, U_\alpha = \phi$.

(e) For every family $\{B_\alpha : \alpha \in \Lambda\}$ of non-empty sets in X with $\bigcap_{\alpha\in\Lambda} p(\theta) - \text{cl}B_\alpha = \phi$, there exists a finite subset Λ_0 of Λ such that $\bigcap_{\alpha\in\Lambda_0} B_\alpha = \phi$.

(f) Every net in X pre-θ-adheres at some point of X.

(g) Every ultranet in X, pre-θ-converges.

(h) Every net in X has a pre-θ-convergent subnet.

THEOREM 2.14 : A space X is p-closed if and only if every filterbase on X with at most one pre-θ-adherent point pre-θ-converges.

PROOF : Let X be p-closed and $\mathscr{F}$ be a filterbase on X with at most one pre-θ-adherent point. By theorem 2.13, $\mathscr{F}$ has then a unique pre-θ-adherent point x_0 (say) in X. Let $\mathscr{F}$ do not pre-θ-converge to x_0. Then for some preopen set U containing x_0, and for each $F \in \mathscr{F}$, $F \cap (X \setminus \text{pcl}U) \neq \phi$. So, $\mathscr{G} = \{F \cap (X \setminus \text{pcl}U) : F \in \mathscr{F}\}$is a filterbase on X and hence has a pre-θ-adherent point x in X. Since U is a preopen set containing x_0 such that $(\text{pcl}U) \cap G = \phi$, for all $G \in \mathscr{G}$, we have $x \neq x_0$. Now, for each preopen set V containing x and each $F \in \mathscr{F}$, $F \cap (\text{pcl}V) \supseteq F \cap \text{pcl}V \cap (X \setminus \text{pcl}U) \neq \phi$ i.e., $F \cap (\text{pcl}V) \neq \phi$. Thus, x is a pre-θ-adherent point of $\mathscr{F}$, where $x \neq x_0$. This contradicts that x_0 is the only pre-θ-adherent point of $\mathscr{F}$. The converse is clear in view of Theorem 2.13 ((c) ⇒ (a)) and the fact that a point x is necessarily a pre-θ-adherent point of a filterbase $\mathscr{F}$ if $\mathscr{F} \xrightarrow{p(\theta)} x$.

DEFINITION 2.15 : A family $\mathscr{U}$ of preclosed subsets of a space X will be called ***a precover of X*** if for each $x \in X$, there is some $U \in \mathscr{U}$ such that U is a prenbd of x (i.e., $x \in V \subset U$), for some preopen set V).

THEOREM 2.16 : A space X is p-closed iff every precover of X has a finite subcover.

PROOF : Let X be p-closed and $\mathscr{U}$ be any precover of X. Then, for each each $x \in X$, there are $U_x \in \mathscr{U}$ and a preopen set V_x such that $x \in V_x \subset U_x$. It then follows that $\{V_x : x \in X\}$ is a cover of X by preopen sets of X. By p-closedness of X, $X = \bigcup_{i=1}^{n} pclV_{x_i}$, for a finite subset $\{x_1,..., x_n\}$ of X. Thus, $X = \bigcup_{i=1}^{n} U_{x_i}$. Conversely, if U is any preopen cover of X, then $\{pclU : U \in \mathscr{U}\}$ is precover of X and hence the rest follows trivially.

§ 3. p(θ)-CONTINUITY, p(θ)-SUBCLOSED GRAPH AND STRONG p(θ)-CLOSEDNESS

According to our proposed scheme, we introduce in this section a sort of functions, termed p(θ)-continuous ones, and the concept of p(θ)-subclosedness of the graphs of functions, with the tacit aim of characterizing p-closedness of topological spaces. We start with the definition of p(θ)-continuity followed by some of its equivalent formulations and certain relevant properties.

DEFINITION 3.1 : A function $f : X \to Y$ is called ***p(θ)-continuous*** if for each filterbase $\mathscr{F}$ on X, $f(ad\mathscr{F}) \subseteq p(\theta)\text{-}adf(\mathscr{F})$, where as usual, ***ad$\mathscr{F}$*** denotes the adherence of $\mathscr{F}$, i.e., $ad\mathscr{F} = \cap\{clF : F \in \mathscr{F}\}$.

THEOREM 3.2 : A function $f : X \to Y$ is called p(θ)-continuous iff for each $x \in X$ and each preopen set W containing f(x), there is an open nbd U of x such that $f(U) \subseteq pclW$.

