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PERMANENCE CRITERIA OF REPLICATOR DYNAMICS ON A 2-SPHERE WITH BOUNDARY THROUGH VECTOR OPTIMIZATION TECHNIQUE

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ABSTRACT. The paper discusses a sufficient criteria for permanence of replicator dynamics on a 2- sphere with boundary via vector optimization technique. The paper obtains an equivalent formulation of the problem of getting a permanence criteria of replicator dynamics on a 2- sphere with boundary with the help of vector optimization technique. In fact, this amounts to obtaining a properly efficient solution of a vector minimization problem. This is equivalent to solving a suitable scalar minimization problem, which is nothing but a Kuhn-Tucker type of saddle point nonlinear minimization programming problem. So finally, the existence of solution of such a problem ensures the permanence of the replicator dynamics on a 2- sphere with boundary.

Key words and phrases: Replicator system, Permanence, Vector optimization problem, Properly efficient solution.

0. Introduction

Study of replicator system of dynamics on a n- simplex or on a concentration simplex is important from experimental as well as theoretical point of view. The experimental importance is well known in the literature [2,11]. The theoretical importance is also worth mentioning. In fact, such study answers two types of drawbacks normally dealing with

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dynamical systems in general, referred to its stability. In general, for a metric space or a locally metric space the question of existence of flow and also its property to keep the boundary invariant are not clear. Further existence of global flow, which is the most vital part of stability is not assured on such a space. Replicator dynamics on a simplex or a concentration simplex is free from all such drawbacks. Every such study of generalized replicator system on a 2- sphere with boundary [4] was free from those drawbacks. Incidentally it may be noted that the standard sufficient conditions for permanence of replicator system as obtained in [5,6,9], were taken up in [7] to obtain an equivalent sufficient criteria for obtaining solution of a linear programming problem. The necessary condition was justified only numerically and this was possible as the linear programming problem could be solved by standard algorithms. Looking back to obtaining equivalent sufficient criteria for the existence of permanence of a generalized replicator system given in [4], the first observation is that it is not a standard nonlinear optimization problem, not to speak of a linear optimization problem. Rather it is a **vector minimization problem**. So the present problem was studied with the help of the technique of obtaining solution for vector minimization problem.

We divide the whole matter of this paper into four different articles. Article- 1 gives some known definitions and results of replicator systems. Article-2 discusses a permanence criteria of generalized replicator system on a 2- sphere with boundary. Article-3 discusses interior equilibrium point of replicator dynamics on a 2-sphere with boundary and hence finds three equilibrium points. Article-4 discusses an equivalent vector optimization (minimization) problem and gives an alternative criteria of permanence of a generalized replicator system on a 2- sphere with boundary through vector minimization technique.

1. Some known Definitions and Results [4,5,6,9]

Definition 1.1 [5] Let a dynamical system on

$S_n = \{x = (x_1, x_2, ..., x_n) \in R^n : \sum_{i=1}^{n} x_i = 1, x_i \geq 0, \forall i\}$ be given by

$$(I) \qquad \dot{x}_i = x_i[f_i(x) - \bar{f}]$$

where $\bar{f} = \sum_{i=1}^{n} x_i f_i(x)$. (I) is called a replicator equation. It leaves the boundary and all faces of S_n invariant.
In general, the function $f_i(x)$ is linear in x_i for $i = 1, 2, ..., n$.

Definition 1.2 [5] The dynamical system (I) on S_n is called permanent, if bdS_n is a repeller i.e. there exists $M > 0$ such that if $x_i(0) > 0$, $\forall i$, then $x_i(t) > M$, for all sufficiently large t and $\forall\ i = 1, 2, ..., n$.

Definition 1.3 [5] Let $x(t)$ be the orbit of (I) with $x(0) = x$. Let $P : S_n \rightarrow R$ be a differentiable function on S_n which satisfies
(1) $\qquad P(x) = 0 \qquad \forall x \in bdS_n$
(2) $\qquad P(x) > 0 \qquad \forall x \in intS_n$
and further
(3) $\qquad \dot{P} = P.\Psi(x)$
where Ψ is continuous function on S_n. Such a function P is called an average Lyapunov function on S_n.

Result 1.1 [5,6] The replicator equation (I) on S_n is permanent if there exists a differentiable function $P : S_n \rightarrow R$ such that the following two conditions hold:
(i) For $x \in S_n$, $\frac{\dot{P}(x)}{P(x)} = \Psi(x)$
(ii) For $x \in bdS_n$, $\frac{1}{T}\int_0^T \Psi(x(t))dt > 0$ for some $T > 0$.

