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ON THE EXTENSION OF A BILATERAL GENERATING FUNCTION OF JACOBI POLYNOMIALS

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Abstract : In this note, we have obtained a more general generating function of Jacobi polynomial from the existence of quasi – bilinear generating function involving the special function under consideration from the Lie group view – point.

Key Words and phrases : Generating functions, Jacobi polynomials, group theoretic method.

1. **Introduction :**

In their paper [1], Chatterjea and Chakraborty defined the quasi – bilinear generating functions as follows :

$$G(x,u,w)=\sum_{n=0}^{\infty} a_n p_n^{(\alpha)}(x) p_m^{(n)}(u) w^n, \tag{1.1}$$

where $p_n^{(\alpha)}(x)$ is a polynomial of degree (n) and of parameter (α).

In [2], Das has proved the following theorem on bilateral generating function involving Jacobi polynomials :

Theorem 1.

If there exists a generating function of the form

$$G(x,w) = \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(x) w^n \tag{1.2}$$

then

$$\left(1+\frac{1+x}{2}t\right)^{\alpha}\left(1-\frac{1-x}{2}t\right)^{\beta} G\left(x-\frac{1-x^2}{2}t, wt\right) \tag{1.3}$$

$$= \sum_{n=0}^{\alpha} \sigma_n(x,w) t^n,$$

where

$$\sigma_n(x,w)= \sum_{k=0}^{n} a_k \frac{(k+1)_{n-k}}{(n-k)!} P_n^{(\alpha+k-n,\beta+k-n)}(x) w^k . \tag{1.4}$$

In this note we have obtained a more general generating relation of Jacobi polynomials from the existence of quasi – bilinear generating relation by using group – theoretic method. The main result is stated in the form of the following theorem :

Theorem 2.

If there exists a generating relation of the form :

(1.5) $$G(x,u,w)= \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(x) P_m^{(n,\beta)}(u) w^n$$

then

(1.6) $$(1-w\zeta)^{-(1+\beta+m)} \left[\frac{yz-w(1+x)}{z}\right]^{\alpha} \left[\frac{yz+wt(1-x)}{y}\right]^{\beta}$$

$$\times G\left(\frac{xyz+wt(1-x^2)}{yz}, \frac{u+w\zeta}{1-w\zeta}, \frac{wz\zeta v}{1-w\zeta}\right)$$

$$= \sum_{n,p,q=0}^{\infty} a_n \cdot \frac{w^{n+p+q}}{p!q!} v^n (-2)^q (n+1)_q$$

$$\times (1+n+\beta+m)_p y^{\alpha-q} z^{\beta-q} t^{n+q} P_{n+q}^{(\alpha-q,\beta-q)}(x) P_m^{(n+p,\beta)}(u) \zeta^{n+p}.$$

2. **Proof of the theorem :**

Let

(2.1) $$G(x,u,w) = \sum_{n=0}^{\infty} a_m P_n^{(\alpha,\beta)}(x) P_m^{(n,\beta)}(u) w^n.$$

We first consider the operators R_1 *and* R_2 [3 ,4], where

$$R_1 = \left(1-x^2\right) y^{-1} z^{-1} t \frac{\partial}{\partial x} - (1+x) z^{-1} t \frac{\partial}{\partial y} + (1-x) y^{-1} t \frac{\partial}{\partial z}$$

and

$$R_2 = (1+u)\zeta \frac{\partial}{\partial u} + \zeta^2 \frac{\partial}{\partial \zeta} + (1+\beta+m)\zeta$$

such that

(2.2) $$R_1\left(P_n^{(\alpha,\beta)}(x) y^{\alpha} z^{\beta} t^n\right) = -2(n+1) P_{n+1}^{(\alpha-1,\beta-1)}(x) y^{\alpha-1} z^{\beta-1} t^{n+1}$$

and

(2.3) $$R_2\left(P_m^{(n,\beta)}(u)\zeta^n\right) = (1+n+\beta+m)\, P_m^{(n+1,\beta)}(u)\zeta^{n+1}.$$

Also, the extended form of the groups generated by R_1 and R_2 are as follows :

(2.4) $$e^{wR_1} f(x,y,z,t)$$

$$= f\left(\frac{xyz + wt(1-x^2)}{yz}, \frac{yz - wt(1+x)}{z}, \frac{yz + wt(1-x)}{y}, t\right)$$

and

(2.5) $$e^{wR_2} f(u,\zeta) = (1 - w\zeta)^{-1-\beta-m} f\left(\frac{u + w\zeta}{1 - w\zeta}, \frac{\zeta}{1 - w\zeta}\right).$$

Replacing w by $wt\zeta v$ in (2.1) and then multiplying both sides by $y^{\alpha} \cdot z^{\beta}$ we get