PROOF : Let the given condition hold and let $\mathscr{F}$ be a filterbase on X. If ad$\mathscr{F}$ and W is a preopen set containing f(x), there is an open nbd U of x such that $f(U) \subseteq pclW$ and $U \cap F \neq \phi$, for all $F \in \mathscr{F}$. So, $pclW \cap f(F) \neq \phi$, for every $F \in \mathscr{F}$. This shows that $f(x) \in p(\theta)\text{-}adf(\mathscr{F})$ and f becomes p(θ)-continuous.

Conversely, let for some $x \in X$ and some preopen set V containing f(x), $f(U_x) \not\subset pclV$, for every open nbd U_x of x. Then, $\mathscr{F} = \{ U_x \cap [X \setminus f^{-1}(pclV)] : U_x \text{ is an open nbd of } x\}$ is a filter-base on X with x as an adherent point. But $f(x) \notin p(\theta)\text{-}adf(\mathscr{F})$ so that $f(ad\mathscr{F}) \not\subset p(\theta)\text{-}adf(\mathscr{F})$.

THEOREM 3.3 : For a function $f : X \to Y$ the following are equivalent :

(a) f is p(θ)-continuous.

(b) For each $x \in X$ and each filterbase $\mathscr{F}$ on X with $\mathscr{F} \to x$, the filterbase $f(\mathscr{F}) \xrightarrow{p(\theta)} f(x)$.

(c) For each $x \in X$ and each filterbase $\mathscr{F}$ on X with $\mathscr{F} \to x$, $f(x) \in p(\theta)\text{-adf}(\mathscr{F})$.

(d) For each $x \in X$ and every net (x_α) in X with $x_\alpha \to x$, $f(x_\alpha) \xrightarrow{p(\theta)} f(x)$.

PROOF : (a) ⇒ (b) : For any preopen set V containing f(x), there exists, by Theorem 3.2, an open nbd U of x such that $f(U) \subseteq pclV$, and then $F \subseteq U$, for some $F \in \mathscr{F}$. Hence, $f(F) \subseteq pclV$.

(b) ⇒ (c) : Obvious.

(c) ⇒ (a) : If f is not p(θ)-continuous at some point x of X, there exists some preopen set V containing f(x) such that $[X \setminus f^{-1}(pclV)] \cap U \neq \phi$, for each open nbd U of x. Then $\mathscr{F} = \{[X \setminus f^{-1}(pclV)] \cap U : U$ is an open nbd of $x\}$ is a filterbase on X such that $\mathscr{F} \to x$, but $f(x) \notin p(\theta)\text{-adf}(\mathscr{F})$.

(a) ⇔ (d) : First suppose that f is p(θ)-continuous, and x_α is a net such that $x_\alpha \to x$. Consider any preopen set V containing f(x). Then by Theorem 3.2, there exists some open nbd V of x such that $f(U) \subseteq pclV$. Now, U being an open nbd of x, there exists some β such that $x_\alpha \in U$, for all $\alpha \geq \beta$. Consequently, for all $\alpha \geq \beta$, $f(x_\alpha) \in pclV$, i.e., $f(x_\alpha) \xrightarrow{p(\theta)} f(x)$.

Conversely, let f be not p(θ)-continuous at some $x \in X$. Then for any preopen set V containing f(x) and any open nbd U of x, $f(U) \not\subset pclV$. Now, we define '≤' on X as follows :

$U \leq V$ if and only if $U \subseteq V$, for any two open sets U, V in X.

Choose $x_U \in U$ such that $f(x_U) \notin pclV$. Then $\{x_U\}_U$ is a net in X, which converges to x, but $\{f(x_U)\}_U$ does not pre-θ-convergences to f(x).

THEOREM 3.4 : If $f : X \to Y$ is a p(θ)-continuous function and Y is Hausdorff, then the graph G(f) of f is closed in X x Y.

PROOF : Let $(x, y) \in X \times Y \setminus G(f)$, then $y \neq f(x)$. By Hausdorffness of Y, there exist open nbds U, V of y and f(x) respectively in Y such that $U \cap V = \phi$ and hence $U \cap pclV = \phi$. By p(θ)-continuity of f, there exists an open set W in X containing x such that $f(W) \subseteq pclV$. Then W x U is an open nbd of (x, y) in X x Y such that $(W \times U) \cap G(f)$, and hence G(f) is closed X x Y.