Remark 1.1 [5] If S_n is replaced by any compact set X and bdS_n is a closed invariant subset A of X, then also the Result 1.1 holds good. This shows that the Result 1.1 is a general result which is applicable to any compact set with boundary kept invariant under the flow of the given system.

An alternative sufficient condition of permanence of replicator equation is given as follows:

Result 1.2 [5] If P satisfies (1), (2), (3) and if every orbit on the boundary of S_n converges to a fixed point and if $\Psi(x) > 0$ for all such fixed points on the boundary, then (I) is permanent.

Remark 1.2 That the condition of result 1.1 ensures the permanence may be explained as follows:
In fact, this condition implies that $\Psi > 0$ on bdS_n. So $\dot{P}(x) > 0$ in $intS_n$. This means that $P(x)$ is an increasing function. As $P(x)$ denotes

the distance from the boundary, so the orbit will be repelled from the boundary. In otherwords, the system becomes permanent.

Definition 1.4 [9] Let

$S_n^c = \{x = (x_1, x_2, ..., x_n) \in R^n : \sum_{i=1}^{n} x_i = c, x_i \geq 0 \; for \; 1 \leq i \leq n\}$. It is called the concentration simplex.

The dynamics on S_n^c is given by the differential equations

$(II) \qquad \dot{x}_i = x_i[q_i + \sum_{j=1}^{n} k_{ij}x_j - \frac{\phi}{c}]$

where $c > 0$, q_i and $k_{ij} \in R$ and $\phi = \sum_{i=1}^{n} x_i(q_i + \sum_{j=1}^{n} k_{ij}x_j)$. The x_i represents the concentration of the chemical or biological species i and $q_i \in R$ corresponds to the selfreproduction or decay of the species i and $k_{ij}x_j$ represents the effect of the species j on the reproduction of species i. This is of mass action type, it is catalytic if $k_{ij} > 0$ and is inhibiting if $k_{ij} < 0$. (II) is called a replicator system on S_n^c; if it keeps the boundaries and faces of S_n^c invariant.

Remark 1.3 The particular case of (II) is the **inhomogeneous hypercycle** given by

$(III) \qquad \dot{x}_i = x_i[q_i + k_i x_{i-1} - \Phi]$

$k_i > 0, i = 1, 2, ..., n$. Here the species 1 acts catalytically on 2, 2 on 3, 3 on 4 and so on. This equation was studied in [9]. The following result was obtained there: The system is cooperative if the selfreplication terms q_i are sufficiently small i.e. it is only slight perturbations of the homogeneous hypercycle (all $q_i = 0$) for which the cooperation/permanence was shown in [10]. On the other hand (III) and more general systems can be permanent/cooperative if there exists a unique inner equilibrium [9]. Moreover

Result 1.3[5] The inhomogeneous hypercycle (III) is cooperative if and only if there exists a fixed point in $intS_n$.

2. Permanence Criteria of a generalized replicator system on a 2- sphere with boundary

Let $S_+^2 = \{x = (x_1, x_2, x_3) \in R^3 : x_1^2 + x_2^2 + x_3^2 = c^2, x_3 > 0\}$, a sphere of radius c. Let us take a horizontal section by a hyperplane

$\Gamma = \{x \in S^2_+ : x_3 = k, 0 < k < c\}$ in the upper part of S^2_+. Then the 2-sphere with boundary is the subset of S^2_+ obtained by the above section together with small circle $x_1^2 + x_2^2 = c^2 - k^2$, $x_3 = k$. It is denoted by G. In fact, G is a compact subset of R^3, where the boundary of G is given by $B = \{x \in G : x_1^2 + x_2^2 = c^2 - k^2, x_3 = k\}$. For practical purposes the positive half G_1 of G is considered where $G_1 = \{x \in G : x_1 \geq 0, x_2 \geq 0, x_3 = k\}$. Now G_1 has three boundaries namely;

$$
\begin{aligned}
B_1 =& \{x \in B : x_1 = \sqrt{c^2 - k^2} cos\theta, x_2 = \sqrt{c^2 - k^2} sin\theta, x_3 = k\} \\
B_2 =& \{x \in B : x_1 = 0, x_2 = c\, cos\theta, x_3 = c\, sin\theta\} \\
B_3 =& \{x \in B : x_1 = c\, cos\theta, x_2 = 0, x_3 = c\, sin\theta\}
\end{aligned}
$$

where $0 < \theta < \frac{\pi}{2}$. The boundary B_1 is a small semicircle of radius $\sqrt{c^2 - k^2}$ and the boundaries B_2 and B_3 are two curves on the coordinate planes. Then the system of differential equations defined on G_1 is called a **generalized replicator system** [4] if it keeps the boundaries B_1, B_2 and B_3 of G_1 invariant.