(2.6) $$y^{\alpha} z^{\beta} G(x, u, wt\zeta v)$$

$$= \sum_{n=0}^{\infty} a_n \left(P_n^{(\alpha,\beta)}(x) y^{\alpha} z^{\beta} t^n\right) \times \left(P_m^{(n,\beta)}(u)\zeta^n\right)(wv)^n.$$

Operating $e^{wR_1} e^{wR_2}$ on both sides of (2.6) we get

(2.7)
$$e^{wR_1}e^{wR_2}\left[y^{\alpha}z^{\beta}G(x,u,wt\zeta v)\right]$$

$$=e^{wR_1}e^{wR_2}\left[\sum_{n=0}^{\infty}a_n(P_n^{(\alpha,\beta)}(x)y^{\alpha}z^{\beta}t^{n})(P_m^{(n,\beta)}(u)\zeta^{n})(wv)^{n}\right]$$

Now left member of(2.7), with the help of (2.4)and (2.5) becomes

(2.8)
$$e^{wR_1}e^{wR_2}\left[y^{\alpha}z^{\beta}G(x,u,wt\zeta v)\right]$$

$$=(1-w\zeta)^{-(1+\beta+m)}\left[\frac{yz-wt(1+x)}{z}\right]^{\alpha}\left[\frac{yz+wt(1-x)}{y}\right]^{\beta}$$

$$\times G\left(\frac{xyz+wt(1-x^{2})}{yz},\frac{u+w\zeta}{1-w\zeta},\frac{wz\zeta v}{1-w\zeta}\right).$$

The right hand member of(2.7),with the help of (2.2)and(2.3), becomes

(2.9)
$$e^{wR_1}e^{wR_2}\left[\sum_{n=0}^{\infty}a_n\left(P_n^{(\alpha,\beta)}(x)y^{\alpha}z^{\beta}t^{n}\right)\left(P_m^{(n,\beta)}(u)\,\zeta^{n}\right)(wv)^{n}\right]$$

$$=\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}a_n\frac{w^{n+p+q}}{p!q!}v^{n}(-2)^{q}(n+1)_q P_{n+q}^{(n-q,\beta-q)}(x)\;y^{\alpha-q}z^{\beta-q}t^{n+q}$$

$$\times(1+n+\beta+m)_p P_m^{(n+p,\beta)}(u)\;\zeta^{n+p}.$$

Equating (2.8)and (2.9),we get

(2.10)
$$(1-w\zeta)^{-(1+\beta+m)}\left[\frac{yz-w(1+x)}{z}\right]^{\alpha}\left[\frac{yz+wt(1-x)}{y}\right]^{\beta}$$

$$\times G\left(\frac{xyz+wt(1-x^2)}{yz},\frac{u+w\zeta}{1-w\zeta},\frac{wz\zeta v}{1-w\zeta}\right)$$

$$=\sum_{n,p,q=o}^{\infty}a_n.\frac{w^{n+p+q}}{p!q!}v^n(-2)^q(n+1)_q$$

$$\times(1+n+\beta+m)_p\,y^{\alpha-q}z^{\beta-q}t^{n+q}P_{n+q}^{(\alpha-q,\beta-q)}(x)P_m^{(n+p,\beta)}(u)\zeta^{n+p},$$

which is our desired result.

3. Particular Case :

Putting $m=0, y=z=t=\zeta=1$ in (2.10) and then simplifying, we get

(3.1)
$$(1-w)^{-1-\beta}\;[1-w(1+x)]^{\alpha}\;[1+w(1-x)]^{\beta}$$

$$\times G\left(x+w(1-x^2),\frac{wv}{1-w}\right)$$

$$=\sum_{n,p,q=0}^{\infty}a_n\frac{w^{n+p+q}}{p!q!}v^n(-2)^q(n+1)_q\times(1+n+\beta)_p\,P_{n+q}^{(\alpha-q,\beta-q)}(x)$$

$$=(1-w)^{-\beta-1}\sum_{n=0}^{\infty}(-2w)^n\sum_{q=0}^{n}a_q\frac{(1+q)_{n-q}}{(n-q)!}P_n^{(\alpha-n-q,\beta-n+q)}(x)$$

$$\times\left(\frac{v}{-2(1-w)}\right)^q.$$

Now first replacing $-\dfrac{v}{2(1-w)}$ by v on both sides of (3.1) and finally replacing $-2w$ by w, we get

(3.2) $$\left\{1-\frac{w}{2}(1+x)\right\}^{\alpha}\left\{1-\frac{w}{2}(1-x)\right\}^{\beta} G\left(x-\frac{w}{2}(1-x^{2}), wv\right)$$

$$=\sum_{n=0}^{\infty} w^{n} f_{n}(x,v)$$

where

$$f_{n}(x,v) = \sum_{q=0}^{n} a_{q} \frac{(1+q)_{n-q}}{(n-q)!} p_{n}^{(\alpha-n+q,\beta-n+q)}(x) v^{q},$$

which is Theorem 1.

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