THEOREM 3.5 : Suppose $f : X \to Y$ is a function and $g : X \to X \times Y$ is the graph fnction of f, given by $g(x) = (x, f(x))$, for $x \in X$. If g is p(θ)-continuous, then so is f.

PROOF : Let $x \in X$ and V be any preopen set in Y containing $f(x)$. It is easy to see that whenever $U \in PO(X)$ and $V \in PO(Y)$, then $U \times V \in PO(X \times Y)$. Thus $X \times V$ is a preopen set in $X \times Y$ containing $g(x)$. By $p(\theta)$-continuity of g, there is an open nbd U of x such that $g(U) \subseteq pcl(X \times V)$. We can show that $pcl(X \times V) \subseteq X \times pclV$, and thus we have $f(U) \subseteq pclV$, proving the $p(\theta)$-continuity of f.

It is the turn of the notion of $p(\theta)$-subclosedness of graphs of arbitrary functions between topological spaces, which will now be introduced and characterized.

DEFINITION 3.6 : A function $f : X \to Y$ is said to have ***p(θ)-subclosed graph*** if for each $x \in X$ and each filterbase $\mathscr{F}$ on $X \setminus \{x\}$ with $\mathscr{F} \to x$, $p(\theta)\text{-adf}(\mathscr{F}) \subseteq \{f(x)\}$.

The proof of the following theorem, giving an equivalent description of $p(\theta)$-subclosedness of graphs in terms of nets, is quite clear.

THEOREM 3.7 : A function $f : X \to Y$ has a $p(\theta)$-subclosed graph iff for each $x \in X$ and each net (x_α) in $X \setminus \{x\}$ with $x_\alpha \to x$, $p(\theta)\text{-adf}(x_\alpha) \subseteq \{f(x)\}$.

THEOREM 3.8 : A function $f : X \to Y$ has a $p(\theta)$-subclosed graph iff for each $(x, y) \in X \times Y \setminus G(f)$, there exists an open nbd U of x in X and some preopen set V containing y such that $[(U \setminus \{x\}) \times pclV] \cap G(f) = \phi$, where G(f), as usual, denotes the graph of f.

PROOF : First, suppose that $f : X \to Y$ has a $p(\theta)$-subclosed graph, and $(x, y) \in X \times Y \setminus G(f)$. Then, $y \neq f(x)$. Consider $\eta_x^* = \{U \setminus \{x\} : U$ is an open nbd of $x\}$. If it is a filterbase, then $\eta_x^* \to x$ and hence $y \notin p(\theta)\text{-adf}(\eta_x^*)$. So, there are $U \in \eta_x^*$ and a preopen set V containing y such that $pclV \cap f(U) = \phi$. Then $U_0 = U \cup \{x\}$ is a nbd of x and V is a preopen set containing y such that $[(U_0 \setminus \{x\}) \times pclV] \cap G(f) = \phi$. If η_x^* is not a filterbase, then $U = \{x\}$ for some open nbd U of x, and the rest is obvious.

Conversely, suppose $\mathscr{F}$ is a filterbase in $X \setminus \{x\}$ converging to $x \in X$ and the given condition holds. Suppose $y \neq f(x)$, i.e., $(x, y) \in X \times Y \setminus G(f)$. Hence by the given condition, there are an open nbd U of x in X and a preopen set V containing y such that $[(U \setminus \{x\}) \times pclV] \cap G(f) = \phi$, or equivalently, $f(U \setminus \{x\}) \cap pclV = \phi$. Since, $\mathscr{F} \to x$, it follows that $F \subseteq U \setminus \{x\}$ for some $F \in \mathscr{F}$. Hence, $f(F) \cap pclV = \phi$. Then, $y \notin p(\theta)\text{-adf}(\mathscr{F})$. Consequently, f has a $p(\theta)$-subclosed graph.

The concepts which we are pondering upon so far in this section, will now be used to obtain a few characterizing conditions for p-closedness.

THEOREM 3.9 : A space (Y, τ) is p-closed iff for any space X, every function $f : X \to Y$ with $p(\theta)$-subclosed graph is $p(\theta)$-continuous.