The differential equations defined on the compact set G_1 of S^2_+ with boundary B_1 is given by

(IV)
$$
\begin{aligned}
\dot{x}_1 &= (x_1 - l_1)^2[q_1 + k_1 x_2^2 + m_1 x_3^2 - \tfrac{\Phi_1}{c}] \\
\dot{x}_2 &= (x_2 - l_2)^2[q_2 + k_2 x_3^2 + m_2 x_1^2 - \tfrac{\Phi_1}{c}] \\
\dot{x}_3 &= (x_3 - k)^2[q_3 + k_3 x_1^2 + m_3 x_2^2 - \tfrac{\Phi_1}{c}]
\end{aligned}
$$

where $\Phi_1 = \Phi_1(x) = (x_1 - l_1)^2[k_1 x_2^2 + m_1 x_3^2] + (x_2 - l_2)^2[k_2 x_3^2 + m_2 x_1^2] + (x_3 - k)^2[k_3 x_1^2 + m_3 x_2^2]$, $k_i, m_i > 0$ and $(l_1 = \sqrt{c^2 - k^2} cos\theta, l_2 = \sqrt{c^2 - k^2} sin\theta, k)$ is any point on B_1 $(0 < \theta < \frac{\pi}{2})$.

Similarly the differential equations defined on the compact set G_1 of S^2_+ with boundaries B_2 and B_3 are given by

(V)
$$
\begin{aligned}
\dot{x}_1 &= x_1^2[q_1 + k_1 x_2^2 + m_1 x_3^2 - \tfrac{\Phi_2}{c}] \\
\dot{x}_2 &= (x_2 - c\, cos\theta)^2[q_2 + k_2 x_3^2 + m_2 x_1^2 - \tfrac{\Phi_2}{c}] \\
\dot{x}_3 &= (x_3 - c\, sin\theta)^2[q_3 + k_3 x_1^2 + m_3 x_2^2 - \tfrac{\Phi_2}{c}]
\end{aligned}
$$

where $\Phi_2 = \Phi_2(x) = x_1^2[k_1 x_2^2 + m_1 x_3^2] + (x_2 - c\, cos\theta)^2[k_2 x_3^2 + m_2 x_1^2] + (x_3 - c\, sin\theta)^2[k_3 x_1^2 + m_3 x_2^2]$, $k_i > 0$, $m_i > 0$ and $(0, c\, cos\theta, c\, sin\theta)$ is any point on B_2 $(0 < \theta < \frac{\pi}{2})$.

$$(VI) \qquad \begin{aligned} \dot{x}_1 &= (x_1 - c\,cos\theta)^2[q_1 + k_1x_2^2 + m_1x_3^2 - \tfrac{\Phi_3}{c}] \\ \dot{x}_2 &= x_2^2[q_2 + k_2x_3^2 + m_2x_1^2 - \tfrac{\Phi_3}{c}] \\ \dot{x}_3 &= (x_3 - c\,sin\theta)^2[q_3 + k_3x_1^2 + m_3x_2^2 - \tfrac{\Phi_3}{c}] \end{aligned}$$

where $\Phi_3 = \Phi_3(x) = (x_1 - c\,cos\theta)^2[k_1x_2^2 + m_1x_3^2] + x_2^2[k_2x_3^2 + m_2x_1^2] + (x_3 - c\,sin\theta)^2[k_3x_1^2 + m_3x_2^2]$, $k_i > 0$, $m_i > 0$ and $(c\,cos\theta, 0, c\,sin\theta)$ is any point on B_3 $(0 < \theta < \frac{\pi}{2})$.

Result 2.1:[4] Permanence theorem of replicator system on a 2 manifold with boundary:
Let a differential equation defined on the compact set G_1 of S_+^2 with boundaries B_1, B_2 and B_3 be given by (IV), (V) and (VI).
If (IV), (V) and (VI) are permanent then there exists an equilibrium point in $intG_1$ with respect to the boundaries B_1, B_2 and B_3. Conversely, if there exists an equilibrium point of (IV), (V) and (VI) in $intG_1$ and if
$\dfrac{1}{m_1} + \dfrac{1}{k_3} = \dfrac{1}{m_2} + \dfrac{1}{k_1} = \dfrac{1}{m_3} + \dfrac{1}{k_2} = A$, where A is any arbitrary positive constant, then (IV), (V) and (VI) are permanent.