PROOF : First, we suppose that Y is $p(\theta)$-closed and suppose $f : X \to Y$ is a function with $p(\theta)$-subclosed graph, X being an arbitrary topological space. Let $\mathscr{F}$ be a filterbase on X and $y \in f(ad\mathscr{F})$. Then there is an $x \in ad\mathscr{F}$ such that $y = f(x)$. Let $\mathscr{G} = \{U \cap F \setminus \{x\} : F \in \mathscr{F} \text{ and } U \in \eta(x)\}$, where $\eta(x)$ denotes the system of all open nbds of x in X.

First, suppose that $\mathscr{G}$ is not a filterbase. Then for some $U_1 \in \eta(x)$ and some $F_1 \in \mathscr{F}$, $U_1 \cap F_1 = \{x\}$. We assert that $x \in F$, for each $F \in \mathscr{F}$. If not, then for some $F_2 \in \mathscr{F}$, $x \notin F_2$, Choose $F_3 \in \mathscr{F}$ such that $F_3 \subseteq F_1 \cap F_2$. Then, $(U_1 \cap F_3) \setminus \{x\} \subseteq (U_1 \cap F_1) \setminus \{x\} = \phi$ i.e., $U_1 \cap F_3 = \{x\}$ (as $U_1 \cap F_3 \neq \phi$), i.e., $x \in F_3 \subseteq F_1 \cap F_2$ and hence $x \in F_2$, a contradiction. Thus, $f(x) \in f(F)$, for every $F \in \mathscr{F}$. Hence, $y \in p(\theta)\text{-}adf(\mathscr{F})$.

Next, let $\mathscr{G}$ be a filterbase on $X \setminus \{x\}$. Clearly $\mathscr{G}$ converges to x in X. Since f has a $p(\theta)$-sub-closed graph, $f(\mathscr{G})$ has at most one pre-θ-adherent point, viz. $f(x)$. Since Y is p-closed, it then follows by virtue of Theorem 2.13, that $p(\theta)\text{-}adf(\mathscr{G}) = \{f(x)\}$. Thus, $\{y\} = p(\theta)\text{-}adf(\mathscr{G}) \subseteq p(\theta)\text{-}adf(\mathscr{F})$. Hence, in any case, $f(ad\mathscr{F}) \subseteq p(\theta)\text{-}adf(\mathscr{F})$ and consequently f is $p(\theta)$-continuous.

Conversely, to prove (Y, τ) to be p-closed under the stated condition, it is to be shown, in view of Theorem 2.13, that every filterbase on Y has a pre-θ-adherent point. If possible, let there exist a filterbase $\mathscr{F}$ on Y such that $p(\theta)\text{-}adf(\mathscr{F}) = \phi$. Let us choose and fix some $y_0 \in Y$. Consider the collection $\tau^* = \{A \subseteq Y : y_0 \in Y \setminus A\} \cup \{A \subseteq Y : y_0 \in A \text{ and } F \subseteq A \text{ for some } F \in \mathscr{F}\}$. Clearly, τ^* is a topology on Y. Now, we consider the identity function $f : (Y, \tau^*) \to (Y, \tau)$. We show that f has a $p(\theta)$-subclosed graph. For this let $y \in Y$ and $\mathscr{B}$ be a filterbase on $Y \setminus \{y\}$ such that $\mathscr{B} \to y$ in (Y, τ^*). Then, by definition of τ^*, $y = y_0$. For otherwise, $\{y\}$ is a τ^*-open nbd of y and hence the filterbase $\mathscr{B}$ on $Y \setminus \{y\}$ cannot converge to y. Also, we have $\mathscr{F} \subseteq \mathscr{B}$. In fact, for a given $F \in \mathscr{F}$, $\{y_0\} \cup F$ is a τ^*-open of y_0 and hence contains some member $B \in \mathscr{B}$ (as $\mathscr{B} \to y_0$) and then $B \subseteq F$. Now, in (Y, τ), $p(\theta)\text{-}adf(\mathscr{B}) = p(\theta)\text{-}ad\mathscr{B} \subseteq p(\theta)\text{-}ad\mathscr{F}$ (as $\mathscr{F} \subseteq \mathscr{B}$) $= \phi$. Thus f has a $p(\theta)$-subclosed graph, and consequently by hypothesis, f is p(θ)-continuous. But $y_0 \in ad\mathscr{F}$ in (Y, τ*) whereas $f(y_0) = y_0 \notin$ p(θ)-adf($\mathscr{F}$) (= φ) in (Y, τ). This contradicts the p(θ)-continuity of f. The contradiction proves that (Y, τ) is p-closed.