We set the average lyapunov function with respect to the boundaries B_i $(i = 1, 2, 3)$ as follows:
$P_i(x) = e^{-\frac{1}{m_1k_1(x_1 - l_1^i)}} e^{-\frac{1}{m_2k_2(x_2 - l_2^i)}} e^{-\frac{1}{m_3k_3(x_3 - l_3^i)}}$ for $(i = 1, 2, 3)$,

where $(l_1^1 = \sqrt{c^2 - k^2}cos\theta, l_2^1 = \sqrt{c^2 - k^2}sin\theta, l_3^1 = k)$, $(l_1^2 = 0, l_2^2 = c\cos\theta, l_3^2 = c\sin\theta)$ and $(l_1^3 = c\cos\theta, l_2^3 = 0, l_3^3 = c\sin\theta)$ $(0 < \theta < \frac{\pi}{2})$ being any points on B_1, B_2 and B_3 respectively.

Clearly P_i satisfies

(i) $\qquad P_i(x) = 0 \qquad \forall x \in B_i$, for $i = 1, 2, 3$

(ii) $\qquad P_i(x) > 0 \qquad \forall x \in intG_1$. Thus

$$\Psi_i(x) = \frac{\dot{P}_i(x)}{P_i(x)} = (\tfrac{1}{m_3k_3} + \tfrac{1}{m_1k_1} + \tfrac{1}{m_2k_2})(\tfrac{\Phi_i(p)}{c} - \tfrac{\Phi_i}{c})$$

provided $\dfrac{1}{m_1} + \dfrac{1}{k_3} = \dfrac{1}{m_2} + \dfrac{1}{k_1} = \dfrac{1}{m_3} + \dfrac{1}{k_2} = A$ where A is any arbitrary positive constant. Further $\Psi_i(x)$ is continuous in G_1.
Moreover $\Psi_i(x) = (\frac{1}{m_3k_3} + \frac{1}{m_1k_1} + \frac{1}{m_2k_2})(\frac{\Phi_i(p)}{c} - \frac{\Phi_i}{c}) > 0$ for all steady states on B_1, B_2 and B_3 respectively and Φ_i $(i = 1, 2, 3)$ is defined as in (IV), (V) and (VI) respectively.

Remark 2.1[4] With the help of result 2.1 and remark 1.2 we state that:

If $\frac{1}{m_1}+\frac{1}{k_3} = \frac{1}{m_2}+\frac{1}{k_1} = \frac{1}{m_3}+\frac{1}{k_2} = A$ where A is any arbitrary positive constant and $\Psi_i(x) > 0$ for all steady states on B_i $(i = 1, 2, 3)$ respectively then the boundaries B_i (for $i = 1, 2, 3$) of G_1 are repellers.

3. Interior equilibrium point of a generalized Replicator Dynamics on a 2-sphere with boundary

The rest points of (IV) with respect to the boundary B_1 is respectively given by the system of equations

$$(\alpha) \qquad \begin{aligned} q_1 + k_1y_2 + m_1y_3 - \tfrac{\Phi_1}{c} &= 0 \\ q_2 + k_2y_3 + m_2y_1 - \tfrac{\Phi_1}{c} &= 0 \\ q_3 + k_3y_1 + m_3y_2 - \tfrac{\Phi_1}{c} &= 0 \\ y_1 + y_2 + y_3 &= c^2 \end{aligned}$$

which is linear in y_1, y_2 and y_3 respectively where $y_1 = x_1^2, y_2 = x_2^2, y_3 = x_3^2$.
Subtracting first from second and second from third (α) reduces to

$$(\beta) \qquad \begin{aligned} m_2y_1 - k_1y_2 + (k_2 - m_1)y_3 &= q_1 - q_2 \\ (m_2 - k_3)y_1 - m_3y_2 + k_2y_3 &= q_3 - q_2 \\ y_1 + y_2 + y_3 &= c^2 \end{aligned}$$

(β) can be written as $QY = R$ where

$$Q = \begin{pmatrix} m_2 & -k_1 & k_2 - m_1 \\ m_2 - k_3 & -m_3 & k_2 \\ 1 & 1 & 1 \end{pmatrix}, \; Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \text{ and } R = \begin{pmatrix} q_1 - q_2 \\ q_3 - q_2 \\ c^2 \end{pmatrix}.$$

After row operations the augmented matrix $(Q|R) = Z$, say becomes

$$Z = \begin{pmatrix} m_2 & -k_1 & k_2 - m_1 & q_1 - q_2 \\ 0 & \frac{(k_1-m_3)m_2-k_1k_3}{m_2} & m_1 + \frac{k_2-m_1}{m_2} & q_3 - q_1 + \frac{(q_1-q_2)k_3}{m_2} \\ 0 & 0 & \frac{Em_2^2+Fm_2+G}{m_2[(k_1-m_3)m_2-k_1k_3]} & \frac{Hm_2+I}{m_2[(k_1-m_3)m_2-k_1k_3]} \end{pmatrix}$$

where $E = k_1 - m_3 - m_1$, $F = (k_2 - m_1)(m_3 - k_1 - 1) - k_1(m_1 + k_3)$, $G = k_1(k_2 - m_1)(k_3 - 1)$, $H = (k_1 - m_3)c^2 + (q_1 - q_3)$, $I = (q_2 - q_1)(k_1 - m_3 + k_3) - k_1(k_3c^2 + q_3 - q_1)$.