THEOREM 3.10 : Let X be p-closed. Then for all spaces Y and Z, and all functions f : Y → X and g : Z → X with p(θ)-subclosed graphs, the set $\Delta(f, g) = \{(y, z) \in Y \times Z : f(y) = g(z)\}$ is closed in Y x Z.

PROOF : Let (a, b) be a limit of Δ(f, g). Then there is a filterbase $\mathscr{F}$ on Δ(f, g) \{(a, b)} converging to (a, b). If p_1, p_2 denote the projection maps from Y x Z to Y and Z respectively, we have $f(p_1(F)) = g(p_2(F))$, for all $F \in \mathscr{F}$.

First, suppose that there is some $F_1 \in \mathscr{F}$ such that $p_1(F_1) = \{a\}$. We obtain a filterbase $\mathscr{F}^*$ on Δ(f, g) \ {(a, b)} by replacing only those elements F of $\mathscr{F}$ which contain b as the second co-ordinate of at least one element, by an $F^* \in \mathscr{F}$ where $F^* \subseteq F_1 \cap F$. Clearly, $\mathscr{F}^*$ converges to (a, b). Then $p_2(\mathscr{F}^*)$is a filterbase on Z \ {b} converging to b (since projection maps are continuous). Since g has a p(θ)-subclosed graph, p(θ)-ad(g($p_2(\mathscr{F}^*)$) ⊆ {g(b)}. In this case $a \in p_1(F)$ for all $F \in \mathscr{F}$ so that f(a) ∈ p(θ)-adf($p_1(\mathscr{F})$) = p(θ)-ad(g($p_2(\mathscr{F})$ ⊆ p(θ)-ad(g($p_2(\mathscr{F}^*)$) (since $\mathscr{F}^* \subseteq \mathscr{F}$) ⊆ {g(b)}. Thus, (a, b) ∈ Δ(f, g). In case $p_2(F) = \{b\}$ for some $F \in \mathscr{F}$, we proceed similarly as above.

Finally, suppose that $p_1(F) \neq \{a\}$ and $p_2(F) \neq \{b\}$ for all $F \in \mathscr{F}$. We replace each $F \in \mathscr{F}$ by the subset F* obtained by deleting from F all those elements with first coordinate a, and thereby obtain a filterbase $\mathscr{F}^*$ on Δ(f, g) \ {(a, b)} for every $F \in \mathscr{F}^*$, and $p_1(\mathscr{F}^*)$ is a filterbase on Y \ {a} converging to a. In view of the above, we can suppose without loss of generality that $\{b\} \neq p_2(F)$ for all $F \in \mathscr{F}^*$. Now for each $F^* \in \mathscr{F}^*$, we consider the subset F** of F* by deleting from F* the elements with second coordinate b. Then $\mathscr{F}^{**} = \{ F^{**} \subseteq F^* : F^* \in \mathscr{F}^*\}$is a filterbase on Δ(f, g) \ {(a, b)} converging to (a, b). Clearly, $f(p_1(F)) = g(p_2(F))$ for every $F \in \mathscr{F}^{**}$ and $p_1(\mathscr{F}^{**})$ and $p_2(\mathscr{F}^{**})$ are filterbases on Y \ {a} and Z \ {b} respectively converging to a and b respectively. Since f has p(θ)-subclosed graphs, we have, p(θ)-adf($p_1(\mathscr{F}^{**})$) ⊆ {f(a)}. By Theorem 3.9, g is p(θ)-continuous and hence by Theorem 3.3 we have, g(b) ∈ p(θ)-adg($p_2(\mathscr{F}^{**})$) = p(θ)-adf($p_1(\mathscr{F}^{**})$ ⊆ {f(a)} so that (a, b) ∈ Δ(f, g) .

COROLLARY 3.11 : Let a space Y be p-closed. Then for any space X and any function f : X → Y having p(θ)-subclosed graphs, the set $\Delta(f) = \{(x_1, x_2) \in X \times X : f(x_1) = f(x_2)\}$ is closed in X x X.

THEOREM 3.12 : A space Y is p-closed if for any space X and any function $f : X \to Y$ with $p(\theta)$-subclosed graph, the set $\Delta(f) = \{(x_1, x_2) \in X \times X : f(x_1) = f(x_2)\}$ is closed in X x X.

PROOF : Let the space Y be not p-closed so that there is a net $S = \{S_\alpha : \alpha \in (D, \geq)\}$ in Y ((D, ≥) being a directed set), which has no pre-θ-adherent point. We choose two distinct points y_1 and y_2 in Y and assume without any loss of generality that S is a net in $Y \setminus \{y_1, y_2\}$. Consider a space (X, T), where X = Y, and $T = \{A \subseteq X : A \cap \{y_1, y_2\} = \varphi\} \cup \{A \subseteq X : A \cap \{y_1, y_2\} \neq \varphi$ and $T_\alpha \subseteq A$ for some $\alpha \in D\}$, where $T_\alpha = \{S_\beta : \beta \in D$ and $\beta > \alpha\}$.

Consider the map $f : (X, T) \to Y$ given by

$$f(x) = \begin{cases} x, & \text{if } x \neq y_1, y_2 \\ y_2, & \text{if } x = y_1 \\ y_1, & \text{if } x = y_2. \end{cases}$$

In order to prove that f has a $p(\theta)$-subclosed graph, let $R = \{R_\beta : \beta \in D_1\}$be a net in $X \setminus \{x\}$ converging to x (∈ X). Then $x = y_1$ or y_2 (otherwise {x} is a T-nbd of x and hence R cannot converge to x). For definiteness, let $x = y_1$. Suppose, if possible f(R) pre-θ-adheres at some point $y \in Y$. Since the net f(S) = S has no pre-θ-adherent point in Y, there exists a preopen set U in Y containing y and an $\alpha_0 \in D$ such that for all $\alpha > \alpha_0$ ($\alpha \in D$), $S_\alpha \notin$ pclU, i.e., $T_\alpha \subseteq Y \setminus$ pclU. Now, $\{y_1\} \cup T_\alpha$ is a T-open nbd of y_1 in X. Then $\{R_\beta : \beta \in D_1$ and $\beta \geq \beta_0\} \subseteq \{y_1\} \cup T_\alpha$ (since $R \to y_1$ in X), for some $\beta_0 \in D_1$. Since, $T_\beta \subseteq X \setminus \{y_1, y_2\}$ no R_β can be y_2 for $\beta \geq \beta_0$. Thus, $\{f(R_\beta) : \beta \geq \beta_0\} = \{R_\beta : \beta \geq \beta_0\} \subseteq T_\alpha \subseteq Y \setminus$ pclU so that f(R) cannot pre-θ-adhere at y, a contradiction. Thus, f has a $p(\theta)$-subclosed graph. Now, clearly $(y_1, y_2) \notin \Delta(f)$. To arrive at a contradiction it suffices to show that $(y_1, y_2) \in$ clΔ(f). In fact, let U be an open nbd of (y_1, y_2) in X x X. By definition of T, there exist $\alpha, \beta \in D$ such that $(T_\alpha \cup \{y_1\}) \times (T_\beta \cup \{y_2\}) \subseteq U$. If $\gamma \in D$ such that $\gamma \geq \alpha, \beta$, then $(S_\gamma, S_\gamma) \in U \cap \Delta(f)$ proving that $(y_1, y_2) \in$ clΔ(f).

Combining Corollary 3.11 and Theorem 3.12 we get,

THEOREM 3.13 : A topological space Y is p-closed iff for any space X and any function $f : X \to Y$ with $p(\theta)$-subclosed graph, the set $\Delta(f) = \{(x_1, x_2) \in X \times X : f(x_1) = f(x_2)\}$ is closed in X x X.

Again, from Theorems 3.10 and 3.12, it follows that

THEOREM 3.14 : A topological space Y is p-closed iff for any topological spaces X and Z and all functions $f : X \to Y$ and $g : Z \to Y$ with p(θ)-subclosed graphs, the set $\{(x, z) \in X \times Z : f(x) = g(z)\}$ is closed in X x Z.

THEOREM 3.15 : A space X is p-closed iff for any space Y and any functions f, $g : Y \to X$ with p(θ)-subclosed graphs, the set $\Delta^*(f, g) = \{y \in Y : f(y) = g(y)\}$ is a closed subset of Y.