The above system of equations (β) has
(i) unique solution (i.e rank of the augmented matrix $(Q|R) = 3 =$ rank

of Q) if

$$Em_2^2 + Fm_2 + G \neq 0$$

(ii) no solution (i.e. rank of $(Q|R) = 3 \neq 2 =$ rank of Q) if

$$Em_2^2 + Fm_2 + G = 0$$

$$Hm_2 + I \neq 0$$

(iii) infinite no. of solution (i.e rank of $(Q|R) = 2 =$ rank of Q) if

$$Em_2^2 + Fm_2 + G = 0$$

$$Hm_2 + I = 0$$

If (β) has more than one solutions then the given system is equivalent to

$\frac{(k_1-m_3)m_2-k_1k_3}{m_2}y_2 + \frac{m_1m_2+k_2-m_1}{m_2}y_3 = \frac{(q_3-q_1)m_2+(q_1-q_2)k_3}{m_2}$ (1)

$m_2y_1 - k_1y_2 + (k_2 - m_1)y_3 = q_1 - q_2$ (2)

Since $m_2 > 0$ solving (1) and (2), we get,

$y_2 = c_1 + d_1y_3$ (3)

$y_1 = a_1 + b_1y_3$ (4)

where $c_1 = \frac{(q_3-q_1)m_2+(q_1-q_2)k_3}{(k_1-m_3)m_2-k_1k_3}$, $d_1 = -\frac{(k_2-m_1)-m_1m_2}{(k_1-m_3)m_2-k_1k_3}$,
$a_1 = \frac{q_1-q_2+k_1c_1}{m_2}$, $b_1 = \frac{k_1d_1-(k_2-m_1)}{m_2}$.

Thus y_2 and y_1 are both expressed in terms of y_3, so any solution y_1, y_2, y_3 must also satisfy

$q_1 + k_1y_2 + m_1y_3 = \frac{\Phi_1}{c}$ (5)

where $\frac{\Phi_1}{c} = \frac{(x_1-l_1)^2[k_1x_2^2+m_1x_3^2]+(x_2-l_2)^2[k_2x_3^2+m_2x_1^2]+(x_3-k)^2[k_3x_1^2+m_3x_2^2]}{c}$, where l_1, l_2 given in (IV). As $\frac{\Phi_1}{c}$ is also ultimately expressed in terms of y_1, y_2, y_3, so by the above result it is also found to be expressed in terms of y_3 only. Hence from the above equality y_3 can be found out expressed in terms of the parameters $m_1, m_2, m_3, k_1, k_2, k_3$. With this value of y_3, the values of y_1 and y_2 can be calculated from their aforesaid relations. This gives rise to equilibrium points like $P^1 = (p_1^1, p_2^1, p_3^1) \in intG_1$ for the system (IV) with respect to the boundary B_1.

For the equilibrium point of the system (V), equations (3) and (4) will remain the same but the final equation to be solved is

$q_1 + k_1y_2 + m_1y_3 = \frac{\Phi_2}{c}$ (6)

where $\frac{\Phi_2}{c} = \frac{x_1^2[k_1x_2^2+m_1x_3^2]+(x_2-c\ cos\theta)^2[k_2x_3^2+m_2x_1^2]+(x_3-c\ sin\theta)^2[k_3x_1^2+m_3x_2^2]}{c}$. The solution gives rise to equilibrium points like $P^2 = (p_1^2, p_2^2, p_3^2) \in intG_1$ for the system (V) with respect to the boundary B_2.

Lastly by solving $(3), (4)$ and

$$q_1 + k_1y_2 + m_1y_3 = \frac{\Phi_3}{c} \qquad (7)$$

where $\frac{\Phi_3}{c} = \frac{(x_1-c\ cos\theta)^2[k_1x_2^2+m_1x_3^2]+x_2^2[k_2x_3^2+m_2x_1^2]+(x_3-c\ sin\theta)^2[k_3x_1^2+m_3x_2^2]}{c}$, we get the equilibrium points like $P^3 = (p_1^3, p_2^3, p_3^3) \in intG_1$ for the system (VI) with respect to the boundary B_3.

4. An alternative way of looking at the condition of permanence of a generalized replicator system on a 2-sphere with boundary

In this connection we require the notion of vector optimization problem and its nature of solution. So we state below the necessary type of problem and its solution which will be needed for our present problem.