PROOF : Let X be p-closed. Putting Y = Z in Theorem 3.10, we see that for any given space Y and any functions f, $g : Y \to X$ with p(θ)-subclosed graphs, the set $\Delta(f, g) = \{(y_1, y_2) \in Y \times Y : f(y_1) = g(y_2)\}$ is closed in Y x Y. Now, clearly $\Delta^*(f, g) = p_1[\Delta(f, g) \cap \Delta_Y]$, where Δ_Y is the diagonal in Y x Y and $p_1 : Y \times Y \to Y$ is the first projection map Since, $p_1|_{\Delta Y}$ is a homeomorphism, $\Delta^*(f, g)$ is closed in Y.

Conversely, suppose X is not p-closed. Then for some filterbase $\mathscr{F}$ on X, p(θ)-ad$\mathscr{F} = \phi$. Select any two distinct points a, $b \in X$ and put Y = X. Let, $T_1 = \{A \subseteq X : b \in Y \setminus A$ or $F \subseteq A$ for some $F \in \mathscr{F}\}$. Then (Y, T_1) is a topological space.

Consider the maps f, $g : (Y, T_1) \to X$ given by f(x) = x, for $x \in Y$ and

$$g(x) = x, \quad \text{for } x \in Y \setminus \{b\}$$
$$= a, \quad \text{for } x = b.$$

We show that each f and g has a p(θ)-subclosed graph. Let $y \in Y$ and $\mathscr{G}$ be any filterbase on $Y \setminus \{y\}$such that $\mathscr{G} \to y$ in Y. If $y \neq b$, then {y} being a T_1-open nbd of y, $\mathscr{G}$ cannot converge to y. Hence, y = b. Now, for each $F \in \mathscr{F}$, $F \cup \{b\} \in T_1$, so that p(θ)-ad($\mathscr{G}$) = p(θ)-ad$\mathscr{G} \subseteq$ p(θ)-ad$\mathscr{F} = \phi \subseteq \{f(y)\}$. Hence, f has a p(θ)-subclosed graph. Similarly, g has a p(θ)-subclosed graph. Now, $\Delta^*(f, g) = Y \setminus \{b\}$, and we see that for any open nbd V of b in Y, there is an $F \in \mathscr{F}$ such that $F \subseteq V$ and $V \cap \Delta^*(f, g) \supseteq F \cap \Delta^*(f, g) \neq \phi$ (as $F \neq \{b\}$, for each $F \in \mathscr{F}$). Hence $\Delta^*(f, g)$ is not closed in Y.

§ 4. STRONGLY p(θ)-CLOSED GRAPH AND p-CLOSEDNESS

DEFINITION 4.1 : A function $f : X \to Y$ is said to have ***strongly p(θ)-closed graph*** G(f) if whenever a net $x_\alpha \to x$ in X and $f(x_\alpha) \xrightarrow{p(\theta)} y$ in Y, it follows that y = f(x).

THEOREM 4.2 : A function $f : X \to Y$ has a strongly $p(\theta)$-closed graph $G(f)$ iff whenever a filterbase $\mathscr{F} \to x$ in X and $f(\mathscr{F}) \xrightarrow{p(\theta)} y$ in Y, it follows that $y = f(x)$.

PROOF : Straightforward and left.

LEMMA 4.3 : A function $f : X \to Y$ has a strongly $p(\theta)$-closed graph $G(f)$ iff for each point $(x, y) \notin G(f)$, there exist an open nbd U of x in X and a preopen set V in Y containing y such that $(U \times pclV) \cap G(f) = \phi$.

PROOF : Let the given condition hold for a function $f : X \to Y$, and let x_α be a net in X such that $x_\alpha \to x$ and $f(x_\alpha) \xrightarrow{p(\theta)} y$ in Y. It then follows that for each open nbd V of x and each preopen set W in Y containing y, $(V \times pclW) \cap G(f) \neq \phi$. Hence, $(x, y) \in G(f)$ and so, $y = f(x)$. Hence, f has a strongly $p(\theta)$-closed graph.

Conversely, suppose that a function $f : X \to Y$ does not satisfy the stated condition of the theorem. Then for some $(a, b) \in X \times Y \setminus G(f)$, we have $(U \times pclV) \cap G(f) \neq \phi$, for every open nbd U of a and each preopen set V containing b. Suppose, $\mathscr{F}_1 = \{U_\alpha : \alpha \in I_1$ and U_α is an open nbd of $a\}$ and $\mathscr{F}_2 = \{pclV_\beta : \beta \in I_2$ and V_β is a preopen set containing $b\}$, and put $\mathscr{F}_3 = \{V_{(\alpha, \beta)} = (U_\alpha \times pclV_\beta) \cap G(f) : (\alpha, \beta) \in I_1 \times I_2\}$. Then $\mathscr{F} = \{F_{(\alpha, \beta)} : (\alpha, \beta) \in I_1 \times I_2\}$ where $F_{(\alpha, \beta)} = \{x \in U_\alpha : (x, f(x)) \in W_{(\alpha, \beta)}\}$, is a filterbase on X such that $\mathscr{F}$ converges to the point a in X, $f(\mathscr{F}) \xrightarrow{p(\theta)} b$ and $f(a) \neq b$. Thus in view of Theorem 4.2, f does not have a strongly $p(\theta)$-closed graph.

In view of Theorems 3.8 and 3.9 and Lemma 4.3, it now follows that

THEOREM 4.4 : If Y is a p-closed space, then every function f from any space X to Y with a strongly $p(\theta)$-closed graph is $p(\theta)$-continuous.

DEFINITION 4.5 : A space X is called ***pre-Hausdorff*** if corresponding to any two distinct points x, y of X, there exist disjoint preopen sets U, V such that $x \in U$, $y \in V$.

THEOREM 4.6 : A pre-Hausdorff space Y is p-closed if each function f from any topological space X to Y with a strongly $p(\theta)$-closed graph is $p(\theta)$-continuous.

PROOF : Let, $x_0 \in Y$ and let $(x_\alpha)_{\alpha\in D}$ be a net in $Y \setminus \{x_0\}$ with no pre-Θ-adherent point in $Y \setminus \{x_0\}$. Let $X = \{x_\alpha : \alpha \in D\} \cup \{x_0\}$, and let T be the topology on X generated by $\{\{x_\alpha\} : \alpha \in D\} \cup \{T_\mu \cup \{x_0\} : \mu \in D\}$ as basic open sets, where $T_\mu = \{x_\alpha : \alpha \geq \mu, \alpha \in D\}$. Let $f : X \rightarrow Y$ be the identity map and let $(x, y) \in X \times Y \setminus G(f)$. If $x \neq x_0$, then $\{x\}$ is open in X. Choose, by pre-Hausdorffness of Y, a preopen set V in Y, containing y with $x \notin pclV$. Then clearly $(\{x\} \times pclV) \cap G(f) = \phi$. If $x = x_0$, then $y \neq x_0$, so there is a preopen set V containing y satisfying $x_0 \notin pclV$, as Y is pre-Hausdorff. Again $y \neq x_0$, so y is not a pre-Θ-adherent point of (x_α) in Y. Hence, there is a $\mu \in D$ such that $T_\mu \cap pclV = \phi$. Thus, $T_\mu \cup \{x_0\}$ is an open nbd of x in X and V a preopen set in Y containing y such that $[(T_\mu \cup \{x_0\}) \times pclV] \cap G(f) = \phi$. Hence, the graph of f is strongly p(θ)-closed and by hypothesis, f is p(θ)-continuous. So for any preopen set V containing x_0 in Y, there exists $\mu \in D$ satisfying $T_\mu \subseteq pclV$. Thus, $x_\alpha \xrightarrow{p(\theta)} x_0$ in Y. Hence, Y is p-closed.

That the assumption of pre-Hausdorffness on Y is essential can be observed from the following result :

THEOREM 4.7 : If a surjection $f : X \rightarrow Y$ has strongly p(θ)-closed graph, then Y is pre-Hausdorff.

PROOF : Let y and z be any two distinct points of Y. Then since f is onto, there is $x \in X$ such that $f(x) = y$. Hence, $(x, z) \notin G(f)$. Since f has a strongly p(θ)-closed graph, there exist an open nbd U of x and a preopen set V containing z such that $f(U) \cap pclV = \phi$. Put $W = Y \setminus pclV$, then W is a preopen set containing y and $W \cap V = \phi$. Hence Y is pre-Hausdorff.

Combining Theorems 4.4 and 4.6 we obtain :

THEOREM 4.8 : A pre-Hausdorff space Y is p-closed iff each function from any space X into Y with strongly p(θ)-closed graph is p(θ)-continuous.

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