Vector Optimization problem and its solution[1]

Let us consider the vector maximization problem (P) Max $\chi(s) = (\chi_1(s), \chi_2(s), \chi_3(s), ..., \chi_p(s))$ subject to $s \in X$, where
$X = \{s \in R^n : g(s) \leq 0, h(s) = 0\}$
where $g = (g_1, g_2, g_3, ..., g_m)$ and $h = (h_1, h_2, h_3, ..., h_r)$ are m- dimensional and r- dimensional vector valued function on R^n.

Some Known Definitions and Results [1]

Definition 4.1: s^0 is said to be an efficient solution of (P) if $s^0 \in X$ and $\chi_i(s) > \chi_i(s^0)$ for some $s \in X$ and some $i \in J$, where $J = \{1, 2, 3, ..., p\}$ implies that there exists at least one $j \in J - \{i\}$, such that $\chi_j(s) < \chi_j(s^0)$.

Definition 4.2: s^0 is said to be a properly efficient solution of (P) when it is efficient for (P) and there exists a scalar $M > 0$ such that for each $i \in J$ and each $s \in X$ satisfying $\chi_i(s) > \chi_i(s^0)$, there exists at least one $j \in J - \{i\}$ with $\chi_j(s) < \chi_j(s^0)$ and $\frac{[\chi_i(s)-\chi_i(s^0)]}{[\chi_j(s^0)-\chi_j(s)]} \leq M$.

Definition 4.3: s^0 is said to be a k^{th} entry efficient solution of (P) where $k \in J$, if $s^0 \in X$ and $\chi_k(s) > \chi_k(s^0)$ for some $s \in X$ implies that there exists at least one $j \in J - \{k\}$, such that $\chi_j(s) < \chi_j(s^0)$.

Definition 4.4: s^0 is said to be a properly k^{th} entry efficient solution of (P), where $k \in J$, when it is k^{th} entry efficient for (P) and there exists a scalar $M_k > 0$ such that for each $s \in X$ satisfying $\chi_k(s) > \chi_k(s^0)$, there exists at least one $j \in J - \{k\}$ with $\chi_j(s) < \chi_j(s^0)$ and $\frac{[\chi_k(s) - \chi_k(s^0)]}{[\chi_j(s^0) - \chi_j(s)]} \leq M_k$.

From the above definitions the following results can be verified:

Result 4.1: A point s^0 is an efficient solution of (P) if and only if it is a k^{th} entry efficient solution of (P) for each $k \in J$.

Result 4.2: A point s^0 is a properly efficient solution of (P) if and only if it is a properly k^{th} entry efficient solution of (P) for each $k \in J$.

Scalar maximization problem corresponding to (P)

$$(P_u) \quad \max_{s \in X} u_1\chi_1(s) + u_2\chi_2(s) + u_3\chi_3(s) + ... + u_p\chi_p(s)$$

and $u = (u_1, u_2, ..., u_p) \in R_p^+$. Let X_u^* denote the set of maximum solutions for (P_u). Then the following results are true:

Result 4.3: If $s^0 \in X_u^*$ for some $(u_1, u_2, u_3, ..., u_p) = (u_1^0, u_2^0, u_3^0, ..., u_p^0) \in R_+^p$, then s^0 is a properly k^{th} entry efficient solution for (P).

Vector minimization problem

(P_1) $\min \chi(s) = (\chi_1(s), \chi_2(s), \chi_3(s), ..., \chi_p(s))$ subject to $s \in X$, where $X = \{s \in R^n : g(s) \leq 0, h(s) = 0\}$, $g = (g_1, g_2, g_3, ..., g_m)$ and $h = (h_1, h_2, h_3, ..., h_r)$ are m- dimensional and r- dimensional vector valued function on R^n.

For this problem definitions 4.1 to 4.4 may be similarly given by replacing maximum efficient and properly efficient solutions by minimum efficient and properly efficient solutions.

Scalar minimization problem

(P_{1u}) $\min_{s \in X} \Lambda(s)$ where
$\Lambda(s) = [u_1\chi_1(s) + u_2\chi_2(s) + u_3\chi_3(s) + ... + u_p\chi_p(s)]$ and $u = (u_1, u_2, ..., u_p) \in R_p^+$, $s \in X = \{s \in R^n : g(s) \leq 0, h(s) = 0\}$.

Let X_{1u}^* denote the set of optimal solutions for (P_{1u}). Then we have the following result:

Result 4.4: If $s^0 \in X_{1u}^*$ for some $(u_1, u_2, u_3, ..., u_p) = (u_1^0, u_2^0, u_3^0, ..., u_p^0) \in R_+^p$, then s^0 is a properly k^{th} efficient solution for (P_1).

For solution of (P_{1u}) we recollect Kuhn-Tucker Saddle Point Problem (KTSP) given as follows:

Kuhn Tucker Saddle point problem:

Let $X = \{s : s \in X, g(s) \leq 0, h(s) = 0\}$, where $g = (g_1, g_2, g_3, ..., g_m)$ and $h = (h_1, h_2, h_3, ..., h_r)$ are m- dimensional and r- dimensional vector functions on $X \subset R^n$ and $\Lambda(s)$ is real valued function on R^n.
The problem is to find $\bar{s} \in X$, $\bar{\lambda} \in R^m$, $\bar{\lambda} \geq 0$, $\bar{\mu} \in R^r$ such that

$$\Theta(\bar{s}, \lambda, \mu) \leq \Theta(\bar{s}, \bar{\lambda}, \bar{\mu}) \leq \Theta(s, \bar{\lambda}, \bar{\mu})$$

for all $\lambda \geq 0$, $\lambda \in R^m$, all $\mu \in R^r$ where

$$\Theta(s, \lambda, \mu) = \Lambda(s) + \lambda g(s) + \mu h(s)$$

In this case $(\bar{s}, \bar{\lambda}, \bar{\mu})$ is called a solution of KTSP.
By sufficient optimality theorem [8], the following result holds:

Result 4.5 If $(\bar{s}, \bar{\lambda}, \bar{\mu})$ is a solution of KTSP then $\bar{s}$ is the solution of (P_{1u}).

Let $\Theta(s, \lambda, \mu) = \Lambda(s) + \lambda g(s) + \mu h(s)$ for $s \in X^0$, $\lambda \in R^m$, $\mu \in R^r$. If there exists $\bar{s} \in X$, $\bar{\lambda} \in R^m$, $\bar{\lambda} \geq 0$, $\bar{\mu} \in R^r$ such that

$$\Theta(\bar{s}, \lambda, \mu) \leq \Theta(\bar{s}, \bar{\lambda}, \bar{\mu}) \leq \Theta(s, \bar{\lambda}, \bar{\mu})$$

for all $\lambda \geq 0$, $\lambda \in R^m$, all $\mu \in R^r$ and all $s \in X$, then $(\bar{s}, \bar{\lambda}, \bar{\mu})$ is called a solution of KTSP.

Remark 4.2 To solve (P_1), it is enough to solve (P_{1u}) and hence finally to solve KTSP corresponding to (P_{1u}).

Alternative criteria of permanence of a generalized replicator system on a 2- sphere with boundary

We note that the sufficient criteria of permanence of a generalized replicator system on a 2- sphere with boundary practically needs $\Psi_i(x) > 0$ where $\Psi_i(x) = [\frac{1}{m_1k_1} + \frac{1}{m_2k_2} + \frac{1}{m_3k_3}][\frac{\Phi_i(P^i)}{c} - \frac{\Phi_i(x)}{c}]$ for $i = 1, 2, 3$. Now, let us take $i = 1, 2, 3$ separately. Even for the boundary B_1, the point P^1 depends mainly on the parameters $m_1, m_2, m_3, k_1, k_2, k_3$; so for only $\Psi_1(x) > 0$ the problem is a programming problem in the sense that the positivity is to be judged over the choice of $(m_1, m_2, m_3, k_1, k_2, k_3) \in R^6$. In fact, if we can show that $\min\{\Phi_1(P^1) - \Phi_1(x)\} > 0$ for P^1 corresponding to one set of values of the parameters then $\Phi_1(P^1) - \Phi_1(x) > 0$ for all possible values of $(m_1, m_2, m_3, k_1, k_2, k_3) \in R^6$. The same judgement is also to be made for $i = 2, 3$. Naturally looking at the feasibility for $\Psi_i(x) > 0$ we are to depend on two types of choices, one on the choice of Ψ_i $(i = 1, 2, 3)$ and also on the choice of the point $(m_1, m_2, m_3, k_1, k_2, k_3) \in R^6$. This means that the criteria of permanence of our system is equivalent to the solution of a **vector minimization problem**.

In this connection we prove the following main theorem.

Theorem 4.1 Result 2.1 giving the sufficient condition of permanence of generalized replicator system given by (IV), (V) and (VI) with respect to the boundaries B_i $(i = 1, 2, 3)$ reduces to the sufficient conditions for the existence of properly efficient solutions of the following vector minimization problem:
$(P_{\min})$ $\min \Psi(s) = (\Psi_1(s), \Psi_2(s), \Psi_3(s))$ subject to $s \in X$, $s \geq 0$, where $X = \{s = (m_1, m_2, m_3, k_1, k_2, k_3) \in R^6 : g(s) \leq 0, h(s) = 0\}$, $g = (g_1, g_2, g_3)$ and $h = (h_1, h_2, h_3)$ are 3- dimensional vector valued function on R^6 for suitable functions g and h.

Proof:
For $s = (m_1, m_2, m_3, k_1, k_2, k_3) \in R^6$, we have

$$\Psi_1(s) = [\frac{1}{m_1k_1} + \frac{1}{m_2k_2} + \frac{1}{m_3k_3}][\frac{\Phi_1(P^1)}{c} - \frac{\Phi_1(x)}{c}]$$

where $P^1 = (p_1^1, p_2^1, p_3^1)$ is obtained as solution of (3), (4) and (5).

Similarly, $\Psi_2(s) = [\frac{1}{m_1k_1} + \frac{1}{m_2k_2} + \frac{1}{m_3k_3}][\frac{\Phi_2(P^2)}{c} - \frac{\Phi_2(x)}{c}]$
where $P^2 = (p_1^2, p_2^2, p_3^2)$ is obtained as solution of (3), (4) and (6) and

$\Psi_3(s) = [\frac{1}{m_1k_1} + \frac{1}{m_2k_2} + \frac{1}{m_3k_3}][\frac{\Phi_3(P^3)}{c} - \frac{\Phi_3(x)}{c}]$
where $P^3 = (p_1^3, p_2^3, p_3^3)$ is obtained as solution of (3), (4) and (7).

Thus we have the vector minimization problem given as follows:

$(P_{\min})$ $\min \Psi(s) = (\Psi_1(s), \Psi_2(s), \Psi_3(s))$ subject to $s \in X$, $s \geq 0$, where $X = \{s = (m_1, m_2, m_3, k_1, k_2, k_3) \in R^6 : g(s) \leq 0, h(s) = 0\}$, $g = (g_1, g_2, g_3)$ and $h = (h_1, h_2, h_3)$ are 3- dimensional vector valued function on R^6 given by

$g_1(s) = (p_1^1)^2 + (p_2^1)^2 + (p_3^1)^2 - (c^2 - k^2)$
$g_2(s) = (p_1^2)^2 + (p_2^2)^2 + (x_p^2)^2 - (c^2 - k^2)$
$g_3(s) = (p_1^3)^2 + (p_2^3)^2 + (p_3^3)^2 - (c^2 - k^2)$
$h_1(s) = \frac{1}{m_1} + \frac{1}{k_3} - A$
$h_2(s) = \frac{1}{m_2} + \frac{1}{k_1} - A$
$h_3(s) = \frac{1}{m_3} + \frac{1}{k_2} - A$

In order to find a properly efficient solution to this minimization problem let us consider the scalar programming problem
$(P_{\min 1})$ $\min_{s\in X}[u_1\Psi_1(s) + u_2\Psi_2(s) + u_3\Psi_3(s)]$ subject to $s \in X$.
Then by result 4.3, s^0 is a properly k^{th} entry efficient solution for $(P_{\min})$ if $s^0 \in X_u^*$ for some $(u_1, u_2, u_3) = (u_1^0, u_2^0, u_3^0) \in R_+^3$, where X_u^* denotes the optimal solution of $(P_{\min 1})$. If we now denote
$\Lambda(s) = [u_1\Psi_1(s) + u_2\Psi_2(s) + u_3\Psi_3(s)]$, then our problem $(P_{\min 1})$ reduces to the following Kuhn Tucker saddle Point Problem:
To find $\bar{s} \in X$, $\bar{\lambda} \in R^3$, $\bar{\lambda} \geq 0$, $\bar{\mu} \in R^3$ such that

$$\Theta(\bar{s}, \lambda, \mu) \leq \Theta(\bar{s}, \bar{\lambda}, \bar{\mu}) \leq \Theta(s, \bar{\lambda}, \bar{\mu})$$

for all $\lambda \geq 0$, $\lambda \in R^3$, all $\mu \in R^3$ and all $s \in X^0$, where

$$\Theta(s, \lambda, \mu) = \Lambda(s) + \lambda g(s) + \mu h(s).$$

So if $(\bar{s}, \bar{\lambda}, \bar{\mu})$ of this KTSP exists then $\bar{s}$ solves $(P_{\min 1})$.

Consequently, by result 4.3 $\bar{s}$ is a k^{th} entry properly efficient solution of $(P_{\min})$. Again by result 4.2, $\bar{s}$ is a properly efficient solution of $(P_{\min})$. This completes the proof of the theorem. □

Discussion:

The sufficient criteria for replicator system on a n- simplex has an alternative form given by solutions of a linear programming problem. This is quite natural as the surface of a n- simplex is a plane surface. Obviously for replicator system on a curve surface the alternative criteria of permanence may be given (if possible) by a non-linear programming problem only. In our case, the most interesting feature is that the problem is a **vector nonlinear minimization problem**.

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